

MONOTONE POINTS IN ORLICZ-BOCHNER SEQUENCE SPACES

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Abstract. In Orlicz-Bochner sequence spaces endowed with Orlicz norm and Luxemburg norm, points of lower monotonicity, upper monotonicity, lower local uniform monotonicity and upper local uniform monotonicity are characterized.

Key words: *Banach lattice, Orlicz-Bochner space, Luxemburg norm, Orlicz norm, upper (lower) monotone point, upper (lower) locally uniformly monotone point*

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1 Introduction

A Banach lattice X with a lattice norm $\|\cdot\|$ is said to be strictly monotone if for any $x \in X^+$ (positive cone in X) and any $y \in X^+ \setminus \{0\}$, we have $\|x + y\| > \|x\|$. A point $x \in S(X^+) := S(X) \cap X^+$ is said to be upper monotone (*UM* for short) if for any $y \in X^+ \setminus \{0\}$, $\|x + y\| > 1$. A point $x \in S(X^+)$ is said to be lower monotone (*LM*) if for any $y \in X^+ \setminus \{0\}$ and $y \leq x$, $\|x - y\| < 1$. An equivalent condition for X being strictly monotone^[15] is that any point $x \in S(X^+)$ is lower monotone. But lower monotone points and upper monotone points are different, see [12]. X is called upper locally uniformly monotone^[10] if for any $\varepsilon > 0$ and $x \in S(X^+)$, there exists $\delta(x, \varepsilon) > 0$ such that $y \in X^+$ and $\|y\| \geq \varepsilon$ imply $\|x + y\| \geq 1 + \delta(x, \varepsilon)$. If for any $\varepsilon > 0$ and $x \in S(X^+)$, there is $\delta(x, \varepsilon) > 0$ such that $\|x - y\| \leq 1 - \delta(x, \varepsilon)$ whenever $y \in X^+$, $\|y\| \geq \varepsilon$ and $y \leq x$, then X is said to be lower locally uniformly monotone^[10]. We can analogously

define point of lower local uniform monotonicity (LLUM) and point of upper local uniform monotonicity (ULUM).

It is well known that some rotundity properties of Banach spaces have been widely applied in ergodic theory, fixed pointed theory, probability theory and approximation theory etc, and in many cases these rotundity properties can be replaced by respective monotonicity properties when restrict ourselves to Banach space being Banach lattice^[1, 2, 3, 10]. Moreover, there are close relationships between monotonicity points and rotundity points^[7]. Therefore in recent years monotonicity properties and monotonicity points have been widely investigated in Musielak-Orlicz, Orlicz-Lorentz, Köthe-Bochner, Calderón-Lozanovskii spaces^[5, 9–14, 16]. In this paper we mainly give the criteria for a point in Orlicz-Bochner sequence spaces being UM, LM, ULUM, and LLUM.

Let \mathbf{R}, \mathbf{N} stand for the set of all real and natural numbers respectively. A function $M : \mathbf{R} \rightarrow \mathbf{R}^+$ is called a \mathcal{N} -function if M is convex, even, $M(0) = 0$, $M(u) > 0$ ($u \neq 0$) and $\lim_{u \rightarrow 0} \frac{M(u)}{u} = 0$, $\lim_{u \rightarrow \infty} \frac{M(u)}{u} = \infty$. M is said to satisfy the δ_2 -condition for small u ($M \in \delta_2$) if for some K and $u_0 > 0$, $M(2u) \leq KM(u)$ as $|u| \leq u_0$.

In the sequel, M and N denote a pair of complemented \mathcal{N} - functions, p the right-hand derivative of M . For a real sequence $u = (u(1), u(2), \dots)$, we call $\rho_M(u) = \sum_{i=1}^{\infty} M(u(i))$ the modular of u . The linear space $\{u : \rho_M(\lambda u) < \infty \text{ for some } \lambda > 0\}$ equipped with the Orlicz norm

$$\|u\|_M = \sup_{\rho_N(v) \leq 1} \sum_{i=1}^{\infty} u(i)v(i) = \inf_{k > 0} \frac{1}{k} \{1 + \rho_M(ku)\} = \frac{1}{k} \{1 + \rho_M(ku)\}, \quad \forall k \in K(u),$$

where $K(u) = [k^*, k^{**}]$, $k^* = \inf\{k > 0 : \rho_N(p(k|u|)) \geq 1\}$, $k^{**} = \sup\{k > 0 : \rho_N(p(k|u|)) \leq 1\}$, or the Luxemburg norm

$$\|u\|_{(M)} = \inf \left\{ \lambda > 0 : \rho_M \left(\frac{u}{\lambda} \right) \leq 1 \right\}$$

are Banach spaces. They are called Orlicz sequence spaces, and denoted by l_M and $l_{(M)}$ respectively. Their subspace $h_M = \{u : \rho_M(\lambda u) < \infty \text{ for all } \lambda > 0\}$ endowed with the norms, denoted by h_M and $h_{(M)}$ respectively, are also Banach spaces.

If $u = (u(1), u(2), \dots)$, where $u(i) \in X_i$ and X_i is a Banach space for any $i \in \mathbf{N}$, we denote by $l_M(X_i)$ and $l_{(M)}(X_i)$. We call such spaces Orlicz-Bochner sequence spaces. Moreover, we have known that if X_i is a Banach lattice for each $i \in \mathbf{N}$, then $l_M(X_i)$ and $l_{(M)}(X_i)$ are Banach lattices (with $u \leq v$ if and only if $x(i) \leq y(i)$) as well. In this paper, for $u = (u(1), u(2), \dots)$, denote $S_u := \{i \in \mathbf{N} : u(i) \neq 0\}$. For more details about Orlicz and Orlicz-Bochner spaces, we refer to [4, 8, 14].

2 Lemmas

Lemma 1. *Suppose $M \in \delta_2$ and X_i is a Banach space for each $i \in \mathbf{N}$. If $u_n, u \in l_{(M)}(X_i)$ satisfy $u_n(i) \rightarrow u(i)$ and $\rho_M(u_n) \rightarrow \rho_M(u)$ as $n \rightarrow \infty$, then $\|u_n - u\|_{(M)} \rightarrow 0$.*

Proof. Since $M \in \delta_2$, it suffices to show $\rho_M\left(\frac{1}{2}(u_n - u)\right) \rightarrow 0$. From $M \in \delta_2$ we know $\rho_M(u) < \infty, \rho_M(u_n) < \infty, \forall n \in \mathbf{N}$. So for any $\varepsilon_0 > 0$ there exists a $k \in \mathbf{N}$ such that

$$\sum_{i=k+1}^{\infty} M(\|u(i)\|) < \frac{\varepsilon_0}{4}.$$

Owing to $u_n(i) \rightarrow u(i)$, there holds

$$\sum_{i=1}^k M(\|u_n(i)\|) \rightarrow \sum_{i=1}^k M(\|u(i)\|),$$

consequently

$$\begin{aligned} \sum_{i=k+1}^{\infty} M(\|u_n(i)\|) &= \sum_{i=1}^{\infty} M(\|u_n(i)\|) - \sum_{i=1}^k M(\|u_n(i)\|) \\ &\rightarrow \sum_{i=1}^{\infty} M(\|u(i)\|) - \sum_{i=1}^k M(\|u(i)\|) = \sum_{i=k+1}^{\infty} M(\|u(i)\|). \end{aligned}$$

Therefore, there is $m \in \mathbf{N}$ such that for any $n > m$,

$$\sum_{i=k+1}^{\infty} M(\|u_n(i)\|) \leq \sum_{i=k+1}^{\infty} M(\|u(i)\|) + \frac{\varepsilon_0}{12} \leq \frac{\varepsilon_0}{3}.$$

By the continuity of M and $u_n(i) \rightarrow u(i)$, one gets $M\left(\frac{\|u_n(i) - u(i)\|}{2}\right) \rightarrow 0$ and

$$\sum_{i=1}^k M\left(\frac{\|u_n(i) - u(i)\|}{2}\right) \rightarrow 0.$$

Hence there is $l \in \mathbf{N}$ such that for any $n > l$,

$$\sum_{i=1}^k M\left(\frac{\|u_n(i) - u(i)\|}{2}\right) \leq \frac{\varepsilon_0}{3}.$$

Set $n_0 = \max\{m, l\}$. Then for any $n > n_0$,

$$\begin{aligned} \rho_M\left(\frac{1}{2}(u_n - u)\right) &= \sum_{i=1}^{\infty} M\left(\frac{\|u_n(i) - u(i)\|}{2}\right) \\ &= \sum_{i=1}^k M\left(\frac{\|u_n(i) - u(i)\|}{2}\right) + \sum_{i=k+1}^{\infty} M\left(\frac{\|u_n(i) - u(i)\|}{2}\right) \end{aligned}$$

$$\begin{aligned}
&< \frac{\varepsilon_0}{3} + \frac{1}{2} \sum_{i=k+1}^{\infty} M(\|u_n(i)\|) + \frac{1}{2} \sum_{i=k+1}^{\infty} M(\|u(i)\|) \\
&< \frac{\varepsilon_0}{3} + \frac{1}{2} \cdot \frac{\varepsilon_0}{3} + \frac{1}{2} \cdot \frac{\varepsilon_0}{4} < \varepsilon_0,
\end{aligned}$$

that is $\rho_M\left(\frac{1}{2}(u_n - u)\right) \rightarrow 0$.

Corollary 1. Suppose $M \in \delta_2$ and X_i is a Banach space for each $i \in \mathbf{N}$. If $u_n, u \in l_{(M)}(X_i)$ satisfy $u_n(i) \rightarrow u(i)$ and $\|u_n\|_{(M)} \rightarrow \|u\|_{(M)}$ as $n \rightarrow \infty$, then $\|u_n - u\|_{(M)} \rightarrow 0$.

Proof. It is obviously holds when $u = 0$. If $u \neq 0$, then there exists a $n_0 \in \mathbf{N}$ such that for any $n > n_0$, $\|u_n\|_{(M)} > 0$. For $n > n_0$, denote

$$v_n := \frac{u_n}{\|u_n\|_{(M)}}, \quad v := \frac{u}{\|u\|_{(M)}}.$$

Then $\|v_n\|_{(M)} = \|v\|_{(M)} = 1$, and consequently $\rho_M(v_n) = \rho_M(v)$. Combining $u_n(i) \rightarrow u(i)$ with $\|u_n\|_{(M)} \rightarrow \|u\|_{(M)}$, we have $v_n(i) \rightarrow v(i), i \in \mathbf{N}$. Whence by lemma 1, $\|v_n - v\|_{(M)} \rightarrow 0$. And by

$$\begin{aligned}
\|v_n - v\|_{(M)} &= \left\| \frac{u_n}{\|u_n\|_{(M)}} - \frac{u_n}{\|u\|_{(M)}} + \frac{u_n}{\|u\|_{(M)}} - \frac{u}{\|u\|_{(M)}} \right\| \\
&\geq \left| \|u_n\|_{(M)} \cdot \frac{\|u\|_{(M)} - \|u_n\|_{(M)}}{\|u_n\|_{(M)} \cdot \|u\|_{(M)}} - \frac{\|u_n - u\|_{(M)}}{\|u\|_{(M)}} \right|,
\end{aligned}$$

we get $\|u_n - u\|_{(M)} \rightarrow 0$.

Lemma 2. Suppose $M \in \delta_2$ and X_i is a Banach space for each $i \in \mathbf{N}$. If $u_n, u_0 \in l_M(X_i)$ satisfy $u_n(i) \rightarrow u_0(i)$ and $\|u_n\|_M \rightarrow \|u_0\|_M$ as $n \rightarrow \infty$, then $\|u_n - u_0\|_M \rightarrow 0$.

Proof. It is evidently holds when $u = 0$. If $u \neq 0$, then there exists a $n_0 \in \mathbf{N}$ such that for any $n > n_0$, $\|u_n\|_M > 0$. For $n > n_0$ set

$$v_n := \frac{u_n}{\|u_n\|_M}, \quad v := \frac{u_0}{\|u_0\|_M}.$$

By the proof of Corollary 1, it is enough to check $\|v_n - v\|_M \rightarrow 0$.

For $v_n \in S(l_M(X_i))$ there exists a $k_n > 1$ satisfying

$$1 = \|v_n\|_M = \frac{1}{k_n}(1 + \rho_M(k_n v_n)).$$

By $M \in \delta_2$ we know that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho_M(u) < \delta$ implies $\|u\|_M < \varepsilon$. Due to $\rho_M(v) < \infty$ there is $i_1 \in \mathbf{N}$ such that $\sum_{i=i_1+1}^{\infty} M(\|v(i)\|) < \delta$, which follows that

$$\left\| \sum_{i=i_1+1}^{\infty} v(i)e_i \right\|_M < \varepsilon.$$

Note that $\|[v]_n\|_M \nearrow \|v\|_M$, where $[v]_n = (v(1), v(2), \dots, v(n), 0, \dots)$, there exists $i_2 \in \mathbf{N}$ such that

$$\left\| \sum_{i=1}^{i_2} v(i)e_i \right\|_M > 1 - \delta.$$

Set $i_0 := \max\{i_1, i_2\}$. Then

$$\left\| \sum_{i=i_0+1}^{\infty} v(i)e_i \right\|_M < \varepsilon \quad \text{and} \quad \left\| \sum_{i=1}^{i_0} v(i)e_i \right\|_M > 1 - \delta.$$

Certainly $\sum_{i=1}^{i_0} M(\|v_n(i)\|) \rightarrow \sum_{i=1}^{i_0} M(\|v(i)\|)$ by $v_n(i) \rightarrow v(i)$. From Lemma 1 and $M \in \delta_2$ one gets

$$\left\| \sum_{i=1}^{i_0} v_n(i)e_i - \sum_{i=1}^{i_0} v(i)e_i \right\|_M \rightarrow 0, \text{ and by the equivalent of } \|\cdot\|_{(M)} \text{ and } \|\cdot\|_M,$$

$$\left\| \sum_{i=1}^{i_0} v_n(i)e_i - \sum_{i=1}^{i_0} v(i)e_i \right\|_M \rightarrow 0. \tag{1}$$

So $\left\| \sum_{i=1}^{i_0} v_n(i)e_i \right\|_M \rightarrow \left\| \sum_{i=1}^{i_0} v(i)e_i \right\|_M$, and there is an $n_1 \in \mathbf{N}$ such that for any $n > n_1$,

$$\left\| \sum_{i=1}^{i_0} v_n(i)e_i \right\|_M > 1 - \delta.$$

Hence for $n > n_1$,

$$\begin{aligned} 1 &= \frac{1}{k_n}(1 + \rho_M(k_n v_n)) = \frac{1}{k_n} \left(1 + \sum_{i=1}^{i_0} M(k_n \|v_n(i)\|) + \sum_{i=i_0+1}^{\infty} M(k_n \|v_n(i)\|) \right) \\ &\geq \frac{1}{k_n} \left(1 + \sum_{i=1}^{i_0} M(k_n \|v_n(i)\|) \right) + \sum_{i=i_0+1}^{\infty} M(\|v_n(i)\|) \\ &\geq \left\| \sum_{i=1}^{i_0} v_n(i)e_i \right\|_M + \sum_{i=i_0+1}^{\infty} M(\|v_n(i)\|) \\ &> 1 - \delta + \sum_{i=i_0+1}^{\infty} M(\|v_n(i)\|), \end{aligned}$$

i.e., $\sum_{i=i_0+1}^{\infty} M(\|v_n(i)\|) < \delta$. Thus

$$\left\| \sum_{i=i_0+1}^{\infty} v_n(i)e_i \right\|_M < \varepsilon.$$

From (1) we know that there exists $n_2 \in \mathbf{N}$ such that for any $n > n_2$,

$$\left\| \sum_{i=1}^{i_0} (v_n(i) - v(i))e_i \right\|_M < \varepsilon.$$

Set $n_3 := \max\{n_0, n_1, n_2\}$. Then for $n > n_3$,

$$\begin{aligned} \|v_n - v\|_M &= \left\| \sum_{i=1}^{i_0} (v_n(i) - v(i))e_i + \sum_{i=i_0+1}^{\infty} v_n(i)e_i - \sum_{i=i_0+1}^{\infty} v(i)e_i \right\|_M \\ &\leq \left\| \sum_{i=1}^{i_0} (v_n(i) - v(i))e_i \right\|_M + \left\| \sum_{i=i_0+1}^{\infty} v_n(i)e_i \right\|_M + \left\| \sum_{i=i_0+1}^{\infty} v(i)e_i \right\|_M < 3\varepsilon, \end{aligned}$$

that is $\|v_n - v\|_M \rightarrow 0$.

Lemma 3. (I) If $0 \leq u_n \leq u \in h_M(X_i)$ (or $0 \leq u_n \leq u \in h_{(M)}(X_i)$) with $\rho_M(u_n) \rightarrow 0$, then $u_n \rightarrow 0$;

(II) If $0 \leq u_n \leq u \in h_M(X_i)$ (or $0 \leq u_n \leq u \in h_{(M)}(X_i)$) with $u_n(i) \rightarrow u(i)$ for any $i \in \mathbf{N}$, then $u_n \rightarrow u$.

Proof. (I) By $\rho_M(u_n) \rightarrow 0$ one get $\|u_n(i)\| \rightarrow 0$ for all $i \in \mathbf{N}$. According to which and $\|u_n(i)\| \leq \|u(i)\|$, $\rho_M(\frac{1}{\varepsilon}u) < \infty$ ($\forall \varepsilon > 0$), we obtain $\rho_M(\frac{1}{\varepsilon}u_n) \rightarrow 0$. Consequently, by $\|\cdot\|_M \leq 1 + \rho_M(\cdot)$ it follows that $\|\frac{1}{\varepsilon}u_n\|_M \leq 1 + \rho_M(\frac{1}{\varepsilon}u_n) \rightarrow 1$. Therefore $\|u_n\|_M \rightarrow 0$, and the equivalent of $\|\cdot\|_M$ and $\|\cdot\|_{(M)}$ yields $\|u_n\|_{(M)} \rightarrow 0$.

(II) For any $\varepsilon > 0$ there exists $k = k(\varepsilon) \in \mathbf{N}$ such that $\sum_{i=k+1}^{\infty} M(\|u(i)\|) < \frac{\varepsilon}{4}$. Combining $u_n(i) \rightarrow u(i)$ with $0 \leq u_n \leq u$, we know $\sum_{i=1}^k M(\frac{\|u_n(i) - u(i)\|}{2}) \rightarrow 0$. So there is $n_0 = n_0(k, \varepsilon) = n_0(\varepsilon)$ such that for any $n > n_0$, $\sum_{i=1}^k M(\frac{\|u_n(i) - u(i)\|}{2}) \leq \frac{\varepsilon}{3}$. Hence for $n > n_0$,

$$\begin{aligned} \rho_M\left(\frac{u_n - u}{2}\right) &= \sum_{i=1}^{\infty} M\left(\frac{\|u_n(i) - u(i)\|}{2}\right) \\ &= \sum_{i=1}^k M\left(\frac{\|u_n(i) - u(i)\|}{2}\right) + \sum_{i=k}^{\infty} M\left(\frac{\|u_n(i) - u(i)\|}{2}\right) \\ &< \frac{\varepsilon}{3} + \frac{1}{2} \sum_{i=k}^{\infty} M(\|u(i)\|) < \frac{\varepsilon}{3} + \frac{1}{2} \cdot \frac{\varepsilon}{4} < \varepsilon, \end{aligned}$$

that is $\rho_M(\frac{u_n - u}{2}) \rightarrow 0$. Consequently $u_n \rightarrow u$ by part (I).

3 Monotone Points in $l_{(M)}(X_i)$ and $l_M(X_i)$

Theorem 1. Let X_i be a Banach Lattice for each $i \in \mathbf{N}$. Then $u \in S(l_{(M)}^+(X_i))$ is an upper monotone point (or a lower monotone point, resp.) if and only if

(I) $\|u(\cdot)\|$ is an upper monotone point^[17], i. e., $\rho_M(u) = 1$ (or a lower monotone point, i. e., $\xi_M(u) < 1$ ^[17], resp.);

(II) $\frac{u(i)}{\|u(i)\|}$ is an upper monotone point (or a lower monotone point, resp.) for all $i \in S_u$.

Proof. We only prove the case for upper monotone point, the other one can be obtained similarly.

Necessity. (I) Let $\omega \in l_{(M)}$ with $\omega \geq 0$ satisfy

$$\| \|u(\cdot)\| + \omega \|_{(M)} = 1.$$

Define

$$v(i) = \begin{cases} \frac{\omega(i)u(i)}{\|u(i)\|}, & i \in S_u, \\ \omega(i)e_i, & i \in \mathbf{N} \setminus S_u, \end{cases}$$

where $e_i \in S(X_i^+)$. Then $v \in l_{(M)}^+(X_i)$ and

$$\|u + v\|_{(M)} = \| \|u(\cdot) + v(\cdot)\| \|_{(M)} = \| \|u(\cdot)\| + \omega \|_{(M)} = 1.$$

In view of u being an upper monotone point, we know $\|\omega\|_{(M)} = \|v\|_{(M)} = 0$. Therefore $\|u(\cdot)\|$ is an upper monotone point.

(II) If the result does not hold, then there exist $i_0 \in S_u$ and $y \in X_{i_0}^+$ with $y \succeq 0$ such that

$$\left\| \frac{u(i_0)}{\|u(i_0)\|} + y \right\| = 1.$$

Define $v := \|u(i_0)\|ye_{i_0}$. Then $v \in l_{(M)}^+(X_{i_0})$ and $v \neq 0$, but for any $i \in \mathbf{N}$ there holds $\|u(i) + v(i)\| = \|u(i)\|$. So $\|u + v\|_{(M)} = \|u\|_{(M)}$, a contradiction with the condition.

Sufficiency. Let $v \in l_{(M)}^+(X_i)$ satisfy $\|u + v\|_{(M)} = 1$. Define $\omega := \|u(\cdot) + v(\cdot)\| - \|u(\cdot)\|$. Then $\omega \geq 0$ and $\| \|u(\cdot)\| + \omega \|_{(M)} = \|u + v\|_{(M)} = 1$. In view of the condition (I), we have $\omega = 0$, that is, $\|u(\cdot) + v(\cdot)\| = \|u(\cdot)\|$. Hence $\|u(i) + v(i)\| = \|u(i)\|$ for all $i \in \mathbf{N}$. Therefore for $i \in \mathbf{N}$,

$$\left\| \frac{u(i)}{\|u(i)\|} + \frac{v(i)}{\|u(i)\|} \right\| = 1.$$

Thus for $i \in S_u$, $v(i) = 0$ by the condition (II). Certainly this is true for $i \in \mathbf{N} \setminus S_u$, i.e. $v(i) = 0$ for any $i \in \mathbf{N}$, that is $v = 0$, which yields that u is an upper monotone point.

Using the similar argument, we get

Theorem 2. Let X_i be a Banach Lattice for each $i \in \mathbf{N}$. Then $u \in S(l_M^+(X_i))$ is an upper monotone point (or a lower monotone point, resp.) if and only if $\frac{u(i)}{\|u(i)\|}$ is upper monotone point (or a lower monotone point, resp.) for all $i \in S_u$.

Theorem 3. Let X_i be Banach Lattice for all $i \in \mathbf{N}$. Then $u \in S(l_{(M)}^+(X_i))$ is upper locally uniformly monotone if and only if

(I) $\|u(\cdot)\|$ is upper locally uniformly monotone of $S(l_{(M)}^+)$, i. e., $M \in \delta_2^{[17]}$;

(II) $\frac{u(i)}{\|u(i)\|}$ is upper locally uniformly monotone for all $i \in S_u$.

Proof. Necessity. (I) Similar to the proof of Theorem 1.

(II) If it does not hold, then there exist $i_0 \in S_u$ and $y_n \in X_{i_0}^+$ with $\|y_n\| \geq \varepsilon_0$ for some $\varepsilon_0 > 0$, such that $\|\frac{u(i_0)}{\|u(i_0)\|} + y_n\| \rightarrow 1$.

Define $v_n = \|u(i_0)\|y_n e_{i_0}$. Then $v_n \in l_{(M)}^+(X_i)$ and $\|v_n\|_{(M)} = \frac{\|u(i_0)\| \cdot \|y_n\|}{M^{-1}(1)} \geq \frac{\|u(i_0)\| \cdot \varepsilon_0}{M^{-1}(1)}$. Since

$$\rho_M(u + v_n) = \sum_{i \neq i_0} M(\|u(i)\|) + M(\|u_{i_0}\|y_n + u(i_0)) \rightarrow \sum_i M(\|u(i)\|) = 1,$$

$\|u + v_n\|_{(M)} \rightarrow 1 = \|u\|_{(M)}$, a contradiction with $u \in S(l_{(M)}^+(X_i))$ which is upper locally uniformly monotone.

Sufficiency. Suppose $v_n \in l_{(M)}^+(X_i)$ with $v_n \geq u$ and $\|v_n\|_{(M)} \rightarrow \|u\|_{(M)}$. Since $M \in \delta_2$,

$$\sum_{i=1}^{\infty} M(\|v_n(i)\|) \rightarrow \sum_{i=1}^{\infty} M(\|u(i)\|).$$

In view of $v_n \geq u$, one gets

$$0 = \sum_{i=1}^{\infty} [M(\|v_n(i)\|) - M(\|u(i)\|)] \geq \sum_{i=1}^{\infty} [M(\|v_n(i)\| - \|u(i)\|)] \geq 0,$$

and $\|v_n(i)\| \rightarrow \|u(i)\|$ for any $i \in \mathbf{N}$. Hence

$$\left\| \frac{v_n(i)}{\|u(i)\|} \right\| \rightarrow \left\| \frac{u(i)}{\|u(i)\|} \right\|$$

for any $i \in S_u$, and we get by the condition (II),

$$\frac{v_n(i)}{\|u(i)\|} \rightarrow \frac{u(i)}{\|u(i)\|}$$

for any $i \in S_u$. Certainly $v_n(i) \rightarrow u(i)$. In virtue of Corollary 1, there holds $v_n \rightarrow u$, thus the proof is completed.

Theorem 4. Let X_i be a Banach Lattice for each $i \in \mathbf{N}$. Then $u \in S(l_M^+(X_i))$ is upper locally uniformly monotone if and only if

(I) $\|u(\cdot)\|$ is upper locally uniformly monotone of $S(l_M^+)$, i. e., $M \in \delta_2^{[17]}$;

(II) $\frac{u(i)}{\|u(i)\|}$ is upper locally uniformly monotone for all $i \in S_u$.

Proof. In view of Lemma 2, the proof of Theorem 3 and Theorem 1, the sufficiency and the necessity condition (I) is obviously. So we only need to check the necessity part (II). As the

proof of Theorem above, if not, then there exist $i_0 \in S_u$ and $y_n \in X_{i_0}^+$ with $\|y_n\| \geq \varepsilon_0$ for some $\varepsilon_0 > 0$ such that

$$\left\| \frac{u(i_0)}{\|u(i_0)\|} + y_n \right\| \rightarrow 1.$$

Define $v_n = \|u(i_0)\|y_n e_{i_0}$. Then $v_n \in l_M^+(X_{i_0})$ and

$$\|v_n\|_M = \|u(i_0)\| \cdot \|y_n\| N^{-1}(1) \geq \|u(i_0)\| \cdot \varepsilon_0 N^{-1}(1).$$

Owing to

$$\begin{aligned} \|u\|_M &\leq \|u + v_n\|_M \leq \frac{1}{k} \left(1 + \sum_{i \neq i_0} M(k\|u(i)\|) + M(k\|u(i_0)\|y_n + u_{i_0}) \right) \\ &\rightarrow \frac{1}{k} \left(1 + \sum_{i \neq i_0} M(k\|u(i)\|) + M(k\|u_{i_0}\|) \right) = \|u\|_M, \end{aligned}$$

we have $\|u + v_n\|_M \rightarrow \|u\|_M$, a contradiction with u being upper locally uniformly monotone.

Theorem 5. *Let X_i be a Banach Lattice for each $i \in \mathbf{N}$. Then $u \in S(l_M^+(X_i))$ (or $S(l_M^+)$, resp.) is lower locally uniformly monotone if and only if*

(I) $\|u(\cdot)\|$ is lower locally uniformly monotone of $S(l_M^+)$ (or $S(l_M^+)$, resp.), i. e., $\xi_M(u) = 0^{[17]}$;

(II) $\frac{u(i)}{\|u(i)\|}$ is lower locally uniformly monotone for all $i \in S_u$.

Proof. Sufficiency. Noticing that $\xi_M(u) = 0$ implies $u \in S(h_M^+(X_i))$, in virtue of Lemma 3, we can get the result similarly to the proof of Theorem 3.

Necessity. We only need to prove (II). For any $i \in S_u$ and $n \in \mathbf{N}$, define

$$\varepsilon_n(i) = \sup \left\{ \|y\| : y \in X_i^+; y \leq \frac{u(i)}{\|u(i)\|}; \left\| \frac{u(i)}{\|u(i)\|} - y \right\| > 1 - \frac{1}{n} \right\}. \tag{2}$$

Then there exists $\varepsilon(i) \geq 0$ such that $\varepsilon_n(i) \searrow \varepsilon(i)$. Certainly

$$\varepsilon(i) \equiv \lim_{n \rightarrow \infty} \varepsilon_n(i) = 0 \Leftrightarrow \frac{u(i)}{\|u(i)\|} \text{ is a lower locally uniformly monotone point.} \tag{3}$$

For any $i \in S_u$ and any $n \in \mathbf{N}$, by (2) there exists a $y_i^n \in X_i^+$ such that

$$y_i^n \leq \frac{u(i)}{\|u(i)\|}, \|y_i^n\| > (1 - \frac{1}{n})\varepsilon_n(i) \quad \text{and} \quad \left\| \frac{u(i)}{\|u(i)\|} - y_i^n \right\| > 1 - \frac{1}{n}.$$

Define

$$e'_n := \sum_{i \in S_u} \|y_i^n\| e_i \quad \text{and} \quad v_n := \|u(\cdot)\| \cdot \sum_{i \in S_u} y_i^n e_i.$$

Then $0 \leq v_n \leq u$ and $\varepsilon_n(i) \geq \varepsilon'_n(i) = \|y_i^n\| \geq (1 - \frac{1}{n})\varepsilon_n(i)$. Hence $\varepsilon(i) = \lim_n \varepsilon_n(i) = \lim_n \varepsilon'_n(i)$. Consequently for any $i \in S_u$,

$$\|u(i) - v_n(i)\| = \|u(i) - \|u(i)\|y_i^n\| = \|u(i)\| \cdot \left\| \frac{u(i)}{\|u(i)\|} - y_i^n \right\| \geq (1 - \frac{1}{n})\|u(i)\|,$$

which follows that $\|u - v_n\|_{(M)} \rightarrow 1$. By the condition, we know $v_n \rightarrow 0$. Hence

$$\varepsilon'_n(i) = \frac{\|v_n(i)\|}{\|u(i)\|} \rightarrow 0,$$

i.e., $\varepsilon(i) = 0$. Thus complete the proof completed by (3).

Corollary 2. (I) ^[8, 14] $l_{(M)}(X_i)$ is strictly monotone if and only if $M \in \delta_2$ and X_i is strictly monotone for each $i \in \mathbf{N}$; $l_M(X_i)$ is strictly monotone if and only if X_i is strictly monotone for each $i \in \mathbf{N}$;

(II) $l_{(M)}(X_i)$ (or $l_M(X_i)$) is upper (lower) locally uniformly monotone if and only if $M \in \delta_2$ and X_i is upper (lower) locally uniformly monotone for each $i \in \mathbf{N}$.

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