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CERTAIN SUBCLASS OF $P-$ VALENT MEROMORPHIC ANALYTIC FUNCTIONS INVOLVING CERTAIN INTEGRAL OPERATOR

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Abstract. The purpose of the present paper is to introduce a new subclass of $p-$valent meromorphic functions by using certain integral operator and to investigate various properties for this subclass.

Key words: *analytic function, meromorphic function, hypergeometric function, linear operator*

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1 Introduction

Let $\sum\limits_{p}$ denote the class of functions f of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k, \qquad p \in \mathbf{N} = \{1,2,3,\cdots\}, \tag{1.1}$$

which are analytic and $p-$valent in the punctured unit disc $U^* = \{z \in \mathbf{C} : 0 < |z| < 1\} = U \setminus \{0\}$.

For a function f in the class $\sum_p$ given by (1.1), Aqlan et al.[1] introduced the following one parameter families of integral operator

$$\mathcal{P}_p^{\alpha} f(z) = \frac{1}{z^{p+1}\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} t^{\alpha-1} f(t)\,\mathrm{d}t, \qquad \alpha > 0; \quad p \in \mathbf{N} \tag{1.2}$$

Using an elementary integral calculus, it is easy to verify that

$$\mathcal{P}_p^{\alpha} f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \left(\frac{1}{k+p+1}\right)^{\alpha} a_k z^k, \qquad \alpha \geq 0; \quad p \in \mathbf{N}. \tag{1.3}$$

Also, it is easily verified from (1.3) that

$$z\left(\mathcal{P}_p^{\alpha} f(z)\right)' = \mathcal{P}_p^{\alpha-1} f(z) - (1+p)\,\mathcal{P}_p^{\alpha} f(z). \tag{1.4}$$

Definition. Let $\sum_p^{\alpha}(\eta,\delta,\mu,\lambda)$ be the class of functions $f\in\sum_p$ which satisfy the following inequality:

$$\Re\left\{(1-\lambda)\left(\frac{\mathcal{P}_p^{\alpha} f(z)}{\mathcal{P}_p^{\alpha} g(z)}\right)^{\mu} + \lambda\frac{\mathcal{P}_p^{\alpha-1} f(z)}{\mathcal{P}_p^{\alpha} g(z)}\left(\frac{\mathcal{P}_p^{\alpha} f(z)}{\mathcal{P}_p^{\alpha} g(z)}\right)^{\mu}\right\} > \eta, \tag{1.5}$$

where $g\in\sum_p$ satisfies the following inequality:

$$\Re\left\{\frac{\mathcal{P}_p^{\alpha} g(z)}{\mathcal{P}_p^{\alpha-1} g(z)}\right\} > \delta, \qquad 0\le\delta<1, z\in U, \tag{1.6}$$

and η,δ and μ are real numbers such that $0\le\eta,\delta<1$ and $\lambda\in\mathbf{C}$ with $\Re\{\lambda\}>0$.

To establish our main results we need the following lemmas.

Lemma 1[5]**.** *Let Ω be a set in the complex plane $\mathbf{C}$ and let the function $\psi:\mathbf{C}^2\to\mathbf{C}$ satisfy the condition $\psi(ir_2,s_1)\notin\Omega$ for all real $r_2,s_1\le-\frac{1+r_2^2}{2}$. If q is analytic in U with $q(0)=1$ and $\psi(q(z),zq'(z))\in\Omega, z\in U$, then*

$$\Re\{q(z)\}>0\ (z\in U).$$

Lemma 2[6]**.** *If q is analytic in U with $q(0)=1$, and if $\lambda\in C\backslash\{0\}$ with $\Re\{\lambda\}\geqslant 0$, then*

$$\Re\{q(z)+\lambda zq'(z)\}>\alpha, \qquad 0\le\alpha<1$$

implies

$$\Re\{q(z)\}>\alpha+(1-\alpha)(2\gamma-1),$$

where γ is given by

$$\gamma=\gamma(\Re\{\lambda\})=\int_0^1(1+t^{\Re\{\lambda\}})^{-1}\mathrm{d}t$$

which is increasing function of $\Re\{\lambda\}$ and $\frac{1}{2}\le\gamma<1$. The estimate is sharp in the sense that the bound cannot be improved.

For real or complex numbers a,b and $c\,(c\neq 0,-1,-2,\cdots)$, the Gaussian hypergeometric function is defined by

$${}_2F_1(a,b;c;z)=1+\frac{a.b}{c}\frac{z}{1!}+\frac{a(a+1).b(b+1)}{c(c+1)}\frac{z^2}{2!}+\cdots. \tag{1.7}$$

We note that the series (1.9) converges absolutely for $z\in U$ and hence represents an analytic function in U (see, for details, [7, Ch. 14]). Each of the identities (asserted by Lemma 3 below) is fairly well known (cf., e.g., [7, Ch. 14]).

Lemma 3[7]**.** *For real or complex numbers a, b and c $(c \neq 0, -1, -2, \cdots)$,*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\, {}_2F_1(a,b;c;z), \qquad \Re(c) > \Re(b) > 0; \tag{1.8}$$

$${}_2F_1(a,b;c;z) = (1-z)^{-a}_2 F_1(a, c-b; c; \frac{z}{z-1}); \tag{1.9}$$

$${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z); \tag{1.10}$$

and

$${}_2F_1(1,1;2;\frac{1}{2}) = 2\ln 2. \tag{1.11}$$

In the present paper, we investigate various properties for the class $\sum_p^\alpha(\eta, \delta, \mu, \lambda)$. A similar problem for $p-$valent meromorphic functions was studied by Aouf and Mostafa[2] , EL-Ashwah[3] and EL-Ashwah and Aouf[4].

2 Main Results

Unless otherwise mentioned, we assume throughout this paper that $p \in \mathbf{N}, \alpha \geqslant 0, \mu > 0, 0 \leq \eta < 1$ and $\lambda \geqslant 0$.

Theorem 1. *Let*

$$f \in \sum_p^\alpha(\eta, \delta, \mu, \lambda).$$

Then

$$\Re\left\{\left(\frac{\mathcal{P}_p^\alpha f(z)}{\mathcal{P}_p^\alpha g(z)}\right)^\mu\right\} > \frac{2\mu\eta + \lambda\delta}{2\mu + \lambda\delta} \ (z \in U), \tag{2.1}$$

where the function $g \in \sum_p$ satisfies the condition (1.6).

Proof. Let

$$\gamma = \frac{2\mu\eta + \lambda\delta}{2\mu + \lambda\delta},$$

and we define the function q by

$$q(z) = \frac{1}{1-\gamma}\left[\left(\frac{\mathcal{P}_p^\alpha f(z)}{\mathcal{P}_p^\alpha g(z)}\right)^\mu - \gamma\right]. \tag{2.2}$$

Then q is analytic in U and $q(0) = 1$. If we set

$$h(z) = \frac{\mathcal{P}_p^\alpha g(z)}{\mathcal{P}_p^{\alpha-1} g(z)}, \qquad z \in U, \tag{2.3}$$

then by the hypothesis (1.5), $\Re\{h(z)\} > \delta$. Differentiating (2.2) with respect to z and using the identity (1.4), we have

$$\begin{aligned}(1-\lambda)\left(\frac{\mathcal{P}_p^\alpha f(z)}{\mathcal{P}_p^\alpha g(z)}\right)^\mu + \lambda\frac{\mathcal{P}_p^{\alpha-1} f(z)}{\mathcal{P}_p^\alpha g(z)}\left(\frac{\mathcal{P}_p^\alpha f(z)}{\mathcal{P}_p^\alpha g(z)}\right)^\mu \\ = (1-\gamma)q(z) + \gamma + \frac{\lambda(1-\gamma)}{\mu} zq'(z)h(z).\end{aligned} \tag{2.4}$$

Let us define the function $\psi(r,s)$ by

$$\psi(r,s)=(1-\gamma)r+\gamma+\frac{\lambda(1-\gamma)}{\mu}sh(z). \tag{2.5}$$

Using (2.5) and the fact that $f\in\sum_p^{\alpha}(\eta,\delta,\mu,\lambda)$, we obtain

$$\{\psi(q(z),zq'(z));z\in U)\}\subset\Omega=\{w\in C:\Re(w)>\eta\}.$$

Now for all real

$$r_2,s_1\le-\frac{1+r_2^2}{2},$$

we have

$$\begin{aligned}\Re\{\psi(ir_2,s_1)\}\} &= \gamma+\frac{\lambda(1-\gamma)s_1}{\mu}\Re\{h(z)\}\le\gamma-\frac{\lambda(1-\gamma)\delta(1+r_2^2)}{2\mu}\\ &\le \gamma-\frac{\lambda(1-\gamma)\delta}{2\mu}=\eta.\end{aligned}$$

Hence for each $z\in U,\ \psi(ir_2,s_1)\notin\Omega$. Thus by Lemma 1, we have

$$\Re\{q(z)\}>0\ (z\in U)$$

and hence

$$\Re\left\{\left(\frac{\mathcal{P}_p^{\alpha}f(z)}{\mathcal{P}_p^{\alpha}g(z)}\right)^{\mu}\right\}>\gamma\,(z\in U).$$

This proves Theorem 1.

Corollary 1. *Let the functions f and g be in $\sum_p$ and let g satisfy the condition* (1.6). *If* $\lambda\geqslant1$ *and*

$$\Re\left\{(1-\lambda)\frac{\mathcal{P}_p^{\alpha}f(z)}{\mathcal{P}_p^{\alpha}g(z)}+\lambda\frac{\mathcal{P}_p^{\alpha-1}f(z)}{\mathcal{P}_p^{\alpha-1}g(z)}\right\}>\eta,\qquad 0\le\eta<1;z\in U, \tag{2.6}$$

then

$$\Re\left\{\frac{\mathcal{P}_p^{\alpha-1}f(z)}{\mathcal{P}_p^{\alpha-1}g(z)}\right\}>\gamma=\frac{\eta\,(2+\delta)+\delta(\lambda-1)}{2+\delta\lambda},\qquad z\in U. \tag{2.7}$$

Proof. We have

$$\lambda\frac{\mathcal{P}_p^{\alpha-1}f(z)}{\mathcal{P}_p^{\alpha-1}g(z)}=\left\{(1-\lambda)\frac{\mathcal{P}_p^{\alpha}f(z)}{\mathcal{P}_p^{\alpha}g(z)}+\lambda\frac{\mathcal{P}_p^{\alpha-1}f(z)}{\mathcal{P}_p^{\alpha-1}g(z)}\right\}+(\lambda-1)\frac{\mathcal{P}_p^{\alpha}f(z)}{\mathcal{P}_p^{\alpha}g(z)},\qquad z\in U.$$

Since $\lambda\geqslant1$, making use of (2.6) and (2.1) (for $\mu=1$), we deduce that

$$\Re\left\{\frac{\mathcal{P}_p^{\alpha-1}f(z)}{\mathcal{P}_p^{\alpha-1}g(z)}\right\}>\gamma=\frac{\eta\,(2+\delta)+\delta(\lambda-1)}{2+\delta\lambda},\qquad z\in U.$$

this completes the proof of the corollary.

Corollary 2. *Let* $\lambda \in \mathbf{C}^* = \mathbf{C}\backslash\{0\}$ *with* $\Re\{\lambda\} \geqslant 0$. *If* $f \in \sum_p$ *satisfies the following condition :*

$$\Re\{(1-\lambda)(z^p\mathcal{P}_p^\alpha f(z))^\mu + \lambda z^p\mathcal{P}_p^{\alpha-1} f(z)(z^p\mathcal{P}_p^\alpha f(z))^{\mu-1}\} > \eta, \qquad z \in U,$$

then

$$\Re\{(z^p\mathcal{P}_p^\alpha f(z))^\mu\} > \frac{2\mu\eta + \Re\{\lambda\}}{2\mu + \Re\{\lambda\}}, \qquad z \in U. \tag{2.8}$$

Further, if $\lambda \geqslant 1$ *and* $f \in \sum_p$ *satisfies*

$$\Re\{(1-\lambda)z^p\mathcal{P}_p^\alpha f(z) + \lambda z^p\mathcal{P}_p^{\alpha-1} f(z)\} > \eta \ (z \in U), \tag{2.9}$$

then

$$\Re\{z^p\mathcal{P}_p^{\alpha-1} f(z)\} > \frac{2\eta + \lambda - 1}{2+\lambda}, \qquad z \in U. \tag{2.10}$$

Proof. The results (2.8) and (2.10) follows by putting $g(z) = z^{-p}$ in Theorem 1 and Corollary 1, respectively.

Remark 1. Choosing α, λ and μ appropriately in Corollary 2, we have

(i) For $\alpha = 0$ and $\lambda = 1$ in Corollary 2, we have that

$$\Re\left\{\left(p+1+\frac{zf'(z)}{f(z)}\right)(z^p f(z))^\mu\right\} > \eta, \qquad z \in U$$

implies

$$\Re\left\{(z^p f(z))^\mu\right\} > \frac{2\mu\eta + 1}{2\mu + 1}, \qquad z \in U.$$

(ii) For $\alpha = 0, \mu = 1$ and $\lambda \in \mathbf{C}^*$ with $\Re\{\lambda\} \geqslant 0$ in Corollary 2, we have that

$$\Re\left\{(1+\lambda p)z^p f(z) + \lambda z^{p+1} f'(z)\right\} > \eta$$

implies

$$\Re\{z^p f(z)\} > \frac{2\eta + \Re\{\lambda\}}{2 + \Re\{\lambda\}}, \qquad z \in U.$$

(iii) Replacing $f(z)$ by $-\dfrac{zf'(z)}{p}$ in the result (ii) we have that

$$-\Re\left\{[1+\lambda(p+1)]\frac{z^{p+1}f'(z)}{p} + \tfrac{\lambda}{p}z^{p+1}f''(z)\right\} > \eta, \qquad 0 \le \eta < 1; z \in U$$

implies

$$-\Re\left\{\frac{z^{p+1}f'(z)}{p}\right\} > \frac{2\eta + \Re\{\lambda\}}{2 + \Re\{\lambda\}}, \qquad z \in U.$$

Theorem 2. *Let $\lambda \in \mathbf{C}$ with $\Re\{\lambda\} > 0$. If $f \in \sum_p$ satisfies the following condition :*

$$\Re\{(1-\lambda)(z^p\mathcal{P}_p^{\alpha} f(z))^{\mu} + \lambda z^p\mathcal{P}_p^{\alpha-1} f(z)(z^p\mathcal{P}_p^{\alpha} f(z))^{\mu-1}\} > \eta, \qquad z \in U, \tag{2.11}$$

then

$$\Re\left\{(z^p\mathcal{P}_p^{\alpha} f(z))^{\mu}\right\}\} > \eta + (1-\eta)(2\rho - 1), \tag{2.12}$$

where

$$\rho = \frac{1}{2}\ {}_2F_1(1,1;\frac{\mu}{\Re\{\lambda\}} + 1; \frac{1}{2}). \tag{2.13}$$

Proof. Let

$$q(z) = (z^p\mathcal{P}_p^{\alpha} f(z))^{\mu}. \tag{2.14}$$

Then q is analytic with $q(0) = 1$. Differentiating (2.14) with respect to z and using the identity (1.4), we have

$$(1-\lambda)(z^p\mathcal{P}_p^{\alpha} f(z))^{\mu} + \lambda z^p\mathcal{P}_p^{\alpha-1} f(z)(z^p\mathcal{P}_p^{\alpha} f(z))^{\mu-1} = q(z) + \frac{\lambda}{\mu} zq'(z),$$

so that by the hypothesis (2.11), we have

$$\Re\left\{q(z) + \frac{\lambda}{\mu} zq'(z)\right\} > \eta, \qquad z \in U.$$

In view of Lemma 2, this implies that

$$\Re\{q(z)\} > \eta + (1-\eta)(2\rho - 1),$$

where

$$\rho = \rho(\Re\{\lambda\}) = \int_0^1 \left(1 + t^{\frac{\Re\{\lambda\}}{\mu}}\right)^{-1} \mathrm{d}t.$$

Putting

$$\Re\{\lambda\} = \lambda_1 > 0,$$

we have

$$\rho = \int_0^1 \left(1 + t^{\frac{\lambda_1}{\mu}}\right)^{-1} dt = \frac{\mu}{\lambda_1} \int_0^1 (1+u)^{-1} u^{\frac{\mu}{\lambda_1} - 1} \mathrm{d}u$$

Using $(1.8) - (1.11)$, we obtain

$$\rho = \frac{1}{2}\ {}_2F_1(1,1;\frac{\mu}{\lambda_1} + 1; \frac{1}{2}).$$

This completes the proof of Theorem 2.

Corollary 3. *Let $\lambda \in \mathbf{R}$ with $\lambda \geqslant 1$. If $f \in \sum_p$ satisfies*

$$\Re\left\{(1-\lambda) z^p\mathcal{P}_p^{\alpha} f(z) + \lambda z^p\mathcal{P}_p^{\alpha-1} f(z)\right\} > \eta, \qquad z \in U, \tag{2.15}$$

then

$$\Re\left\{z^p\mathcal{P}_p^{\alpha-1}f(z)\right\} > \eta+(1-\eta)(2\rho_1-1)(1-\frac{1}{\lambda}), \qquad z\in U,$$

where

$$\rho_1 = \frac{1}{2}\,{}_2F_1(1,1;\frac{1}{\lambda}+1;\frac{1}{2}).$$

Proof. The result follows by using the identity

$$\lambda z^p\mathcal{P}_p^{\alpha-1}f(z) = (1-\lambda)z^p\mathcal{P}_p^{\alpha}f(z)+\lambda z^p\mathcal{P}_p^{\alpha-1}f(z)+(\lambda-1)z^p\mathcal{P}_p^{\alpha}f(z). \tag{2.16}$$

Remark 2. We note that, for $\alpha=0$ and $\lambda=\mu>0$ in Corollary 2, that is, if

$$\Re\left\{(1-\lambda)(z^pf(z))^{\lambda}+\lambda(z^{p+1}f(z))'(z^pf(z))^{\lambda-1}\right\} > \eta, \qquad z\in U, \tag{2.17}$$

then (2.8) implies

$$\Re\{(z^pf(z))^{\lambda}\} > \frac{2\eta+1}{3}, \qquad z\in U, \tag{2.18}$$

whereas if $f\in\sum_p$ satisfies the condition (2.17) then by using Theorem 2, we have

$$\Re\{(z^pf(z))^{\lambda}\} > 2(1-\ln 2)\eta+(2\ln 2-1)\ (z\in U),$$

which is better than (2.18).

Remark 3. The results in Remark 2 also obtained by Aouf and Mostafa[2, Remark2].

Theorem 3. *Suppose that the functions f and g are in $\sum_p$ and g satisfies the condition* (1.6). *If*

$$\Re\left\{\frac{\mathcal{P}_p^{\alpha-1}f(z)}{\mathcal{P}_p^{\alpha-1}g(z)}-\frac{\mathcal{P}_p^{\alpha}f(z)}{\mathcal{P}_p^{\alpha}g(z)}\right\} > -\frac{(1-\eta)\delta}{2}, \qquad z\in U, \tag{2.19}$$

for some $\eta\,(0\le\eta<1)$, *then*

$$\Re\left\{\frac{\mathcal{P}_p^{\alpha}f(z)}{\mathcal{P}_p^{\alpha}g(z)}\right\} > \eta, \qquad z\in U, \tag{2.20}$$

and

$$\Re\left\{\frac{\mathcal{P}_p^{\alpha-1}f(z)}{\mathcal{P}_p^{\alpha-1}g(z)}\right\} > \frac{\eta\,(2+\delta)-\delta}{2}, \qquad z\in U. \tag{2.21}$$

Proof. Let

$$q(z) = \frac{1}{1-\eta}\left[\frac{\mathcal{P}_p^{\alpha}f(z)}{\mathcal{P}_p^{\alpha}g(z)}-\eta\right]. \tag{2.22}$$

Then q is analytic in U with $q(0)=1$. Setting

$$\phi(z) = \frac{\mathcal{P}_p^{\alpha}g(z)}{\mathcal{P}_p^{\alpha-1}g(z)}, \qquad z\in U, \tag{2.23}$$

we observe that from (1.5), we have

$$\Re\{\phi(z)\} > \delta \ \ (0 \le \delta < 1)$$

in U. A simple computation shows that

$$(1-\eta)zq'(z).\phi(z) = \frac{\mathcal{P}_p^{\alpha-1} f(z)}{\mathcal{P}_p^{\alpha-1} g(z)} - \frac{\mathcal{P}_p^{\alpha} f(z)}{\mathcal{P}_p^{\alpha} g(z)} = \psi(q(z), zq'(z)),$$

where

$$\psi(r,s) = (1-\eta)s\phi(z).$$

Using the hypothesis (2.19), we obtain

$$\left\{\psi(q(z),zq'(z)); z\in U\right\} \subset \Omega = \left\{w \in \mathbf{C} : \Re\{w\} > -\frac{(1-\eta)\delta}{2}\right\}.$$

Now, for all real r_2, $s_1 \le -\frac{1+r_2^2}{2}$, we have

$$\Re\{\psi(ir_2,s_1)\} = s_1(1-\eta)\Re\{\phi(z)\} \le \frac{-(1-\eta)\delta(1+r_2^2)}{2} \le \frac{-(1-\eta)\delta}{2}.$$

This shows that $\psi(ir_2,s_1) \notin \Omega$ for each $z\in U$. Hence by Lemma 1, we have $\Re\{q(z)\} > 0$ $(z\in U)$. This proves (2.20). The proof of (2.21) follows by (2.20) and (2.21) in the identity :

$$\Re\left\{\frac{\mathcal{P}_p^{\alpha-1} f(z)}{\mathcal{P}_p^{\alpha-1} g(z)}\right\} = \Re\left\{\frac{\mathcal{P}_p^{\alpha-1} f(z)}{\mathcal{P}_p^{\alpha-1} g(z)} - \frac{\mathcal{P}_p^{\alpha} f(z)}{\mathcal{P}_p^{\alpha} g(z)}\right\} + \Re\left\{\frac{\mathcal{P}_p^{\alpha} f(z)}{\mathcal{P}_p^{\alpha} g(z)}\right\}.$$

This completes the proof of Theorem 3.

Remark 4. (i) For $\alpha = 0$ and $g(z) = z^{-p}$ in Theorem 3, we have

$$\Re\left\{z^{p+1} f'(z) + pz^p f(z)\right\} > \frac{-(1-\eta)\delta}{2}, \qquad z\in U$$

implies

$$\Re\{z^p f(z)\} > \eta, \qquad z \in U$$

and

$$\Re\left\{z^{p+1} f'(z) + (p+1)z^p f(z)\right\} > \frac{\eta\,(2+\delta)-\delta}{2}, \qquad z\in U.$$

(ii) Putting $\alpha = 0$ in Theorem 3, we get that, if

$$\Re\left\{\frac{zf'(z)+(p+1)f(z)}{zg'(z)+(p+1)g(z)} - \frac{f(z)}{g(z)}\right\} > \frac{-(1-\eta)\delta}{2}, \qquad z\in U,$$

then

$$\Re\left\{\frac{f(z)}{g(z)}\right\} > \eta, \qquad z\in U$$

and

$$\Re\left\{\frac{zf'(z)+(p+1)f(z)}{zg'(z)+(p+1)g(z)}\right\} > \frac{\eta(2+\delta)-\delta}{2}, \qquad z\in U.$$

Remark 5. The results in Remark 4 also obtained by Aouf and Mostafa[2, Remak3].

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