EXISTENCE PROBLEMS OF ADDITIVE SELECTION MAPS FOR ANOTHER TYPE SUBADDITIVE SET-VALUED MAP

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Abstract. In this paper, we consider the following subadditive set-valued map $F : X \rightarrow P_0(Y)$:

$$F(\sum_{i=1}^{r} x_i + \sum_{j=1}^{s} x_{r+j}) \subseteq rF\left(\frac{\sum_{i=1}^{r} x_i}{r}\right) + sF\left(\frac{\sum_{j=1}^{s} x_{r+j}}{s}\right), \quad \forall x_i \in X, \quad i = 1, 2, \ldots, r+s,$$

where $r$ and $s$ are two natural numbers. And we discuss the existence and unique problem of additive selection maps for the above set-valued map.

Key words: additive selection map, subadditive, additive selection, cone

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1 Introduction and Preliminaries

The stability problem of functional equations was originated from a question of Ulam\cite{1} concerning the stability of group homomorphisms. In 1941, D.H. Hyers\cite{2} gave a first affirmative partial answer to the question of Ulam for Banach spaces. The famous stability theorem is as follows:

**Theorem 0.** Let $E_1$ be a normed vector space and $E_2$ a Banach space. Suppose that the mapping $f : E_1 \rightarrow E_2$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in E_1$, with $\varepsilon > 0$ a constant. Then the limit

$$g(x) = \lim_{n \rightarrow \infty} 2^{-n} f(2^n x)$$

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exists for each \( x \in E_1 \) and \( g \) is the unique additive mapping satisfying
\[
\| f(x) - g(x) \| \leq \varepsilon
\]
for all \( x \in E_1 \).

Later, Hyers’ Theorem has been generalized by many authors\([3–8]\). Let \( X \) be a real vector space. We denote by \( P_0(X) \) the family of all nonempty subsets of \( X \).

If \( Y \) is a topological vector space, the family of all closed convex subsets of \( Y \) denoted by \( \text{ccl}(Y) \).

Let \( A \) and \( B \) are two nonempty subsets of the real vector space \( X \), \( \lambda \) and \( \mu \) are two real numbers. Define
\[
A + B = \{ x | x = a + b, a \in A, b \in B \}; \\
\lambda A = \{ x | x = \lambda a, a \in A \}.
\]

The next properties are obvious:

**Lemma.** If \( A \) and \( B \) are two nonempty subsets of the real vector space \( X \), \( \lambda \) and \( \mu \) are two real numbers, then
\[
\lambda (A + B) = \lambda A + \lambda B; \quad (\lambda + \mu)A \subseteq \lambda A + \mu A.
\]

Furthermore, if \( A \) is a convex subset and \( \lambda \mu \geq 0 \), then we have the following formula\([9]\):
\[
(\lambda + \mu)A = \lambda A + \mu A.
\]

A subset \( A \subset X \) is said to be a cone if \( A + A \subseteq A \), and \( \lambda A \subseteq A \) for all \( \lambda > 0 \).

If the zero in \( X \) belongs to \( A \), we say that \( A \) is a zero cone.

Let \( X \) and \( Y \) be two real vector spaces, \( f : X \rightarrow Y \) a single-valued map, and \( F : X \rightarrow P_0(Y) \) a set-valued map. \( f \) is called an additive selection of \( F \), if \( f(x+y) = f(x) + f(y) \) for all \( x, y \in X \), and \( f(x) \in F(x) \) for all \( x \in X \).

Let \( B(0, \varepsilon) \) denote the open ball with center 0 and radius \( \varepsilon \) in \( E_2 \) in Theorem 0, then the inequality (0) may be written as
\[
f(x+y) \in B(0, \varepsilon) + f(x) + f(y),
\]
and hence
\[
f(x+y) + B(0, \varepsilon) \subseteq f(x) + B(0, \varepsilon) + f(y) + B(0, \varepsilon),
\]
where \( B(0, \varepsilon) + x \) denote the open ball with center \( x \) and radius \( \varepsilon \) in \( E_2 \).

Thus, if we define a set-valued mapping \( F \) by \( F(x) = f(x) + B(0, \varepsilon) \) for each \( x \in E_1 \), then we get
\[
F(x+y) \subseteq F(x) + F(y)
\]
and
\[
g(x) \in F(x)
\]
for all \( x, y \in E_1 \).

Hence, Theorem 0 shows that \( g(x) \) is the unique additive selection of the set-valued mapping \( F(x) \) with the property \( F(x + y) \subseteq F(x) + F(y) \), where \( F \) is determined by \( f \).

In [10], the author introduced the concept of subadditive set-valued map and proved that such a map has a unique additive selection.

The result improves and generalizes the corresponding conclusions in [11] and [12]. The definition of this map is stated as follows:

Let \( X \) and \( Y \) be two real vector spaces, \( K \subseteq X \) be a zero cone, \( r \in \mathbb{N} \) with \( r > 1 \), \( \alpha_1, \alpha_2, \ldots, \alpha_r > 0 \) and \( \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_r \geq 0 \) with \( \mathcal{F}_1 + \mathcal{F}_2 + \cdots + \mathcal{F}_r > 0 \). A set-valued map \( F : K \rightarrow P_0(Y) \) is called \((\alpha_1, \alpha_2, \ldots, \alpha_r) - (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_r)\)-type subadditive set-valued map, if for any \( x_1, x_2, \ldots, x_r \in K \), the following holds:

\[
F(\sum_{i=1}^r \alpha_i x_i) \subseteq \sum_{i=1}^r \mathcal{F}_i F(x_i).
\]

In this paper, we define a new subadditive set-valued mapping satisfying some inclusion relation on a zero cone in a real vector space, and then prove that the map has a unique additive selection map.

## 2 Main Results

**Theorem 1.** Let \( K \) be a zero cone of a real vector space \( X \), \( Y \) a Banach space, \( r \) and \( s \) two given positive integers. If a set-valued map \( F : K \rightarrow \operatorname{ccl}(Y) \) satisfies that for any \( x_1, x_2, \ldots, x_r, x_{r+1}, \ldots, x_{r+s} \in K \), the following holds

\[
F(\sum_{i=1}^r x_i + \sum_{j=1}^s x_{r+j}) \subseteq rF\left(\frac{\sum_{i=1}^r x_i}{r}\right) + sF\left(\frac{\sum_{j=1}^s x_{r+j}}{s}\right),
\]

and for each \( x \in K \), \( \sup\{\operatorname{diam}(F(x)) : x \in K\} < +\infty \), then \( F \) has a unique additive selection map.

**Proof.** Take an element \( x \in K \) and let \( x_1 = x_2 = \cdots = x_{r+1} = \cdots = x_{r+s} = x \), then (1) becomes the following

\[
F((r+s)x) \subseteq rF(x) + sF(x) = (r+s)F(x).
\]

For any fixed \( n \in \mathbb{N} \), replacing \( x \) by \((r+s)^n x\), then the above formula becomes

\[
F((r+s)^{n+1}x) \subseteq (r+s)F((r+s)^n x),
\]

hence we obtain

\[
F((r+s)^{n+1} x) \subseteq \frac{F((r+s)^n x)}{(r+s)^n}.
\]

Let \( F_n(x) = \frac{F((r+s)^n x)}{(r+s)^n} \) for each \( x \in K \) and \( n \in \mathbb{N} \), then for each fixed \( x \in K \), \( \{F_n(x)\}_{n \in \mathbb{N}} \) is a decreasing sequence of closed convex subsets of a Banach space \( Y \), and the following holds

\[
\operatorname{diam}(F_n(x)) = \frac{1}{(r+s)^n} \operatorname{diam}F((r+s)^n x), \quad \forall x \in K, \quad n \in \mathbb{N}.
\]
Hence by given condition, \( \lim_{n \to \infty} \text{diam}(F_n(x)) = 0 \) for all \( x \in K \). Using Cantor theorem for the sequence \( \{F_n(x)\}_{n \in \mathbb{N}\cup\{0\}} \), we can conclude that for each \( x \in K \), the intersection \( \bigcap_{n=0}^{\infty} F_n(x) \) is a singleton set. Let \( f(x) \) denote the intersection for each \( x \in K \), then we can obtain a single valued map \( f : K \to Y \), and \( f \) is also a selection of \( F \) since \( f(x) \in F_0(x) = F(x) \) for all \( x \in K \).

For any \( x_1, x_2, \cdots, x_r, x_{r+1}, x_{r+2}, \cdots, x_{r+s} \in K \), by the definition of \( F_n \),

\[
F_n(x_1 + x_2 + \cdots + x_r + x_{r+1} + x_{r+2} + \cdots + x_{r+s}) = F((r+s)^n(x_1 + x_2 + \cdots + x_r + x_{r+1} + x_{r+2} + \cdots + x_{r+s})) = rF((r+s)^n x_1 + (r+s)^n x_2 + \cdots + (r+s)^n x_r + (r+s)^n x_{r+1} + \cdots + (r+s)^n x_{r+s}) \leq rF_n \left( \frac{x_1 + x_2 + \cdots + x_r}{r} \right) + sF_n \left( \frac{x_{r+1} + x_{r+2} + \cdots + x_{r+s}}{s} \right).
\]

Hence

\[
f(\sum_{i=1}^{r} x_i + \sum_{j=1}^{s} x_{r+j}) = \bigcap_{n=0}^{\infty} F_n(\sum_{i=1}^{r} x_i + \sum_{j=1}^{s} x_{r+j}) \subseteq \bigcap_{n=0}^{\infty} [rF_n(\frac{\sum_{i=1}^{r} x_i}{r}) + sF_n(\frac{\sum_{j=1}^{s} x_{r+j}}{s})].
\]

On the other hand, for each \( n \in \mathbb{N}\cup\{0\} \),

\[
f(\sum_{i=1}^{r} x_i) \in F_n(\sum_{i=1}^{r} x_i),
\]

\[
f(\sum_{j=1}^{s} x_{r+j}) \in F_n(\sum_{j=1}^{s} x_{r+j}),
\]

hence we obtain

\[
|| f(\sum_{i=1}^{r} x_i + \sum_{j=1}^{s} x_{r+j}) - [rf(\frac{\sum_{i=1}^{r} x_i}{r}) + sf(\frac{\sum_{j=1}^{s} x_{r+j}}{s})] || \leq r \text{diam} \left[ F_n(\frac{\sum_{i=1}^{r} x_i}{r}) \right] + s \text{diam} \left[ F_n(\frac{\sum_{j=1}^{s} x_{r+j}}{s}) \right] \to 0, \quad \text{as } n \to 0.
\]

And therefore, we obtain the following equation

\[
f(\sum_{i=1}^{r} x_i + \sum_{j=1}^{s} x_{r+j}) = rf(\frac{\sum_{i=1}^{r} x_i}{r}) + sf(\frac{\sum_{j=1}^{s} x_{r+j}}{s}).
\]
If \( r = s = 1 \), then it is easy to know from (2) that \( f \) is additive. From now on, suppose that \( r \geq 2 \) or \( s \geq 2 \).

Let \( x_1 = x_2 = \cdots = x_r = x_{r+1} = \cdots = x_{r+s} = 0 \), then (2) becomes \( f(0) = rf(0) + sf(0) \), hence \( f(0) = 0 \). For any \( x \in K \), take \( x_1 = x_2 = \cdots = x_r = x \) and \( x_{r+1} = \cdots = x_{r+s} = x \), then (2) becomes \( f(rx) = rf(x) \), and replacing \( x \) by \( \frac{x}{r} \), then we obtain \( rf\left(\frac{x}{r}\right) = f(x) \), and therefore

\[
r^{2}f\left(\frac{x}{r^2}\right) = r\left[rf\left(\frac{x}{r}\right)\right] = rf\left(\frac{x}{r}\right) = f(x).
\]

Repeating the process, we obtain its general form \( r^{k}f\left(\frac{x}{r^k}\right) = f(x) \) for any \( k \in \mathbb{N} \) and \( x \in K \). Similarly, let \( x_1 = x_2 = \cdots = x_r = 0 \) and \( x_{r+1} = \cdots = x_{r+s} = x \), then we obtain from (2) that \( f(x) = sf\left(\frac{x}{s}\right) \), hence we have its general form \( s^{k}f\left(\frac{x}{s^k}\right) = f(x) \) for any \( k \in \mathbb{N} \) and \( x \in K \).

If \( r \geq 2 \), then we will obtain from (2) that

\[
f\left(\frac{x_1 + x_2 + \cdots + x_r}{r}\right) = f\left(\frac{x_1 + x_2 + \cdots + x_{r-1} + 0 + x_r + 0 + \cdots + 0}{r}\right) \quad \text{(the number of 0 is s)}
\]

\[
= rf\left(\frac{x_1 + x_2 + \cdots + x_{r-1}}{r^2}\right) + sf\left(\frac{x_r}{sr}\right)
\]

\[
= rf\left(\frac{x_1 + x_2 + \cdots + x_{r-1}}{r^2}\right) + f\left(\frac{x_r}{sr}\right)
\]

\[
= f\left(\frac{x_1 + x_2 + \cdots + x_{r-2} + 0 + 0 + x_{r-1} + 0 + \cdots + 0}{r}\right) + f\left(\frac{x_r}{r}\right) \quad \text{(the number of 0 is s)}
\]

\[
= [rf\left(\frac{x_1 + x_2 + \cdots + x_{r-2}}{r^2}\right) + sf\left(\frac{x_{r-1}}{sr}\right)] + f\left(\frac{x_r}{r}\right)
\]

\[
= f\left(\frac{x_1 + x_2 + \cdots + x_{r-2}}{r}\right) + f\left(\frac{x_{r-1}}{r}\right) + f\left(\frac{x_r}{r}\right)
\]

\[
= \cdots
\]

\[
= f\left(\frac{x_1}{r}\right) + f\left(\frac{x_2}{r}\right) + \cdots + f\left(\frac{x_{r-1}}{r}\right) + f\left(\frac{x_r}{r}\right),
\]

hence

\[
rf\left(\frac{x_1 + x_2 + \cdots + x_r}{r}\right) = r\left[f\left(\frac{x_1}{r}\right) + f\left(\frac{x_2}{r}\right) + \cdots + f\left(\frac{x_{r-1}}{r}\right) + f\left(\frac{x_r}{r}\right)\right],
\]

and therefore,

\[
f(x_1 + x_2 + \cdots + x_r) = f(x_1) + f(x_2) + \cdots + f(x_{r-1}) + f(x_r).
\]

Thus, \( f \) is additive.
Similarly, if \( s \geq 2 \), then we have
\[
\begin{align*}
  f(\frac{x_{r+1} + \cdots + x_{r+s}}{s}) &= f\left(\frac{0 + 0 + \cdots + 0 + x_{r+1}}{s} + \frac{0 + x_{r+2} + \cdots + x_{r+s}}{s}\right) \quad \text{(the number of 0 is } r) \\
  &= rf\left(\frac{x_{r+1}}{rs}\right) + sf\left(\frac{x_{r+2} + x_{r+3} + \cdots + x_{r+s}}{s^2}\right) \\
  &= f\left(\frac{x_{r+1}}{s}\right) + f\left(\frac{x_{r+2} + x_{r+3} + \cdots + x_{r+s}}{s}\right) \\
  &= f\left(\frac{x_{r+1}}{s}\right) + rf\left(\frac{x_{r+2}}{rs}\right) + sf\left(\frac{x_{r+3} + \cdots + x_{r+s}}{s^2}\right) \\
  &= f\left(\frac{x_{r+1}}{s}\right) + f\left(\frac{x_{r+2}}{s}\right) + f\left(\frac{x_{r+3} + \cdots + x_{r+s}}{s}\right) \\
  &= \ldots \\
  &= f\left(\frac{x_{r+1}}{s}\right) + f\left(\frac{x_{r+2}}{s}\right) + \cdots + f\left(\frac{x_{r+s}}{s}\right).
\end{align*}
\]

Hence
\[
  sf\left(\frac{x_{r} + x_{r+1} + \cdots + x_{r+s}}{s}\right) = s\left[f\left(\frac{x_{r+1}}{s}\right) + f\left(\frac{x_{r+2}}{s}\right) + \cdots + f\left(\frac{x_{r+s}}{s}\right)\right],
\]
that is,
\[
  f\left(x_{r} + x_{r+1} + \cdots + x_{r+s}\right) = f\left(x_{r+1}\right) + f\left(x_{r+2}\right) + \cdots + f\left(x_{r+s}\right).
\]

This shows that \( f \) is additive.

Next, let us prove the uniqueness of the additive selection maps of \( F \).

Suppose that \( f_1 \) and \( f_2 \) are two additive selection maps of \( F \), then for each \( n \in \mathbb{N} \) and \( x \in K \), we have
\[
nf_i(x) = f_i(nx) \in F(nx), \quad i = 1, 2,
\]
hence \( n \| f_1(x) - f_2(x) \| = \| f_1(nx) - f_2(nx) \| \leq \text{diam} F(nx) \), i.e., \( \| f_1(x) - f_2(x) \| \leq \frac{\text{diam} F(nx)}{n} \).

Let \( n \to +\infty \), then by (ii), \( f_1(x) = f_2(x) \) for each \( x \in K \). This shows that the additive selection map of \( F \) is unique.

Using the same method as in Theorem 1, we can obtain more general form than Theorem 1, but we omit its proof.

**Theorem 2.** Let \( K \) be a zero cone of a real vector space \( X \), \( Y \) a Banach space and \( r_1, r_2, \ldots, r_k \) given positive integers. If a set-valued map \( F : K \to \text{ccl}(Y) \) satisfies that for any \( x_1, x_2, \ldots, x_{r_1}, x_{r_1+1}, \ldots, x_{r_1+r_2}, \ldots, x_{r_1+r_2+\cdots+r_k} \in K \), the following holds
\[
  F\left(\sum_{i=1}^{r_k} x_{r_1+i} + \sum_{i=1}^{r_2} x_{r_1+i+1} + \cdots + \sum_{i=1}^{r_1} x_{r_1+r_2+\cdots+r_{k-1}+i}\right) \\
  \subseteq r_1 F\left(\frac{\sum_{i=1}^{r_k} x_{r_1+i}}{r_1}\right) + r_2 F\left(\frac{\sum_{i=1}^{r_2} x_{r_1+i}}{r_2}\right) + \cdots + r_k F\left(\frac{\sum_{i=1}^{r_k} x_{r_1+r_2+\cdots+r_{k-1}+i}}{r_k}\right),
\]

\( F \) is additive.
and for each $x \in K$, $\sup \{ \text{diam}(F(x) : x \in K) < +\infty \}$, then $F$ has an unique additive selection map.

References


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