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WEIGHTED ESTIMATES FOR MULTIVARIATE HAUSDORFF OPERATORS

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Abstract. In this paper, some weighted estimates for the multivariate Hausdorff operators are obtained. It is proved that the multivariate Hausdorff operators are bounded on L^p spaces with power weights, which is based on the boundedness of multivariate Hausdorff operators on Herz spaces, and are bounded on weighted L^p spaces with the weights satisfying the homogeneity of degree zero.

Key words: *multivariate Hausdorff operator, weighted L^p space, Herz space*

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1 Introduction

The notion of the Hausdorff operator with respect to a positive measure on the unit interval $[0,1]$ is introduced by Hardy in [1]. The operator with respect to a complex measure in the real line $\mathbf{R}$ is defined and studied by Brown and Móricz in [2]. Following that, the multivariate Hausdorff operator with respect to complex Borel measures on $\mathbf{R}^n$ is introduced in a more general framework in [3].

Let μ be a $\sigma-$finite complex Borel measure on $\mathbf{R}^n$ and c be a Borel measurable function on $\mathbf{R}^n$, which is nonzero μ-a.e. Assume that $\mathcal{A} := [a_{jk}]$ is an $n\times n$ matrix whose entries $a_{jk} : \mathbf{R}^n \to \mathbf{C}$ are all Borel measurable functions and such that $\mathcal{A}$ is nonsingular μ-a.e. For a measurable complex valued function f on $\mathbf{R}^n$, the multivariate Hausdorff operator $\mathcal{H} = \mathcal{H}(\mu, c, \mathcal{A})$ is defined

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by:

$$\mathcal{H}f(x) := \int_{\mathbf{R}^n} c(s)f(\mathcal{A}(s)x)\mathrm{d}\mu(s). \tag{1.1}$$

The operator $\mathcal{H}^*$ adjoint to $\mathcal{H}$ is given by

$$\mathcal{H}^* f(x) := \int_{\mathbf{R}^n} c(s)|\det\mathcal{A}^{-1}(s)|f(\mathcal{A}^{-1}(s)x)\mathrm{d}\mu(s). \tag{1.2}$$

Both the above two integrals on the right hand side exist as Lebesgue-Stieltjes integrals [4,5]. It is obvious that $\mathcal{H}^*$ is also a Hausdorff operator corresponding to the triple $\mu(s), c(s)|\det\mathcal{A}^{-1}(s)|$, $\mathcal{A}^{-1}(s)$, that is

$$\mathcal{H}^* = \mathcal{H}(\mu, c|\det\mathcal{A}^{-1}|, \mathcal{A}^{-1}). \tag{1.3}$$

In [3], Brown and Móricz obtained the boundedness of the multivariate Hausdorff operator on $L^p(\mathbf{R}^n)$:

Theorem A. *If μ is a complex measure on $\mathbf{R}^n$ and*

$$k_p := \int_{\mathbf{R}^n} |c(s)||\det\mathcal{A}^{-1}(s)|^{\frac{1}{p}}\mathrm{d}|\mu|(s) < \infty \tag{1.4}$$

for some $1 \le p \le \infty$, then the Hausdorff operator $\mathcal{H} = \mathcal{H}(\mu, c, \mathcal{A})$ defined in (1.1) *is bounded on $L^p(\mathbf{R}^n)$:*

$$\|\mathcal{H}f\|_p \le k_p\|f\|_p, \tag{1.5}$$

where $|\mu|$ is the total variation of μ.

In [6], Móricz proved that the multivariate Hausdorff operator is bounded on the real Hardy space $H^1(\mathbf{R}^n)$ and BMO$(\mathbf{R}^n)$.

In this paper, we will generalize some results in [3] to the weighted L^p space and obtain some useful estimates for multivariate Hausdorff operators.

Note that the Herz space is a natural generalization of the L^p space with power weights (see [7]). We will firstly consider the boundedness of the multivariate Hausdorff operator on the Herz space. As a direct corollary of it, we can obtain the estimates for the operator on the L^p space with power weights. Next, we will estimate the multivariate Hausdorff operator on the weighted L^p space, where the weight functions are homogeneous of degree zero.

2 Main Results

Assume $1 \le p \le \infty$ and denote the exponent conjugate to p by p^*, that is, let $\dfrac{1}{p} + \dfrac{1}{p^*} = 1$ with the agreement that $\dfrac{1}{\infty} = 0$. Let $k \in \mathbf{Z}$, $B_k = \{x \in \mathbf{R}^n : |x| \le 2^k\}$, $D_k = B_k \backslash B_{k-1}$, and $\chi_k = \chi_{D_k}$ is the characteristic function of D_k.

Definition 2.1. Let $-\infty < \alpha < \infty$, $0 < p \leq \infty$ and $0 < q \leq \infty$.

(1) The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbf{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbf{R}^n) = \{f \in L^q_{loc}(\mathbf{R}^n\backslash\{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbf{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbf{R}^n)} = \left\{\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbf{R}^n)}\right\}^{\frac{1}{p}} < \infty.$$

(2) The nonhomogeneous Herz space $K_q^{\alpha,p}(\mathbf{R}^n)$ is defined by

$$K_q^{\alpha,p}(\mathbf{R}^n) = \{f \in L^q_{loc}(\mathbf{R}^n : \|f\|_{K_q^{\alpha,p}(\mathbf{R}^n)} < \infty\},$$

where

$$\|f\|_{K_q^{\alpha,p}(\mathbf{R}^n)} = \left\{\|f\chi_{B_0}\|^p_{L^q(\mathbf{R}^n)} + \sum_{k=1}^{\infty} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbf{R}^n)}\right\}^{\frac{1}{p}} < \infty.$$

With the usual modification made when $p = \infty$ or $q = \infty$ (See [7] for more information of Herz space).

Our first result is stated as follows.

Theorem 2.1. *Let $-\infty < \alpha < \infty$, $1 \leq p \leq \infty$ and $1 \leq q < \infty$. Assume μ is a complex measure on $\mathbf{R}^n$ and $\mathcal{A}(s) := \mathrm{diag}(a(s), \cdots, a(s))$, where $a(s) : \mathbf{R}^n \to \mathbf{C}$ is a Borel measurable function and* $a(s) \neq 0$ μ-a. e. *If*

$$C(\alpha,q) = \int_{\mathbf{R}^n} |a(s)|^{-n\alpha-\frac{n}{q}}|c(s)|\mathrm{d}|\mu|(s) < \infty, \tag{2.1}$$

then the Hausdorff operator $\mathcal{H} = \mathcal{H}(\mu,c,\mathcal{A})$ is bounded on $\dot{K}_q^{\alpha,p}(\mathbf{R}^n)$.

In particular, when $\alpha = 0$ and $p = q$, it is clear that $C(\alpha,q)$ reduces to k_p defined in (1.4) and the Herz space $\dot{K}_q^{\alpha,p}$ reduces to $L^p(\mathbf{R}^n)$. So, Theorem 2.1 implies Theorem A.

Note that $L^q(\mathbf{R}^n, |x|^\beta) = \dot{K}_q^{\frac{\beta}{q},q}(\mathbf{R}^n)$, where $\beta \in \mathbf{R}$. The following weighted estimate for Hausdorff operators is an immediate consequence of Theorem 2.1, which generalizes the result in [3] to L^p spaces with power weights.

Corollary 2.1. *Let $-\infty < \beta < \infty$ and $1 \leq q < \infty$. Assume μ and $\mathcal{A}(s)$ are the same as those of Theorem 2.1. If*

$$C(\frac{\beta}{q},q) = \int_{\mathbf{R}^n} |a(s)|^{-\frac{n\beta}{q}-\frac{n}{q}}|c(s)|\mathrm{d}|\mu|(s) < \infty,$$

then the Hausdorff operator $\mathcal{H} = \mathcal{H}(\mu,c,\mathcal{A})$ is bounded on $L^q(\mathbf{R}^n, |x|^\beta)$.

Since $\mathcal{H}^*$ is also a Hausdorff operator, the following estimate for $\mathcal{H}^*$ is worthy to be formulated.

Corollary 2.2. *Let $-\infty < \alpha < \infty$, $1 \le p \le \infty$ and $1 < q \le \infty$. Assume μ and $\mathcal{A}(s)$ are the same as those of Theorem* 2.1. *If the condition* (2.1) *is satisfied for some α, q, then the operator $\mathcal{H}^*$ is bounded on $\dot{K}_{q^*}^{-\alpha,p}(\mathbf{R}^n)$.*

Proof. By assumption we have

$$\begin{aligned} C(-\alpha,q^*)(\mathcal{H}^*) &= \int_{\mathbf{R}^n} |a(s)|^{-n\alpha+\frac{n}{q^*}} \big|c(s)\big|\big|\det\mathcal{A}^{-1}(s)\big| \mathrm{d}|\mu|(s) \\ &= \int_{\mathbf{R}^n} |a(s)|^{-n\alpha-\frac{n}{q}} \big|c(s)\big| \mathrm{d}|\mu|(s) \\ &= C(\alpha,q)(\mathcal{H}) < \infty. \end{aligned}$$

It follows from Theorem 2.1 that the operator $\mathcal{H}^* = \mathcal{H}(\mu, c|\det\mathcal{A}^{-1}|, \mathcal{A}^{-1})$ is bounded on the Herz space $\dot{K}_{q^*}^{-\alpha,p}(\mathbf{R}^n)$.

There are some similar results for the nonhomogeneous Herz spaces. We omit the details here.

Another weighted estimate for the multivariate Hausdorff operators is stated as follows.

Theorem 2.2. *Let μ be a complex measure on $\mathbf{R}^n$ and $\mathcal{A}(s) = \mathrm{diag}(a(s), a(s), \ldots, a(s))$, where $a(s): \mathbf{R}^n \to \mathbf{C}$ is a Borel measurable function and $a(s) \neq 0$ μ-a.e. Assume that the non-negative weight function $\omega(x)$ satisfies*

$$\omega(\lambda x) = \omega(x), \qquad \lambda \neq 0. \tag{2.2}$$

If the condition (1.4) *is satisfied for some $1 \le p \le \infty$, then the Hausdorff operator $\mathcal{H} = \mathcal{H}(\mu, c, \mathcal{A})$ is bounded on $L^p_\omega(\mathbf{R}^n)$:*

$$\|\mathcal{H}f\|_{L^p_\omega} \le k_p \|f\|_{L^p_\omega}. \tag{2.3}$$

Corollary 2.3. *Assume $\mu, \mathcal{A}(s)$ and $\omega(x)$ are the same as those of Theorem* 2.2. *If the condition* (1.4) *is satisfied for some $1 \le p \le \infty$, then the operator $\mathcal{H}^*$ defined in* (1.2) *is bounded on $L^{p^*}_\omega(\mathbf{R}^n)$.*

Corollary 2.3 can be proved by the same way as that of Corollary 2.2.

The relation of the weighted norm of Hausdorff operator $\mathcal{H}$ and its adjoint operator $\mathcal{H}^*$ is formulated in the following theorem.

Theorem 2.3. *Assume $\mathcal{A}(s)$ and $\omega(x)$ are the same as those of Theorem* 2.2. *If the condition* (1.4) *is satisfied for some $1 \le p \le \infty$, then*

$$\|\mathcal{H}\|_{L^p_\omega} = \|\mathcal{H}^*\|_{L^{p^*}_\omega}. \tag{2.4}$$

From Theorem 2.3, we can also conclude that the operator $\mathcal{H}^* = \mathcal{H}(\mu, c|\det\mathcal{A}^{-1}|, \mathcal{A}^{-1})$ is bounded on the conjugate space $L^{p^*}_\omega(\mathbf{R}^n)$ if $\mathcal{H}$ is bounded on $L^p_\omega(\mathbf{R}^n)$. Corollary 2.3 is demonstrated again.

3 Proof of Theorems

Proof of Theorem 2.1. Using Minkowski's inequality and setting $v = \mathcal{A}(s)x$, we get

$$\begin{aligned}
\|(\mathcal{H}f)\chi_k\|_{L^q} &= \Big\{\int_{D_k} \big|\mathcal{H}f(x)\big|^q \mathrm{d}x\Big\}^{\frac{1}{q}} \\
&= \Big\{\int_{D_k} \Big|\int_{\mathbf{R}^n} c(s) f(\mathcal{A}(s)x)\mathrm{d}\mu(s)\Big|^q \mathrm{d}x\Big\}^{\frac{1}{q}} \\
&\le \int_{\mathbf{R}^n} \Big\{\int_{2^{k-1}<|x|\le 2^k} \big|c(s) f(\mathcal{A}(s)x)\big|^q \mathrm{d}x\Big\}^{\frac{1}{q}} \mathrm{d}|\mu|(s) \\
&= \int_{\mathbf{R}^n} \Big\{\int_{2^{k-1}|a(s)|^n<|v|\le 2^k|a(s)|^n} \big|c(s) f(v)\big|^q \big|\det\mathcal{A}^{-1}(s)\big| \mathrm{d}v\Big\}^{\frac{1}{q}} \mathrm{d}|\mu|(s) \\
&= \int_{\mathbf{R}^n} \big|c(s)\big| \big|\det\mathcal{A}^{-1}(s)\big|^{\frac{1}{q}} \Big\{\int_{2^{k-1}|a(s)|^n<|v|\le 2^k|a(s)|^n} \big|f(v)\big|^q \mathrm{d}v\Big\}^{\frac{1}{q}} \mathrm{d}|\mu|(s).
\end{aligned}$$

For each $s \in \mathbf{R}^n$, there exists an integer m such that $2^{m-1} < |a(s)|^n \le 2^m$. Setting

$$E_m = \{s \in \mathbf{R}^n : 2^{m-1} < |a(s)|^n \le 2^m\},$$

$$A_{k,m} = \{v \in \mathbf{R}^n : 2^{k+m-1} < |v| \le 2^{k+m}\},$$

then we have

$$\begin{aligned}
\|(\mathcal{H}f)\chi_k\|_{L^q} &\le \int_{\mathbf{R}^n} \big|c(s)\big| \big|\det\mathcal{A}^{-1}(s)\big|^{\frac{1}{q}} \Big\{\int_{A_{k-1,m}} \big|f(v)\big|^q \mathrm{d}v + \int_{A_{k,m}} \big|f(v)\big|^q \mathrm{d}v\Big\}^{\frac{1}{q}} \mathrm{d}|\mu|(s) \\
&\le \int_{\mathbf{R}^n} \big|c(s)\big| \big|\det\mathcal{A}^{-1}(s)\big|^{\frac{1}{q}} \big(\|f\chi_{k+m-1}\|_{L^q} + \|f\chi_{k+m}\|_{L^q}\big) \mathrm{d}|\mu|(s).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|\mathcal{H}f\|_{\dot{K}_q^{\alpha,p}} &= \Big\{\sum_{k\in\mathbf{Z}} 2^{k\alpha p} \|(\mathcal{H}f)\chi_k\|_{L^q}^p\Big\}^{\frac{1}{p}} \\
&\le \Big\{\sum_{k\in\mathbf{Z}} 2^{k\alpha p} \Big[\int_{\mathbf{R}^n} \big|c(s)\big| \big|\det\mathcal{A}^{-1}(s)\big|^{\frac{1}{q}} \big(\|f\chi_{k+m-1}\|_{L^q} + \|f\chi_{k+m}\|_{L^q}\big) \mathrm{d}|\mu|(s)\Big]^p\Big\}^{\frac{1}{p}} \\
&= \Big\{\sum_{k\in\mathbf{Z}} 2^{k\alpha p} \Big[\sum_{m\in\mathbf{Z}} \int_{E_m} \big|c(s)\big| \big|\det\mathcal{A}^{-1}(s)\big|^{\frac{1}{q}} \big(\|f\chi_{k+m-1}\|_{L^q} + \|f\chi_{k+m}\|_{L^q}\big) \mathrm{d}|\mu|(s)\Big]^p\Big\}^{\frac{1}{p}} \\
&= \Big\{\sum_{k\in\mathbf{Z}} 2^{k\alpha p} \Big[\sum_{m\in\mathbf{Z}} \big(\|f\chi_{k+m-1}\|_{L^q} + \|f\chi_{k+m}\|_{L^q}\big) \int_{E_m} \big|c(s)\big| \big|\det\mathcal{A}^{-1}(s)\big|^{\frac{1}{q}} \mathrm{d}|\mu|(s)\Big]^p\Big\}^{\frac{1}{p}}.
\end{aligned}$$

If $1 < p < \infty$, then it follows from Minkowski's inequality that

$$
\begin{aligned}
\|\mathcal{H}f\|_{\dot{K}_q^{\alpha,p}} \leq & \Big\{\sum_{k\in\mathbf{Z}} 2^{k\alpha p}\Big[\sum_{m\in\mathbf{Z}}\big(\|f\chi_{k+m-1}\|_{L^q}+\|f\chi_{k+m}\|_{L^q}\big)\int_{E_m}\big|c(s)\big|\big|\det\mathcal{A}^{-1}(s)\big|^{\frac{1}{q}}\mathrm{d}|\mu|(s)\Big]^p\Big\}^{\frac{1}{p}}\\
\leq & \sum_{m\in\mathbf{Z}}\Big[\sum_{k\in\mathbf{Z}} 2^{k\alpha p}\big(\|f\chi_{k+m-1}\|_{L^q}+\|f\chi_{k+m}\|_{L^q}\big)^p\Big(\int_{E_m}\big|c(s)\big|\big|\det\mathcal{A}^{-1}(s)\big|^{\frac{1}{q}}\mathrm{d}|\mu|(s)\Big)^p\Big]^{\frac{1}{p}}\\
\leq & \sum_{m\in\mathbf{Z}}\Big[\sum_{k\in\mathbf{Z}} 2^{k\alpha p}2^p\big(\|f\chi_{k+m-1}\|_{L^q}^p+\|f\chi_{k+m}\|_{L^q}^p\big)\Big(\int_{E_m}\big|c(s)\big|\big|\det\mathcal{A}^{-1}(s)\big|^{\frac{1}{q}}\mathrm{d}|\mu|(s)\Big)^p\Big]^{\frac{1}{p}}\\
= & 2\sum_{m\in\mathbf{Z}}\Big[\sum_{k\in\mathbf{Z}} 2^{k\alpha p}\|f\chi_{k+m-1}\|_{L^q}^p\Big(\int_{E_m}\big|c(s)\big|\big|\det\mathcal{A}^{-1}(s)\big|^{\frac{1}{q}}\mathrm{d}|\mu|(s)\Big)^p\\
& +\sum_{k\in\mathbf{Z}} 2^{k\alpha p}\|f\chi_{k+m}\|_{L^q}^p\Big(\int_{E_m}\big|c(s)\big|\big|\det\mathcal{A}^{-1}(s)\big|^{\frac{1}{q}}\mathrm{d}|\mu|(s)\Big)^p\Big]^{\frac{1}{p}}\\
= & 2\sum_{m\in\mathbf{Z}}\Big[\sum_{k\in\mathbf{Z}} 2^{(k+m-1)\alpha p}\|f\chi_{k+m-1}\|_{L^q}^p\Big(\int_{E_m}2^{(1-m)\alpha}\big|c(s)\big|\big|\det\mathcal{A}^{-1}(s)\big|^{\frac{1}{q}}\mathrm{d}|\mu|(s)\Big)^p\\
& +\sum_{k\in\mathbf{Z}} 2^{(k+m)\alpha p}\|f\chi_{k+m}\|_{L^q}^p\Big(\int_{E_m}2^{-m\alpha}\big|c(s)\big|\big|\det\mathcal{A}^{-1}(s)\big|^{\frac{1}{q}}\mathrm{d}|\mu|(s)\Big)^p\Big]^{\frac{1}{p}}\\
\leq & 2\sum_{m\in\mathbf{Z}}\Big[(1+2^{|\alpha|p})\Big(\int_{E_m}|a(s)|^{-n\alpha}\big|c(s)\big|\big|\det\mathcal{A}^{-1}(s)\big|^{\frac{1}{q}}\mathrm{d}|\mu|(s)\Big)^p\|f\|_{\dot{K}_q^{\alpha,p}}^p\Big]^{\frac{1}{p}}\\
= & 2(1+2^{|\alpha|p})^{\frac{1}{p}}C(\alpha,q)\|f\|_{\dot{K}_q^{\alpha,p}}.
\end{aligned}
$$

In the case $p=1$, the above argument works with Fubini's theorem instead of Minkowski's inequality. The case of $p=\infty$ is trivial.

This finishes the proof of Theorem 2.1.

Proof of Theorem 2.2. For $1<p<\infty$, use Minkowski's inequality and setting $v=\mathcal{A}(s)x$, we have

$$
\begin{aligned}
\|\mathcal{H}f\|_{L^p_\omega} = & \Big\{\int_{\mathbf{R}^n}\Big|\int_{\mathbf{R}^n}c(s)f(\mathcal{A}(s)x)\mathrm{d}\mu(s)\Big|^p\omega(x)\mathrm{d}x\Big\}^{\frac{1}{p}}\\
\leq & \int_{\mathbf{R}^n}\Big\{\int_{\mathbf{R}^n}\big|c(s)f(\mathcal{A}(s)x)\big|^p\omega(x)\mathrm{d}x\Big\}^{\frac{1}{p}}\mathrm{d}|\mu|(s)\\
= & \int_{\mathbf{R}^n}\big|c(s)\big|\Big\{\int_{\mathbf{R}^n}\big|f(v)\big|^p\omega(\mathcal{A}^{-1}(s)v)|\det\mathcal{A}^{-1}(s)|\mathrm{d}v\Big\}^{\frac{1}{p}}\mathrm{d}|\mu|(s)\\
= & \int_{\mathbf{R}^n}\big|c(s)\big|\big|\det\mathcal{A}^{-1}(s)\big|^{\frac{1}{p}}\Big\{\int_{\mathbf{R}^n}\big|f(v)\big|^p\omega(v)\mathrm{d}v\Big\}^{\frac{1}{p}}\mathrm{d}|\mu|(s)\\
= & k_p\|f\|_{L^p_\omega}.
\end{aligned}
$$

If $p=1$, the above argument works with Fubini's theorem instead of Minkowski's inequality. The case of $p=\infty$ is trivial.

The proof of Theorem 2.2 is completed.

To prove Theorem 2.3, we need the following lemma.

Lemma 3.1. *Let*

$$\mathcal{H} = \mathcal{H}(\mu, c, \mathcal{A})$$

be a Hausdorff operator satisfying the condition (1.4) *for some* $1 \le p \le \infty$ *and*

$$\mathcal{H}^* f(x) := \mathcal{H}(\mu, c|\det \mathcal{A}^{-1}|, \mathcal{A}^{-1})$$

be the adjoint operator of $\mathcal{H}$*. Assume that* $\omega(x)$ *is the same as that of Theorem* 2.2*. If* $f \in L^p_\omega(\mathbf{R}^n)$ *and* $g \in L^{p^*}_\omega(\mathbf{R}^n)$*, then*

$$\int_{\mathbf{R}^n} [\mathcal{H} f(x)] g(x) \omega(x) \mathrm{d}x = \int_{\mathbf{R}^n} f(x) [\mathcal{H}^* g(x)] \omega(x) \mathrm{d}x. \tag{3.1}$$

Proof. By the Hölder inequality, we have

$$\int_{\mathbf{R}^n} [\mathcal{H} f(x)] g(x) \omega(x) \mathrm{d}x \le \Big\{ \int_{\mathbf{R}^n} |\mathcal{H} f(x)|^p \omega(x) \mathrm{d}x \Big\}^{\frac{1}{p}} \Big\{ \int_{\mathbf{R}^n} |g(x)|^{p^*} \omega(x) \mathrm{d}x \Big\}^{\frac{1}{p^*}} = \|\mathcal{H} f\|_{L^p_\omega} \|g\|_{L^{p^*}_\omega}.$$

$$\int_{\mathbf{R}^n} f(x) [\mathcal{H}^* g(x)] \omega(x) \mathrm{d}x \le \Big\{ \int_{\mathbf{R}^n} |f(x)|^p \omega(x) \mathrm{d}x \Big\}^{\frac{1}{p}} \Big\{ \int_{\mathbf{R}^n} |\mathcal{H}^* g(x)|^{p^*} \omega(x) \mathrm{d}x \Big\}^{\frac{1}{p^*}} = \|f\|_{L^p_\omega} \|\mathcal{H}^* g\|_{L^{p^*}_\omega}.$$

Applying Fubini's theorem we get

$$\begin{aligned}
\int_{\mathbf{R}^n} [\mathcal{H} f(x)] g(x) \omega(x) \mathrm{d}x &= \int_{\mathbf{R}^n} \Big\{ \int_{\mathbf{R}^n} c(s) f(\mathcal{A}(s)x) \mathrm{d}\mu(s) \Big\} g(x) \omega(x) \mathrm{d}x \\
&= \int_{\mathbf{R}^n} c(s) \Big\{ \int_{\mathbf{R}^n} f(\mathcal{A}(s)x) g(x) \omega(x) \mathrm{d}x \Big\} \mathrm{d}\mu(s) \\
&= \int_{\mathbf{R}^n} c(s) \Big\{ \int_{\mathbf{R}^n} f(v) g(\mathcal{A}^{-1}(s)v) \omega(\mathcal{A}^{-1}(s)v) \big|\det \mathcal{A}^{-1}(s)\big| \mathrm{d}v \Big\} \mathrm{d}\mu(s) \\
&= \int_{\mathbf{R}^n} f(v) \Big\{ \int_{\mathbf{R}^n} c(s) g(\mathcal{A}^{-1}(s)v) \big|\det \mathcal{A}^{-1}(s)\big| \mathrm{d}\mu(s) \Big\} \omega(v) \mathrm{d}v \\
&= \int_{\mathbf{R}^n} f(v) [\mathcal{H}^* g(v)] \omega(v) \mathrm{d}v.
\end{aligned}$$

Now, we prove Theorem 2.3 by Lemma 3.1.

Proof of Theorem 2.3.

$$\begin{aligned}
\|\mathcal{H}\|_{L^p_\omega} &= \sup \Big\{ \|\mathcal{H} f\|_{L^p_\omega} : \|f\|_{L^p_\omega} \le 1 \Big\} \\
&= \sup \Big\{ \sup \Big\{ \int_{\mathbf{R}^n} [\mathcal{H} f(x)] g(x) \omega(x) \mathrm{d}x : \|g\|_{L^{p^*}_\omega} \le 1 \Big\} : \|f\|_{L^p_\omega} \le 1 \Big\} \\
&= \sup \Big\{ \sup \Big\{ \int_{\mathbf{R}^n} f(x) [\mathcal{H}^* g(x)] \omega(x) \mathrm{d}x : \|f\|_{L^p_\omega} \le 1 \Big\} : \|g\|_{L^{p^*}_\omega} \le 1 \Big\} \\
&= \sup \Big\{ \|\mathcal{H}^* g\|_{L^{p^*}_\omega} : \|g\|_{L^{p^*}_\omega} \le 1 \Big\} \\
&= \|\mathcal{H}^*\|_{L^{p^*}_\omega}.
\end{aligned}$$

Theorem 2.3 is proved.

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