THE BOUNDEDNESS FOR A CLASS OF ROUGH FRACTIONAL INTEGRAL OPERATORS ON VARIABLE EXPONENT LEBESGUE SPACES

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Abstract.   In this paper, we will discuss the behavior of a class of rough fractional integral operators on variable exponent Lebesgue spaces,and establish their boundedness from $L^{p_1(\cdot)}(\mathbf{R}^n)$ to $L^{p_2(\cdot)}(\mathbf{R}^n)$.

Key words: *fractional integral, rough kernel, variable exponent Lebesgue space*

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1   Introduction and Main Results

In 1931, the variable exponent Lebesgue space has been first proposed in [1] by Polish mathematician Orlicz. In the last years the space has attracted more and more attention, see for example [1-5]. The main motivation for studying the space is applications to models of elasticity theory, fluid mechanics and differential equation with non-standard growth, see for example [6-8].

Let S^{n-1} denote the unit sphere in the Euclidean n-dimensional space $\mathbf{R}^n$. Suppose that $\Omega \in L^s\left(S^{n-1}\right), s > \dfrac{n}{n-\alpha}$, is homogeneous of degree zero on $\mathbf{R}^n$. Then the fractional integral operator $T_{\Omega,\alpha}$ with a rough kernel is defined by

$$T_{\Omega,\alpha}f(x) = \int_{\mathbf{R}^n} \frac{\Omega(x-y)}{\mid x-y\mid^{n-\alpha}} f(y)\mathrm{d}y,$$

and in the case $\Omega \equiv 1$, $T_{\Omega,\alpha}$ is the fractional integral operator (or Riesz potential operator)

$$I_\alpha f(x) = \int_{\mathbf{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy.$$

The corresponding fractional maximal operator with a rough kernel is defined by

$$M_{\Omega,\alpha} f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |\Omega(x-y)||f(y)| dy.$$

In fact,we can easily see that when $\Omega \equiv 1$, $M_{\Omega,\alpha}$ is just the fractional maximal operator

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| dy,$$

especially in the limiting case $\alpha = 0$, the fractional maximal operator reduces to the Hardy-Littlewood maximal operator.

It is well known that Calderón and Zygmund[9] have proven the fractional integral operator $T_{\Omega,\alpha}$ with a rough kernel is bounded on L^p. It turns out that such kind of operators are much more closely related to elliptic partial differential of second order with variable coefficients. For $0 < \alpha < n$,Muchenhoupt and Wheeden[10] proved the boundedness of $T_{\Omega,\alpha}$ with power weights from L^p to L^q. In [11] Ding, Chen and Fan gave the boundedness properties of $T_{\Omega,\alpha}$ on Hardy Spaces. However, the corresponding results for $T_{\Omega,\alpha}$ have not been proven on variable exponent Lebesgue spaces. Similarly,Diening[2] discovered the Hardy-littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbf{R}^n)$, and Cruz-Uribe[3] investigated the boundedness of M_α on $L^{p(\cdot)}(\mathbf{R}^n)$, but the boundedness of $M_{\Omega,\alpha}$ has not been studied.

Before stating our main results, let us recall some notations and definitions.

Definition 1.　Suppose $p(\cdot) : \mathbf{R}^n \to [1,\infty)$ is a measurable function for some $\lambda > 0$, then the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbf{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbf{R}^n) = \{f \ is \ \text{measurable} : \int_{\mathbf{R}^n} (|f(x)|/\lambda)^{p(x)} dx < \infty\},$$

with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)} = \inf\{\lambda > 0 : \int_{\mathbf{R}^n} (|f(x)|/\lambda)^{p(x)} dx \leq 1\}.$$

We denote

$$p_- = \operatorname{ess\,inf}\{p(x) : x \in \mathbf{R}^n\}, \qquad p_+ = \operatorname{ess\,sup}\{p(x) : x \in \mathbf{R}^n\}.$$

Using this notation we define a class of variable exponent

$$\Phi(R^n) = \{p(\cdot) : \mathbf{R}^n \to [1,\infty), \ p_- > 1, \ p_+ < \infty\}.$$

The exponent $p'(\cdot)$ means the conjugate of $p(\cdot)$, namely $1/p(x)+1/p'(x)=1$ holds.

Definition 2.   Let $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, then the Hardy-Littlewood maximal function of f is defined by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, \mathrm{d}y.$$

It is well known that the boundedness of the Hardy-Littlewood maximal operator on Lebesuge spaces plays a key role in analysis. So does it on the variable exponent Lebesgue spaces.Let $B(\mathbf{R}^n)$ be the set of $p(\cdot) \in \Phi(\mathbf{R}^n)$ such that M is bounded on $L^{p(\cdot)}(\mathbf{R}^n)$. In this paper we state some properties of variable exponents belonging to the the class $B(\mathbf{R}^n)$. Finally, we point out that C will denote positive constants whose value may change at different places.

Proposition 1.[4]   *If $p(\cdot) \in \Phi(\mathbf{R}^n)$ satisfies*

$$|p(x) - p(y)| \leq \frac{-C}{\log(|x-y|)}, \quad |x-y| \leq 1/2, \tag{1}$$

$$|p(x) - p(y)| \leq \frac{C}{\log(e+|x|)}, \quad |y| \geq |x|, \tag{2}$$

then we have $p(\cdot) \in B(\mathbf{R}^n)$.

Recently, Mitsuo Izuki has proved the boundedness of the fractional integral operator I_α as below.

Theorem A[4].   *Suppose that $p_1(\cdot) \in \Phi(\mathbf{R}^n)$ satisfies conditions (1) and (2) in Proposition 1. Let $0 < \alpha < n/(p_1)_+$, and define the variable exponent $p_2(\cdot)$ by*

$$\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\alpha}{n}.$$

Then we have for all $f \in L^{p_1(\cdot)}(R^n)$,

$$\|I_\alpha f\|_{L^{p_2(\cdot)}(R^n)} \leq C \|f\|_{L^{p_1(\cdot)}(R^n)}.$$

Now we are to popularize Theorem A in some conditions. In this paper, we will discuss the boundednedness of $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ on variable exponent Lebesgue spaces. We can get $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ are bounded from $L^{p_1(\cdot)}(\mathbf{R}^n)$ to $L^{p_2(\cdot)}(\mathbf{R}^n)$.

We have the following results.

Theorem 1. *Suppose that $p_1(\cdot) \in \Phi(\mathbf{R}^n)$ satisfies the conditions* (1) *and* (2) *in Proposition 1. Let $0 < \alpha < n/(p_1)_+$, and define the variable exponent $p_2(\cdot)$ by*

$$\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\alpha}{n}.$$

Then we have that for all $f \in L^{p_1(\cdot)}(\mathbf{R}^n)$,

$$\|M_{\Omega,\alpha} f\|_{L^{p_2(\cdot)}(\mathbf{R}^n)} \le C\|f\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}.$$

Theorem 2. *Suppose that $p_1(\cdot) \in \Phi(\mathbf{R}^n)$ satisfies the conditions* (1) *and* (2) *in Proposition 1. Let $0 < \alpha < n/(p_1)_+$, and define the variable exponent $p_2(\cdot)$ by*

$$\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\alpha}{n}.$$

Then we have that for all $f \in L^{p_1(\cdot)}(\mathbf{R}^n)$,

$$\|T_{\Omega,\alpha} f\|_{L^{p_2(\cdot)}(\mathbf{R}^n)} \le C\|f\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}.$$

2   Some Lemmas and proof of Theorems

Lemma 1.   *Given $0 < \alpha < n$, $1 < s' < n/\alpha$, $\Omega \in L^s(S^{n-1})$ is homogeneous of degree zero on R^n. If $1/s + 1/s' = 1$, then*

$$| M_{\Omega,\alpha} f(x) | \le C\|\Omega\|_{L^s(S^{n-1})} \left(M_{\alpha s'}(| f |^{s'})(x) \right)^{\frac{1}{s'}}.$$

Proof of Lemma 1.   By Hölder's inequality, we have

$$| M_{\Omega,\alpha} f(x) | = \sup_{Q \ni x} \frac{1}{| Q |^{1-\frac{\alpha}{n}}} \int_Q | \Omega(y) | | f(x-y) | \, dy$$

$$\le \sup_{Q \ni x} \frac{1}{| Q |^{1-\frac{\alpha}{n}}} \left(\int_Q | \Omega(y) |^s \, dy \right)^{1/s} \times \left(\int_Q | f(x-y) |^{s'} \, dy \right)^{1/s'}$$

$$\le C\|\Omega\|_{L^s(S^{n-1})} \sup_{Q \ni x} \frac{1}{| Q |^{1-\frac{\alpha s'}{n}}} \times \left(\int_Q | f(x-y) |^{s'} \, dy \right)^{1/s'}$$

$$= C\|\Omega\|_{L^s(S^{n-1})} (M_{\alpha s'} | f(|^{s'})(x))^{1/s'}.$$

Lemma 2[3].   *Suppose that $p_1(\cdot) \in \Phi(\mathbf{R}^n)$ satisfies the conditions (1) and (2) in Proposition 1. Let $0 < \alpha < n/(p_1)_+$, and define the variable exponent $p_2(\cdot)$ by*

$$\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\alpha}{n}.$$

Then we have for all $f \in L^{p_1(\cdot)}(\mathbf{R}^n)$,

$$\|M_\alpha f\|_{L^{p_2(\cdot)}(R^n)} \leq C\|f\|_{L^{p_1(\cdot)}(R^n)}.$$

Lemma 3[5].   *Given $p(\cdot) : \mathbf{R}^n \to [1, \infty)$ such that $p_+ < \infty$, then $\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)} < C_1$ if and only if $|f|_{L^{p(\cdot)}(\mathbf{R}^n)} < C_2$. In particular, if either of the constants C_1, C_2 equals 1 we can take the other equal to 1 as well.*

Remark.   We denote $|f|_{L^{p(\cdot)}(\mathbf{R}^n)} = \int_{\mathbf{R}^n} |f(y)|^{p(y)} \, dy$.

Lemma 4[12].   *Given α, $0 < \alpha < n$, fix ε, $0 < \varepsilon < \min(\alpha, n - \alpha)$. Then there exists a constant $C = C(\alpha, n, \varepsilon)$, such that for all $f \in L^1_{loc}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$,*

$$|T_{\Omega,\alpha} f(x)| \leq C[M_{\Omega,\alpha-\varepsilon} f(x)]^{\frac{1}{2}} [M_{\Omega,\alpha+\varepsilon} f(x)]^{\frac{1}{2}}.$$

Lemma 5[5].   *Given $p(\cdot) : \mathbf{R}^n \to [1, \infty)$, we have that for all functions f and g,*

$$\int_{R^n} |f(x)g(x)| \, dx \leq C\|f\|_{L^{p(\cdot)}(R^n)} \|g\|_{L^{p'(\cdot)}(R^n)}.$$

Next we will prove our main results.

Proof of Theorem 1.   By Lemma 1, we have

$$\int_{R^n} \left(\frac{M_{\Omega,\alpha} f}{\lambda}\right)^{p_2(x)} dx \leq C \int_{R^n} \left(\frac{M_{\alpha s'}(|f|^{s'})}{\lambda}\right)^{p_2(x)/s'} dx.$$

Since

$$\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\alpha}{n},$$

we can get

$$\frac{s'}{p_1(x)} - \frac{s'}{p_2(x)} = \frac{\alpha s'}{n},$$

and

$$\left\{\lambda > 0 : \int_{\mathbf{R}^n} \left(\frac{M_{\Omega,\alpha} f}{\lambda}\right)^{p_2(x)} dx \leq 1\right\} \supseteq \left\{\lambda > 0 : C \int_{\mathbf{R}^n} \left(\frac{M_{\alpha s'}(|f|^{s'})}{\lambda}\right)^{p_2(x)/s'} dx \leq 1\right\},$$

such that

$$\inf\left\{\lambda > 0 : \int_{\mathbf{R}^n}\left(\frac{M_{\Omega,\alpha}f}{\lambda}\right)^{p_2(x)}\mathrm{d}x \le 1\right\} \le \inf\left\{\lambda > 0 : C\int_{\mathbf{R}^n}\left(\frac{M_{\alpha s'}(|f|^{s'})}{\lambda}\right)^{p_2(x)/s'}\mathrm{d}x \le 1\right\},$$

hence by Lemma 2,we have

$$\|M_{\Omega,\alpha}f\|_{L^{p_2(\cdot)}(\mathbf{R}^n)} \le C\|M_{\alpha s'}(|f|^{s'})\|_{L^{p_2(\cdot)/s'}(\mathbf{R}^n)}$$

$$\le C\||f|^{s'}\|_{L^{p_1(\cdot)/s'}(\mathbf{R}^n)}.$$

$$\le C\|f\|_{L^{p_1(\cdot)}(\mathbf{R}^n)}.$$

Proof of Theorem 2. Fix $f \in L^{p_1(\cdot)}(\mathbf{R}^n)$, without loss of generality we may assume $\|f\|_{L^{p_1(\cdot)}(\mathbf{R}^n)} = 1$. Since $(p_2)_+ < \infty$, by Lemma 3 it is sufficient to prove that $|T_{\Omega,\alpha}f|_{L^{p_2(R^n)}} \le C$.

Fix ε, $0 < \varepsilon < min(\alpha, n-\alpha)$ such that

$$\frac{2}{\frac{\varepsilon(p_2)_+}{n}+1} > 1, \tag{3}$$

and define $r(\cdot) : \mathbf{R}^n \to [1, +\infty)$ by

$$r(x) = \frac{2}{\frac{\varepsilon p_2}{n}+1}.$$

Then by (3) we have $r_- > 1$. Moreover, by elementary algebra, for all $x \in \mathbf{R}^n$,

$$\frac{1}{p_1(x)} - \frac{1}{\frac{r(x)p_2(x)}{2}} = \frac{\alpha - \varepsilon}{n}, \tag{4}$$

$$\frac{1}{p_1(x)} - \frac{1}{\frac{r'(x)p_2(x)}{2}} = \frac{\alpha + \varepsilon}{n}, \tag{5}$$

so that by Lemma 4

$$\int_{\mathbf{R}^n}|T_{\Omega,\alpha}f(x)|^{p_2(x)}\,\mathrm{d}x \le C\int_{\mathbf{R}^n}[M_{\Omega,\alpha-\varepsilon}f(x)]^{p_2(x)/2}[M_{\Omega,\alpha+\varepsilon}f(x)]^{p_2(x)/2}\mathrm{d}x.$$

By Lemma 5 ,then

$$\int_{\mathbf{R}^n}|T_{\Omega,\alpha}f(x)|^{p_2(x)}\,\mathrm{d}x \le C\|[M_{\Omega,\alpha-\varepsilon}f(x)]^{\frac{p_2(x)}{2}}\|_{L^{r(\cdot)}(\mathbf{R}^n)}\|[M_{\Omega,\alpha+\varepsilon}f(x)]^{\frac{p_2(x)}{2}}\|_{L^{r'(\cdot)}(\mathbf{R}^n)}.$$

Without loss of generality, we may assume that each term is greater than 1, since otherwise there is nothing to prove. In the case, by the definition of each norm we may assume that the

infimum is taken over values of λ greater than 1. But then, since for all $x \in \mathbf{R}^n$ and $\lambda > 1$, $\lambda^{2/p_2(x)} \geq \lambda^{2/(p_2)_+}$, we have

$$\int_{\mathbf{R}^n} \left(\frac{[M_{\Omega,\alpha-\varepsilon}f(x)]^{\frac{p_2(x)}{2}}}{\lambda} \right)^{r(x)} \mathrm{d}x = \int_{\mathbf{R}^n} \left(\frac{M_{\Omega,\alpha-\varepsilon}f(x)}{\lambda^{2/p_2(x)}} \right)^{r(x)p_2(x)/2} \mathrm{d}x$$

$$\leq \int_{\mathbf{R}^n} \left(\frac{M_{\Omega,\alpha-\varepsilon}f(x)}{\lambda^{2/(p_2)_+}} \right)^{r(x)p_2(x)/2} \mathrm{d}x.$$

Therefore, by (4) and Theorem 1, we have

$$\|[M_{\Omega,\alpha-\varepsilon}f(x)]^{\frac{p_2(x)}{2}}\|_{L^{r(x)}(\mathbf{R}^n)} \leq \|M_{\Omega,\alpha-\varepsilon}f(x)\|_{L^{r(x)p_2(x)/2}(\mathbf{R}^n)}^{p_{2+}/2} \leq C\|f\|_{L^{p_1(x)}(\mathbf{R}^n)}^{p_{2+}/2} \leq C.$$

In the same way, we have

$$\int_{\mathbf{R}^n} \left(\frac{[M_{\Omega,\alpha+\varepsilon}f(x)]^{\frac{p_2(x)}{2}}}{\lambda} \right)^{r'(x)} \mathrm{d}x = \int_{\mathbf{R}^n} \left(\frac{M_{\Omega,\alpha+\varepsilon}f(x)}{\lambda^{2/p_2(x)}} \right)^{r'(x)p_2(x)/2} \mathrm{d}x$$

$$\leq \int_{\mathbf{R}^n} \left(\frac{M_{\Omega,\alpha+\varepsilon}f(x)}{\lambda^{2/(p_2)_+}} \right)^{r'(x)p_2(x)/2} \mathrm{d}x.$$

Therefore, by (5) and Theorem 1, we have

$$\|[M_{\Omega,\alpha+\varepsilon}f(x)]^{\frac{p_2(x)}{2}}\|_{L^{r'(x)}(\mathbf{R}^n)} \leq \|M_{\Omega,\alpha+\varepsilon}f(x)\|_{L^{r'(x)p_2(x)/2}(\mathbf{R}^n)}^{p_{2+}/2} \leq C\|f\|_{L^{p_1(x)}(\mathbf{R}^n)}^{p_{2+}/2} \leq C.$$

Hence

$$|T_{\Omega,\alpha}f|_{L^{p_2}(\mathbf{R}^n)} = \int_{\mathbf{R}^n} | T_{\Omega,\alpha}f(x) |^{p_2(x)} \, \mathrm{d}x \leq C.$$

This completes the proof.

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