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# EXISTENCE PROBLEMS OF ADDITIVE SELECTION MAPS FOR ANOTHER TYPE SUBADDITIVE SET-VALUED MAP

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**Abstract.** In this paper, we consider the following subadditive set-valued map $F : X \longrightarrow P_0(Y)$ :

$$F\Big(\sum_{i=1}^{r} x_i + \sum_{j=1}^{s} x_{r+j}\Big) \subseteq rF\Big(\frac{\sum_{i=1}^{r} x_i}{r}\Big) + sF\Big(\frac{\sum_{j=1}^{s} x_{r+j}}{s}\Big), \quad \forall x_i \in X, \quad i = 1, 2, \cdots, r+s,$$

where $r$ and $s$ are two natural numbers. And we discuss the existence and unique problem of additive selection maps for the above set-valued map.

**Key words:** *additive selection map, subadditive, additive selection, cone*

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## 1 Introduction and Preliminaries

The stability problem of functional equations was originated from a question of Ulam[1] concerning the stability of group homomorphisms. In 1941, D.H Hyers[2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. The famous stability theorem is as follows:

**Theorem 0.** *Let $E_1$ be a normed vector space and $E_2$ a Banach space. Suppose that the mapping $f : E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \tag{0}$$

*for all $x, y \in E_1$, with $\varepsilon > 0$ a constant. Then the limit*

$$g(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

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*exists for each $x \in E_1$ and $g$ is the unique additive mapping satisfying*

$$\|f(x) - g(x)\| \leq \varepsilon$$

*for all $x \in E_1$.*

Later, Hyers' Theorem has been generalized by many authors[3−8].

Let $X$ a real vector space. We denote by $P_0(X)$ the family of all nonempty subsets of $X$.

If $Y$ is a topological vector space, the family of all closed convex subsets of $Y$ denoted by $\mathrm{ccl}(Y)$.

Let $A$ and $B$ are two nonempty subsets of the real vector space $X$, $\lambda$ and $\mu$ are two real numbers. Define

$$A + B = \{x | x = a + b, a \in A, b \in B\};$$
$$\lambda A = \{x | x = \lambda a, a \in A\}.$$

The next properties are obvious:

**Lemma.** *If $A$ and $B$ are two nonempty subsets of the real vector space $X$, $\lambda$ and $\mu$ are two real numbers, then*

$$\lambda(A + B) = \lambda A + \lambda B; \qquad (\lambda + \mu)A \subseteq \lambda A + \mu A.$$

*Furthermore, if $A$ is a convex subset and $\lambda\mu \geq 0$, then we have the following formula*[9]*:*

$$(\lambda + \mu)A = \lambda A + \mu A.$$

A subset $A \subset X$ is said to be a cone if $A + A \subseteq A$, and $\lambda A \subseteq A$ for all $\lambda > 0$.

If the zero in $X$ belongs to $A$, we say that $A$ is a zero cone.

Let $X$ and $Y$ be two real vector spaces, $f : X \longrightarrow Y$ a single-valued map, and $F : X \longrightarrow P_0(Y)$ a set-valued map. $f$ is called an additive selection of $F$, if $f(x+y) = f(x) + f(y)$ for all $x, y \in X$, and $f(x) \in F(x)$ for all $x \in X$.

Let $B(0, \varepsilon)$ denote the open ball with center 0 and radius $\varepsilon$ in $E_2$ in Theorem 0, then the inequality (0) may be written as

$$f(x + y) \in B(0, \varepsilon) + f(x) + f(y),$$

and hence

$$f(x + y) + B(0, \varepsilon) \subseteq f(x) + B(0, \varepsilon) + f(y) + B(0, \varepsilon).$$

where $B(0, \varepsilon) + x$ denote the open ball with center $x$ and radius $\varepsilon$ in $E_2$.

Thus, if we define a set-valued mapping $F$ by $F(x) = f(x) + B(0, \varepsilon)$ for each $x \in E_1$, then we get

$$F(x + y) \subseteq F(x) + F(y)$$

and

$$g(x) \in F(x)$$

for all $x, y \in E_1$.

Hence, Theorem 0 shows that $g(x)$ is the unique additive selection of the set-valued mapping $F(x)$ with the property $F(x+y) \subseteq F(x)+F(y)$ , where $F$ is determined by $f$.

In [10], the author introduced the concept of subadditive set-valued map and proved that such a map has a unique additive selection.

The result improves and generalizes the corresponding conclusions in [11] and [12]. The definition of this map is stated as follows:

Let $X$ and $Y$ be two real vector spaces, $K \subseteq X$ be a zero cone, $r \in \mathbf{N}$ with $r > 1$, $\alpha_1, \alpha_2, \cdots, \alpha_r > 0$ and $\overline{\alpha}_1, \overline{\alpha}_2, \cdots, \overline{\alpha}_r \geq 0$ with $\overline{\alpha}_1 + \overline{\alpha}_2 + \cdots + \overline{\alpha}_r > 0$. A set-valued map $F : K \longrightarrow P_0(Y)$ is called $(\alpha_1, \alpha_2, \cdots, \alpha_r)$-$(\overline{\alpha}_1, \overline{\alpha}_2, \cdots, \overline{\alpha}_r)$-type subadditive set-valued map, if for any $x_1, x_2, \cdots, x_r \in K$, the following holds:

$$F(\textstyle\sum_{i=1}^r \alpha_i x_i) \subset \sum_{i=1}^r \overline{\alpha}_i F(x_i).$$

In this paper, we define a new subadditive set-valued mapping satisfying some inclusion relation on a zero cone in a real vector space, and then prove that the map has a unique additive selection map.

## 2 Main Results

**Theorem 1**. *Let $K$ be a zero cone of a real vector space $X$, $Y$ a Banach space, $r$ and $s$ two given positive integers. If a set-valued map $F : K \longrightarrow \mathrm{ccl}(Y)$ satisfies that for any $x_1, x_2, \cdots, x_r, x_{r+1}, \cdots, x_{r+s} \in K$, the following holds*

$$F\Big(\sum_{i=1}^r x_i + \sum_{j=1}^s x_{r+j}\Big) \subseteq rF\Big(\frac{\sum_{i=1}^r x_i}{r}\Big) + sF\Big(\frac{\sum_{j=1}^s x_{r+j}}{s}\Big), \tag{1}$$

*and for each $x \in K$, $\sup\{\mathrm{diam}(F(x) : x \in K\} < +\infty$, then $F$ has a unique additive selection map.*

*Proof.* Take an element $x \in K$ and let $x_1 = x_2 = \cdots = x_{r+1} = \cdots = x_{r+s} = x$, then (1) becomes the following

$$F\big((r+s)x\big) \subseteq rF(x) + sF(x) = (r+s)F(x).$$

For any fixed $n \in \mathbf{N}$, replacing $x$ by $(r+s)^n x$, then the above formula becomes

$$F\big((r+s)^{n+1}x\big) \subseteq (r+s)F\big((r+s)^n x\big),$$

hence we obtain

$$\frac{F\big((r+s)^{n+1}x\big)}{(r+s)^{n+1}} \subseteq \frac{F\big((r+s)^n x\big)}{(r+s)^n}.$$

Let $F_n(x) = \dfrac{F\big((r+s)^n x\big)}{(r+s)^n}$ for each $x \in K$ and $n \in \mathbf{N}$, then for each fixed $x \in K$, $\{F_n(x)\}_{\mathbf{N}\cup\{0\}}$ is a decreasing sequence of closed convex subsets of a Banach space $Y$, and the following holds

$$\mathrm{diam}(F_n(x)) = \frac{1}{(r+s)^n}\,\mathrm{diam}F\big((r+s)^r)x\big), \qquad \forall x \in K, \quad n \in \mathbf{N}.$$

Hence by given condition, $\lim_{n\to+\infty}\operatorname{diam}(F_n(x))=0$ for all $x\in K$. Using Cantor theorem for the sequence $\{F_n(x)\}_{n\in\mathbf{N}\cup\{0\}}$, we can conclude that for each $x\in K$, the intersection $\bigcap_{n=0}^{+\infty}F_n(x)$ is a singleton set. Let $f(x)$ denote the intersection for each $x\in K$, then we can obtain a single valued map $f:K\to Y$, and $f$ is also a selection of $F$ since $f(x)\in F_0(x)=F(x)$ for all $x\in K$.

For any $x_1,x_2,\cdots,x_r,x_{r+1},x_{r+2},\cdots,x_{r+s}\in K$, by the definition of $F_n$,

$$\begin{aligned}
&F_n(x_1+x_2+\cdots+x_r+x_{r+1}+x_{r+2}+\cdots+x_{r+s})\\
=\ &\frac{F\big((r+s)^n(x_1+x_2+\cdots+x_r+x_{r+1}+x_{r+2}+\cdots+x_{r+s})\big)}{(r+s)^n}\\
=\ &\frac{F\big((r+s)^nx_1+(r+s)^nx_2+\cdots+(r+s)^nx_r+(r+s)^nx_{r+1}+\cdots+(r+s)^nx_{r+s}\big)}{(r+s)^n}\\
\subseteq\ &\frac{rF\big(\frac{(r+s)^nx_1+(r+s)^nx_2+\cdots+(r+s)^nx_r}{r}\big)+sF\big(\frac{(r+s)^nx_{r+1}+(r+s)^nx_{r+2}+\cdots+(r+s)^nx_{r+s}}{s}\big)}{(r+s)^n}\\
=\ &rF_n\big(\frac{x_1+x_2+\cdots+x_r}{r}\big)+sF_n\big(\frac{x_{r+1}+x_{r+2}+\cdots+x_{r+s}}{s}\big).
\end{aligned}$$

Hence

$$\begin{aligned}
f\big(\sum_{i=1}^r x_i+\sum_{j=1}^s x_{r+j}\big) &= \bigcap_{n=0}^{+\infty}F_n\big(\sum_{i=1}^r x_i+\sum_{j=1}^s x_{r+j}\big)\\
&\subseteq \bigcap_{n=0}^{+\infty}\big[rF_n\big(\frac{\sum_{i=1}^r x_i}{r}\big)+sF_n\big(\frac{\sum_{j=1}^s x_{r+j}}{s}\big)\big].
\end{aligned}$$

On the other hand, for each $n\in\mathbf{N}\cup\{0\}$,

$$\begin{aligned}
f\big(\frac{\sum_{i=1}^r x_i}{r}\big)&\in F_n\big(\frac{\sum_{i=1}^r x_i}{r}\big),\\
f\big(\frac{\sum_{j=1}^s x_{r+j}}{s}\big)&\in F_n\big(\frac{\sum_{j=1}^s x_{r+j}}{s}\big),
\end{aligned}$$

hence we obtain

$$\begin{aligned}
&\|\, f\big(\sum_{i=1}^r x_i+\sum_{j=1}^s x_{r+j}\big)-\big[rf\big(\frac{\sum\limits_{i=1}^r x_i}{r}\big)+sf\big(\frac{\sum_{j=1}^s x_{r+j}}{s}\big)\big]\,\|\\
&\le r\operatorname{diam}\big[F_n\big(\frac{\sum\limits_{i=1}^r x_i}{r}\big)\big]+s\operatorname{diam}\big[F_n\big(\frac{\sum\limits_{j=1}^s x_{r+j}}{s}\big)\big]\longrightarrow 0,\quad \text{as } n\longrightarrow 0.
\end{aligned}$$

And therefore, we obtain the following equation

$$f\big(\sum_{i=1}^r x_i+\sum_{j=1}^s x_{r+j}\big)=rf\big(\frac{\sum\limits_{i=1}^r x_i}{r}\big)+sf\big(\frac{\sum\limits_{j=1}^s x_{r+j}}{s}\big). \tag{2}$$

If $r=s=1$, then it is easy to know from (2) that $f$ is additive. From now on , suppose that $r\geq 2$ or $s\geq 2$.

Let $x_1=x_2=\cdots=x_r=x_{r+1}=\cdots=x_{r+s}=0$, then (2) becomes $f(0)=rf(0)+sf(0)$, hence $f(0)=0$. For any $x\in K$, take $x_1=x_2=\cdots=x_r=x$ and $x_{r+1}=\cdots_{r+s}=0$, then (2) becomes $f(rx)=rf(x)$, and replacing $x$ by $\frac{x}{r}$, then we obtain $rf\big(\frac{x}{r}\big)=f(x)$, and therefore

$$r^2 f\Big(\frac{x}{r^2}\Big)=r\Big[rf\Big(\frac{\big(\frac{x}{r}\big)}{r}\Big)\Big]=rf\Big(\frac{x}{r}\Big)=f(x).$$

Repeating the process, we obtain its general form $r^k f\big(\frac{x}{r^k}\big)=f(x)$ for any $k\in\mathbf{N}$ and $x\in K$. Similarly, let $x_1=x_2=\cdots=x_r=0$ and $x_{r+1}=\cdots=x_{r+s}=x$, then we obtain from (2) that $f(x)=sf\big(\frac{x}{s}\big)$, hence we have its general form $s^k f\big(\frac{x}{s^k}\big)=f(x)$ for any $k\in\mathbf{N}$ and $x\in K$.

If $r\geq 2$, then we will obtain from (2) that

$$\begin{aligned}
&f\Big(\frac{x_1+x_2+\cdots+x_r}{r}\Big)\\
&=f\Big(\frac{x_1+x_2+\cdots+x_{r-1}+0}{r}+\frac{x_r+0+\cdots+0}{r}\Big)\ \text{(the number of 0 is } s\text{)}\\
&=rf\Big(\frac{x_1+x_2+\cdots+x_{r-1}}{r^2}\Big)+sf\Big(\frac{x_r}{sr}\Big)\\
&=f\Big(\frac{x_1+x_2+\cdots+x_{r-1}}{r}\Big)+f\Big(\frac{x_r}{r}\Big)\\
&=f\Big(\frac{x_1+x_2+\cdots+x_{r-2}+0+0}{r}+\frac{x_{r-1}+0+\cdots+0}{r}\Big)+f\Big(\frac{x_r}{r}\Big)\ \text{(the number of 0 is } s\text{)}\\
&=\Big[rf\Big(\frac{x_1+x_2+\cdots+x_{r-2}}{r^2}\Big)+sf\Big(\frac{x_{r-1}}{sr}\Big)\Big]+f\Big(\frac{x_r}{r}\Big)\\
&=f\Big(\frac{x_1+x_2+\cdots+x_{r-2}}{r}\Big)+f\Big(\frac{x_{r-1}}{r}\Big)+f\Big(\frac{x_r}{r}\Big)\\
&=\cdots\\
&=f\Big(\frac{x_1}{r}\Big)+f\Big(\frac{x_2}{r}\Big)+\cdots+f\Big(\frac{x_{r-1}}{r}\Big)+f\Big(\frac{x_r}{r}\Big),
\end{aligned}\tag{3}$$

hence

$$rf\Big(\frac{x_1+x_2+\cdots+x_r}{r}\Big)=r\Big[f\Big(\frac{x_1}{r}\Big)+f\Big(\frac{x_2}{r}\Big)+\cdots+f\Big(\frac{x_{r-1}}{r}\Big)+f\Big(\frac{x_r}{r}\Big)\Big],$$

and therefore,

$$f\big(x_1+x_2+\cdots+x_r\big)=f\big(x_1\big)+f\big(x_2\big)+\cdots+f\big(x_{r-1}\big)+f\big(x_r\big).$$

Thus, $f$ is additive.

Similarly, if $s \geq 2$, then we have

$$
\begin{aligned}
& f\Big(\frac{x_{r+1}+\cdots+x_{r+s}}{s}\Big)\\
=& f\Big(\frac{0+0+\cdots+0+x_{r+1}}{s}+\frac{0+x_{r+2}+\cdots+x_{r+s}}{s}\Big) \text{ (the number of 0 is } r)\\
=& rf\Big(\frac{x_{r+1}}{rs}\Big)+sf\Big(\frac{x_{r+2}+x_{r+3}+\cdots+x_{r+s}}{s^2}\Big)\\
=& f\Big(\frac{x_{r+1}}{s}\Big)+f\Big(\frac{x_{r+2}+x_{r+3}+\cdots+x_{r+s}}{s}\Big)\\
=& f\Big(\frac{x_{r+1}}{s}\Big)+f\Big(\frac{0+0+\cdots+0+x_{r+2}}{s}+\frac{0+0+x_{r+3}+\cdots+x_{r+s}}{s}\Big) \text{ (the number of 0 is } r)\\
=& f\Big(\frac{x_{r+1}}{s}\Big)+rf\Big(\frac{x_{r+2}}{rs}\Big)+sf\Big(\frac{x_{r+3}+\cdots+x_{r+s}}{s^2}\Big)\\
=& f\Big(\frac{x_{r+1}}{s}\Big)+f\Big(\frac{x_{r+2}}{s}\Big)+f\Big(\frac{x_{r+3}+\cdots+x_{r+s}}{s}\Big)\\
=& \cdots\\
=& f\Big(\frac{x_{r+1}}{s}\Big)+f\Big(\frac{x_{r+2}}{s}\Big)+\cdots+f\Big(\frac{x_{r+s}}{s}\Big). \qquad (4)
\end{aligned}
$$

Hence

$$sf\Big(\frac{x_r+x_{r+1}+\cdots+x_{r+s}}{s}\Big)=s\Big[f\Big(\frac{x_{r+1}}{s}\Big)+f\Big(\frac{x_{r+2}}{s}\Big)+\cdots+f\Big(\frac{x_{r+s}}{s}\Big)\Big],$$

that is,

$$f\big(x_r+x_{r+1}+\cdots+x_{r+s}\big)=f\big(x_{r+1}\big)+f\big(x_{r+2}\big)+\cdots+f\big(x_{r+s}\big).$$

This shows that $f$ is additive.

Next, let us prove the uniqueness of the additive selection maps of $F$.

Suppose that $f_1$ and $f_2$ are two additive selection maps of $F$, then for each $n \in \mathbf{N}$ and $x \in K$, we have

$$nf_i(x)=f_i(nx)\in F(nx), \qquad i=1,2,$$

hence $n \parallel f_1(x)-f_2(x) \parallel = \parallel f_1(nx)-f_2(nx) \parallel \leq \mathrm{diam}F(nx)$, i.e., $\parallel f_1(x)-f_2(x) \parallel \leq \frac{1}{n}\mathrm{diam}F(nx)$. Let $n \to +\infty$, then by (ii), $f_1(x)=f_2(x)$ for each $x \in K$. This shows that the additive selection map of $F$ is unique.

Using the same method as in Theorem 1, we can obtain more general form than Theorem 1, but we omit its proof.

**Theorem 2**. *Let $K$ be a zero cone of a real vector space $X$, $Y$ a Banach space and $r_1, r_2, \cdots, r_k$ given positive integers. If a set-valued map $F: K \longrightarrow \mathrm{ccl}(Y)$ satisfies that for any $x_1, x_2, \cdots, x_{r_1}, x_{r_1+1}, \cdots, x_{r_1+r_2}, \cdots, x_{r_1+r_2+\cdots+r_k} \in K$, the following holds*

$$
\begin{aligned}
& F\Big(\sum_{i=1}^{r_r}x_i+\sum_{i=1}^{r_2}x_{r_1+i}+\cdots+\sum_{i=1}^{r_k}x_{r_1+r_2+\cdots+r_{k-1}+i}\Big)\\
& \subseteq r_1F\Big(\frac{\sum_{i=1}^{r}x_i}{r_1}\Big)+r_2F\Big(\frac{\sum_{i=1}^{r_2}x_{r_1+i}}{r_2}\Big)+\cdots r_kF\Big(\frac{\sum_{i=1}^{r_k}x_{r_1+r_2+\cdots+r_{k-1}+i}}{r_k}\Big),
\end{aligned} \qquad (5)
$$

*and for each* $x \in K$, $\sup\{\mathrm{diam}(F(x) : x \in K\} < +\infty$, *then* $F$ *has an unique additive selection map.*

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