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# ON THE LOCATION OF ZEROS OF A POLYNOMIAL

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**Abstract.** In this paper we extend Enestrom -Kakeya theorem to a large class of polynomials with complex coefficients by putting less restrictions on the coefficients. Our results generalise and extend many known results in this direction.

**Key words:** *polynomial, zero, bound*

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## 1 Introduction and Statement of Results

Let $P(z)$ be a polynomial of degree $n$. A classical result due to Enestrom and Kakeya[8] concerning the bound for the moduli of the zeros of polynomials having positive coefficients is often stated as in the following theorem(see [8]) :

**Theorem A (Enestrom-Kakeya).** *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n whose coefficients satisfy*

$$0 \le a_1 \le a_2 \le \dots \le a_n.$$

*Then P(z) has all its zeros in the closed unit disk* $|z| \le 1$.

In the literature there exist several generalisations of this result (see [1], [3], [4], [7], [8]). Recently Aziz and Zargar[2] relaxed the hypothesis in several ways and proved:

**Theorem B.** *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n such that for some* $k \ge 1$

$$ka_n \ge a_{n-1} \ge \dots \ge a_0.$$

*Then all the zeros of* $P(z)$ *lie in*

$$|z+k-1| \le \frac{ka_n + |a_0| - a_0}{|a_n|}.$$

For ploynomials, whose coeffiencents are not necessarily real, Govil and Rehman[6] proved the following generalisation of Theorem A:

**Theorem C**. *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$ *be a polynomial of degree n with* $\mathrm{Re}(a_j)=\alpha_j$ *and* $\mathrm{Im}(a_j)=\beta_j,\ j=0,\ 1,\ \cdots,\ n$*, such that*

$$\alpha_n\geq\alpha_{n-1}\geq\cdots\geq\alpha_0\geq 0,$$

*where* $\alpha_n>0$*, then* $P(z)$ *has all its zeros in*

$$|z|\leq 1+\left(\frac{2}{\alpha_n}\right)\left(\sum_{j=0}^{n}|\beta_j|\right).$$

More recently, Govil and Mc-tume[5] proved the follwoing generalisations of Theorems B and C:

**Theorem D.** *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$ *be a polynomial of degree n with* $\mathrm{Re}(a_j)=\alpha_j$ *and* $\mathrm{Im}(a_j)=\beta_j,\ j=0,\ 1,\ \cdots,\ n$*. If for some* $k\geq 1$*,*

$$k\alpha_n\geq\alpha_{n-1}\geq\cdots\geq\alpha_0,$$

*then* $P(z)$ *has all its zeros in*

$$|z+k-1|\leq\frac{k\alpha_n-\alpha_0+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

**Theorem E**. *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$ *be a polynomial of degree n with* $\mathrm{Re}(a_j)=\alpha_j$ *and* $\mathrm{Im}(a_j)=\beta_j,\ j=0,\ 1,\ \cdots,\ n$*. If for some* $k\geq 1$*,*

$$k\beta_n\geq\beta_{n-1}\geq\cdots\geq\beta_0,$$

*then* $P(z)$ *has all its zeros in*

$$|z+k-1|\leq\frac{k\beta_n-\beta_0+|\beta_0|+2\sum_{j=0}^{n}|\alpha_j|}{|\beta_n|}.$$

In this paper we shall present some interesting generalizations of Theorems D and E and consequently of Enestrom-Kakeya Theorem. Our first result in this direction is the following:

**Theorem 1**. *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with* $\operatorname{Re}(a_j)=\alpha_j$ *and* $\operatorname{Im}(a_j)=\beta_j,\ j=0,\ 1,\ \cdots,\ n$. *If for some* $\rho\geq 0$,

$$\rho+\alpha_n\geq\alpha_{n-1}\geq\cdots\geq\alpha_0,$$

*then* $P(z)$ *has all its zeros in the disk*

$$\left|z+\frac{\rho}{\alpha_n}\right|\leq\frac{\rho+\alpha_n-\alpha_0+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

*Remark* 1. Taking $\rho=(k-1)\alpha_n$, Theorem 1 reduces to Thoerem D. Theorem C is a special case of theorem 1. To see this we take $\rho=0$, $\alpha_0>0$.

The following corollary is obtained by taking $\rho=\alpha_{n-1}-\alpha_n$ and $\alpha_0\geq 0$ in Thoerem 1.

**Corollary 1.** *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with* $\operatorname{Re}(a_j)=\alpha_j$ *and* $\operatorname{Im}(a_j)=\beta_j,\ j=0,\ 1,\ \cdots,\ n$. *If for some* $k\geq 1$,

$$\alpha_n\geq\alpha_{n-1}\geq\cdots\geq\alpha_0>0,$$

*then* $P(z)$ *has all its zeros in*

$$\left|z+\frac{\alpha_{n-1}}{\alpha_n}-1\right|\leq\frac{\alpha_{n-1}+2\sum_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

Applying Theorem 1 to $P(tz)$, we obtain the following result:

**Corollary 2.** *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with* $\operatorname{Re}(a_j)=\alpha_j$ *and* $\operatorname{Im}(a_j)=\beta_j,\ j=0,\ 1,\ \cdots,\ n$. *If for some real numbers* $\rho\geq 0$ *and* $t>0$,

$$\rho+t^n\alpha_n\geq t^{n-1}\alpha_{n-1}\geq\ldots\geq t\alpha_1\geq\alpha_0,$$

*then* $P(z)$ *has all its zeros in the disk*

$$\left|z+\frac{\rho}{t^{n-1}\alpha_n}\right|\leq\frac{\rho+t^n\alpha_n-\alpha_0+|\alpha_0|+2\sum_{j=0}^{n}|\beta_j|t^j}{t^{n-1}|\alpha_n|}.$$

In Theorem 1, if we take $\alpha_0\geq 0$, we get the following result:

**Corollary 3.** *Let* $P(z)=\sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n with* $\operatorname{Re}(a_j)=\alpha_j$ *and* $\operatorname{Im}(a_j)=\beta_j,\ j=0,\ 1,\ \cdots,\ n$. *If for some real number* $\rho\geq 0$,

$$\rho+\alpha_n\geq\alpha_{n-1}\geq\cdots\geq\alpha_1\geq\alpha_0\geq 0,$$

*then* $P(z)$ *has all its zeros in the disk*

$$\left|z+\frac{\rho}{\alpha_n}\right| \le 1+\frac{\rho+2\sum\limits_{j=0}^{n}|\beta_j|}{|\alpha_n|}.$$

If we apply Theorem 1 to the polynomial $-iP(z)$, we easily get the following result:

**Theorem 2**. *Let* $P(z)=\sum\limits_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with* $\mathrm{Re}(a_j)=\alpha_j$ *and* $\mathrm{Im}(a_j)=\beta_j,\ j=0,\ 1,\ \cdots,\ n$. *If for some* $\rho\ge 0$,

$$\rho+\beta_n \ge \beta_{n-1} \ge \cdots \ge \beta_0,$$

*then* $P(z)$ *has all its zeros in the disk*

$$\left|z+\frac{\rho}{\beta_n}\right| \le \frac{\rho+\beta_n-\beta_0+|\beta_0|+2\sum\limits_{j=0}^{n}|\alpha_j|}{|\beta_n|}.$$

On applying Theorem 2 to the polynomial $P(tz)$, one gets the following result:

**Corollary 4.** *Let* $P(z)=\sum\limits_{j=0}^{n} a_j z^j$ *be a polynomial of degree n with* $\mathrm{Re}(a_j)=\alpha_j$ *and* $\mathrm{Im}(a_j)=\beta_j,\ j=0,\ 1,\ \cdots,\ n$. *If for some* $\rho\ge 0$ *and t > 0*

$$\rho+t^n\beta_n \ge t^{n-1}\beta_{n-1} \ge \cdots \ge t\beta_1 \ge \beta_0,$$

*then* $P(z)$ *has all its zeros in the disk*

$$\left|z+\frac{\rho}{t^{n-1}\beta_n}\right| \le \frac{\rho+t^n\beta_n-\beta_0+|\beta_0|+2\sum\limits_{j=0}^{n}|\alpha_j|t^j}{t^{n-1}|\beta_n|}.$$

## 2 Proofs of the Theorems

*Proof of the Theorem* 1. Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z)=(1-z)(a_nz^n+(a_{n-1}z^{n-1}+\cdots+a_1z+a_0) \\
&= (1-z)(a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0) \\
&= -a_nz^{n+1}+(\alpha_n-\alpha_{n-1})z^n+\cdots+(\alpha_1-\alpha_0)z+\alpha_0-i\beta_nz^{n+1} \\
&\quad +i(\beta_n-\beta_{n-1})z^n+\cdots+i(\beta_1-\beta_0)z+i\beta_0 \\
&= -\alpha_nz^{n+1}-\rho z^n+(\rho+\alpha_n-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+\cdots+(\alpha_1-\alpha_0)z+\alpha_0 \\
&\quad -i\left\{-\beta_nz^{n+1}+(\beta_n-\beta_{n-1})z^n+\cdots+(\beta_1-\beta_0)z+\beta_0\right\}.
\end{aligned}$$

Then,

$$\begin{aligned}
|F(z)| &= |-\alpha_n z^{n+1} - \rho z^n + (\rho+\alpha_n-\alpha_{n-1})z^n + (\alpha_{n-1}-\alpha_{n-2})z^{n-1} + \cdots + (\alpha_1-\alpha_0)z + \alpha_0 \\
&\quad - i\{-\beta_n z^{n+1} + (\beta_n-\beta_{n-1})z^n + \ldots + (\beta_1-\beta_0)z + \beta_0\}| \\
&\geq |z|^n \left\{ |\alpha_n z + \rho| - |\rho+\alpha_n-\alpha_{n-1}| - |\alpha_0|\frac{1}{|z|^n} - \sum_{j=1}^{n-1} |\alpha_j - \alpha_{j-1}| \frac{1}{|z|^{n-j}} \right\} \\
&\quad - |-\beta_n z^{n+1} + \ldots + (\beta_1-\beta_0)z + \beta_0|.
\end{aligned}$$

Thus, for $|z| > 1$,

$$\begin{aligned}
|F(z)| &> |z|^n \{ |\alpha_n z + \rho| - (\rho+\alpha_n-\alpha_{n-1}| - |\alpha_0| - (\alpha_{n-1}-\alpha_{n-2}) \cdots - (\alpha_1-\alpha_0) \} \\
&\quad -(|-\beta_n| + |\beta_0|) - \sum_{j=0}^{n} (\beta_j) + \beta_{j-1}|) \\
&= |z|^n \left\{ |\alpha_n z + \rho| - (\rho + \alpha_n + |\alpha_0| - \alpha_0) - 2\sum_{j=0}^{n} |\beta_j| \right\} > 0
\end{aligned}$$

if

$$|\alpha_n z + \rho| > \rho + \alpha_n + |\alpha_0| - \alpha_0 + 2\sum_{j=0}^{n} |\beta_j|.$$

Hence all the zeros of $F(z)$ whose modulus is greater than 1 lie in the disk

$$\left| z + \frac{\rho}{\alpha_n} \right| \leq \frac{\rho + \alpha_n + |\alpha_0| - \alpha_0 + 2\sum_{j=0}^{n} |\beta_j|}{|\alpha_n|}.$$

But those zeros of $F(z)$ whose modulus is less than or equal to 1 already satisfy the above inequlity. Therefore, all the zeros of $F(z)$ lie in the disk

$$\left| z + \frac{\rho}{\alpha_n} \right| \leq \frac{\rho + \alpha_n + |\alpha_0| - \alpha_0 + 2\sum_{j=0}^{n} |\beta_j|}{|\alpha_n|}.$$

Since all the zeros of $P(z)$ are also the zeros of $F(z)$, it follows that all the zeros of $P(z)$ lie in the disk

$$\left| z + \frac{\rho}{\alpha_n} \right| \leq \frac{\rho + \alpha_n + |\alpha_0| - \alpha_0 + 2\sum_{j=0}^{n} |\beta_j|}{|\alpha_n|}.$$

This completes the proof of Theorem 1.

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