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TOPOLOGICAL ENTROPY AND IRREGULAR RECURRENCE

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Abstract. This paper is devoted to problems stated by Z. Zhou and F. Li in 2009. They concern relations between almost periodic, weakly almost periodic, and quasi-weakly almost periodic points of a continuous map f and its topological entropy. The negative answer follows by our recent paper. But for continuous maps of the interval and other more general one-dimensional spaces we give more results; in some cases the answer is positive.

Key words: *topological entropy, weakly almost periodic point, quasi-weakly almost periodic point*

AMS (2010) subject classification: 37B20, 37B40, 47D45, 37D05

1 Introduction

Let (X,d) be a compact metric space, $I=[0,1]$ the unit interval, and $\mathcal{C}(X)$ the set of continuous maps $f:X\to X$. By $\omega(f,x)$ we denote the *ω-limit set* of x which is the set of limit points of the *trajectory* $\{f^i(x)\}_{i\geq 0}$ of x, where f^i denotes the ith iterate of f. We consider the sets $W(f)$ of *weakly almost periodic points* of f, and $QW(f)$ of *quasi-weakly almost periodic points* of f. They are defined as follows, see [11]:

$$W(f)=\left\{x\in X;\forall\varepsilon\ \exists N>0 \text{ such that } \sum_{i=0}^{nN-1}\chi_{B(x,\varepsilon)}(f^i(x))\geq n,\forall n>0\right\},$$

$$QW(f)=\left\{x\in X;\forall\varepsilon\ \exists N>0,\exists\{n_j\} \text{ such that } \sum_{i=0}^{n_jN-1}\chi_{B(x,\varepsilon)}(f^i(x))\geq n_j,\forall j>0\right\},$$

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where $B(x,\varepsilon)$ is the ε-neighbourhood of x, χ_A the characteristic function of a set A, and $\{n_j\}$ an increasing sequence of positive integers. For $x \in X$ and $t > 0$, let

$$\Psi_x(f,t) = \liminf_{n\to\infty} \tfrac{1}{n}\#\{0 \le j < n; d(x, f^j(x)) < t\}, \tag{1}$$

$$\Psi_x^*(f,t) = \limsup_{n\to\infty} \tfrac{1}{n}\#\{0 \le j < n; d(x, f^j(x)) < t\}. \tag{2}$$

Thus, $\Psi_x(f,t)$ and $\Psi_x^*(f,t)$ are the *lower* and *upper Banach density* of the set $\{n \in \mathbf{N}; f^n(x) \in B(x,t)\}$, respectively. In this paper we make of use more convenient definitions of $W(f)$ and $QW(f)$ based on the following lemma.

Lemma 1. *Lef $f \in \mathcal{C}(X)$. Then*

(i) *$x \in W(f)$ if and only if $\Psi_x(f,t) > 0$, for every $t > 0$,*

(ii) *$x \in QW(f)$ if and only if $\Psi_x^*(f,t) > 0$, for every $t > 0$.*

Proof. It is easy to see that, for every $\varepsilon > 0$ and $N > 0$,

$$\sum_{i=0}^{nN-1} \chi_{B(x,\varepsilon)}(f^i(x)) \ge n \quad \text{if and only if} \quad \#\{0 \le j < nN; f^j(x) \in B(x,\varepsilon)\} \ge n. \tag{3}$$

(i) If $x \in W(f)$ then, for every $\varepsilon > 0$ there is an $N > 0$ such that the condition on the left side in (3) is satisfied for every n. Hence, by the condition on the right, $\Psi_x(f,\varepsilon) \ge 1/N > 0$. If $x \notin W(f)$ then there is an $\varepsilon > 0$ such that for every $N > 0$, there is an $n > 0$ such that the condition on the left side of (3) is not satisfied. Hence, by the condition on the right, $\Psi_x(f,t) < 1/N \to 0$ if $N \to \infty$. Proof of (ii) is similar.

Obviously, $W(f) \subseteq QW(f)$. The properties of $W(f)$ and $QW(f)$ were studied in the nineties by Z. Zhou et al, see [11] for references. The points in $IR(f) := QW(f) \setminus W(f)$ are *irregularly recurrent points*, i.e., the points x such that $\Psi_x^*(f,t) > 0$ for any $t > 0$, and $\Psi_x(f,t_0) = 0$ for *some* $t_0 > 0$, see [7]. Denote by $h(f)$ the *topological entropy* of f and by $R(f)$, $UR(f)$ and $AP(f)$ the set of *recurrent*, *uniformly recurrent* and *almost periodic* points of f, respectively. Thus, $x \in R(f)$ if for every neighborhood U of x, $f^j(x) \in U$ for infinitely many $j \in \mathbf{N}$; $x \in UR(f)$ if for every neighborhood U of x there is a $K > 0$ such that every interval $[n, n+K]$ contains a $j \in \mathbf{N}$ with $f^j(x) \in U$; and $x \in AP(f)$ if for every neighborhood U of x, there is a $k > 0$ such that $f^{kj}(x) \in U$ for every $j \in \mathbf{N}$. Recall that $x \in R(f)$ if and only if $x \in \omega(f,x)$, and $x \in UR(f)$ if and only if $\omega(f,x)$ is a *minimal set*, i.e., a closed set $\emptyset \ne M \subseteq X$ such that $f(M) = M$ and no proper subset of M has this property. Denote by $\omega(f)$ the union of all ω-limit sets of f. The next relations follow by definition:

$$AP(f) \subseteq UR(f) \subseteq W(f) \subseteq QW(f) \subseteq R(f) \subseteq \omega(f) \tag{4}$$

The next theorem will be used in Section 2. Its part (i) is proved in [9] but we are able to give a simpler argument, and extend it to part (ii).

Theorem 1. *If $f \in \mathcal{C}(X)$, then*

(i) $W(f) = W(f^m)$,

(ii) $QW(f) = QW(f^m)$,

(iii) $IR(f) = IR(f^m)$.

Proof. Since $\Psi_x(f,t) \geq \frac{1}{m}\Psi_x(f^m,t)$, $x \in W(f^m)$ implies $x \in W(f)$ and similarly, $QW(f^m) \subseteq QW(f)$. Since (iii) follows by (i) and (ii), it suffices to prove that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every prime integer m,

$$\Psi_x(f^m,\varepsilon) \geq \Psi_x(f,\delta) \text{ and } \Psi^*_x(f^m,\varepsilon) \geq \Psi^*_x(f,\delta). \tag{5}$$

For every $i \geq 0$, denote $\omega_i := \omega(f^m, f^i(x))$ and $\omega_{ij} := \omega_i \cap \omega_j$. Obviously, $\omega(f,x) = \bigcup_{0 \leq i < m} \omega_i$, and $f(\omega_i) = \omega_{i+1}$, where i is taken mod m. Moreover, $f^m(\omega_i) = \omega_i$ and $f^m(\omega_{ij}) = \omega_{ij}$ for every $0 \leq i < j < m$. Hence

$$\omega_i \neq \omega_{ij} \text{ implies } \omega_j \neq \omega_{ij}, \text{ and } f^i(x), f^j(x) \notin \omega_{ij}. \tag{6}$$

Let k be the least period of ω_0. Since m is prime, there are two cases.

(a) If $k = m$ then the sets ω_i are pairwise distinct and, by (6), there is a $\delta > 0$ such that $B(x,\delta) \cap \omega_i = \emptyset$, $0 < i < m$. It follows that if $f^r(x) \in B(x,\delta)$ then r is a multiple of m, with finitely many exceptions. Consequently, (5) is satisfied for $\varepsilon = \delta$, even with $\geq$ replaced by the equality.

(b) If $k = 1$ then $\omega_i = \omega_0$ for every i. Let $\varepsilon > 0$. For every i, $0 \leq i < m$, there is the minimal integer $k_i \geq 0$ such that $f^{mk_i+i}(x) \in B(x,\varepsilon)$. By the continuity, there is a $\delta > 0$ such that $f^{mk_i+i}(B(x,\delta)) \subseteq B(x,\varepsilon)$, $0 \leq i < m$. If $f^r(x) \in B(x,\delta)$ and $r \equiv i (\mathrm{mod}\ m)$, $r = ml + i$, then $f^{m(l+1+k_{m-i})}(x) = f^{r+mk_{m-i}+m-i}(x) \in f^{mk_{m-i}+m-i}(B(x,\delta)) \subseteq B(x,\varepsilon)$. This proves (5).

In 2009 Z. Zhou and F. Li stated, among others, the following problems, see [10].

Problem 1. *Does $IR(f) \neq \emptyset$ imply $h(f) > 0$?*

Problem 2. *Does $W(f) \neq AP(f)$ imply $h(f) > 0$?*

In general, the answer to either problem is negative. In [7] we constructed a skew-product map $F : Q \times I \to Q \times I$, $(x,y) \mapsto (\tau(x), g_x(y))$, where $Q = \{0,1\}^{\mathbf{N}}$ is a Cantor-type set, τ the adding machine (or, odometer) on Q and, for every x, g_x is a nondecreasing mapping $I \to I$, with

$g_x(0) = 0$. Consequently, $h(F) = 0$ and $Q_0 := Q \times \{0\}$ is an invariant set. On the other hand, $IR(F) \neq \emptyset$ and $Q_0 = AP(F) \neq W(F)$. This example answers in the negative both problems.

However, for maps $f \in \mathcal{C}(I)$, $h(f) > 0$ is equivalent to $IR(f) \neq \emptyset$. On the other hand, the answer to Problem 2 remains negative even for maps in $\mathcal{C}(I)$. Instead, we are able to show that such maps with $W(f) \neq AP(f)$ are Li-Yorke chaotic. These results are given in the next section, as Theorems 2 and 3. Then, in Section 3 we show that these results can be extended to maps of more general one-dimensional compact metric space like topological graphs, topological trees, but not dendrites, see Theorems 4 and 5.

2 Relations with Topological Entropy for Maps in $\mathcal{C}(I)$

Theorem 2. *For $f \in \mathcal{C}(I)$, the conditions $h(f) > 0$ and $IR(f) \neq \emptyset$ are equivalent.*

Proof. If $h(f) = 0$ then $UR(f) = R(f)$ (see, e.g., [2], Corollary VI.8). Hence, by (4), $W(f) = QW(f)$. If $h(f) > 0$ then $W(f) \neq QW(f)$; this follows by Theorem 1 and Lemmas 2 and 3 stated below.

Let (Σ_2, σ) be the shift on the set Σ_2 of sequences of two symbols $0, 1$ equipped with a metric ρ of pointwise convergence, say, $\rho(\{x_i\}_{i\geq 1}, \{y_i\}_{i\geq 1}) = 1/k$ where $k = \min\{i \geq 1; x_i \neq y_i\}$.

Lemma 2. *$IR(\sigma)$ is non-empty, and contains a transitive point.*

Proof. Let

$$k_{1,0}, k_{1,1}, k_{2,0}, k_{2,1}, k_{2,2}, k_{3,0}, \cdots, k_{3,3}, k_{4,0}, \cdots, k_{4,4}, k_{5,0}, \cdots$$

be an increasing sequence of positive integers. Let $\{B_n\}_{n\geq 1}$ be a sequence of all finite blocks of digits 0 and 1. Put $A_0 = 10$, $A_1 = (A_0)^{k_{1,0}} 0^{k_{1,1}} B_1$ and in general,

$$A_n = A_{n-1}(A_0)^{k_{n,0}}(A_1)^{k_{n,1}} \cdots (A_{n-1})^{k_{n,n-1}} 0^{k_{n,n}} B_n, \ n \geq 1. \tag{7}$$

Denote by $|A|$ the lenght of a finite block of 0's and 1's, and let

$$a_n = |A_n|, \ b_n = |B_n|, \ c_n = a_n - b_n - k_{n,n}, \ n \geq 1, \tag{8}$$

and

$$\lambda_{n,m} = \left| A_{n-1}(A_0)^{k_{n,0}}(A_1)^{k_{n,1}} \cdots (A_m)^{k_{n,m}} \right|, \ 0 \leq m < n. \tag{9}$$

By induction we can take the numbers $k_{i,j}$ such that

$$k_{n,m+1} = n \cdot \lambda_{n,m}, \ 0 \leq m < n. \tag{10}$$

Let $N(A)$ be the cylinder of all $x \in \Sigma_2$ beginning with a finite block A. Then $\{N(B_n)\}_{n\geq 1}$ is a base of the topology of Σ_2, and $\bigcap_{n=1}^{\infty} N(A_n)$ contains exactly one point; denote it by u.

Since $\sigma^{a_n-b_n}(u) \in N(B_n)$, i.e., since the trajectory of u visits every $N(B_n)$, u is a transitive point of σ. Moreover, $\rho(u,\sigma^j(u)) = 1$, whenever $c_n \leq j < a_n - b_n$. By (10) it follows that $\Psi_u(\sigma,t) = 0$ for every $t \in (0,1)$. Consequently, $u \notin W(\sigma)$.

It remains to show that $u \in QW(\sigma)$. Let $t \in (0,1)$. Fix an $n_0 \in \mathbb{N}$ such that $1/a_{n_0} < t$. Then, by (7),

$$\#\left\{ j < \lambda_{n,n_0}; \rho(u,\sigma^j(u)) < t \right\} \geq k_{n,n_0},\ n > n_0,$$

hence, by (9) and (10),

$$\lim_{n\to\infty} \frac{\#\left\{ j < \lambda_{n,n_0}; \rho(u,\sigma^j(u)) < t \right\}}{\lambda_{n,n_0}} \geq \lim_{n\to\infty} \frac{k_{n,n_0}}{\lambda_{n,n_0}} = \lim_{n\to\infty} \frac{k_{n,n_0}}{\lambda_{n,n_0-1} + a_{n_0} k_{n,n_0}} = \lim_{n\to\infty} \frac{n}{1 + a_{n_0} n} = \frac{1}{a_{n_0}}.$$

Thus, $\Psi_u^*(\sigma,t) \geq 1/a_{n_0}$ and by Lemma 1, $u \in QW(\sigma)$.

Lemma 3. *Let $f \in \mathcal{C}(I)$ have positive topological entropy. Then $IR(f) \neq \emptyset$.*

Proof. When $h(f) > 0$, then f^m is strictly turbulent for some m. This means that there exist disjoint compact intervals K_0, K_1 such that $f^m(K_0) \cap f^m(K_1) \supset K_0 \cup K_1$, see [2], Theorem IX.28. This condition is equivalent to the existence of a continuous map $g : X \subset I \to \Sigma_2$, where X is of Cantor type, such that $g \circ f^m(x) = \sigma \circ g(x)$ for every $x \in X$, and such that each point in Σ_2 is the image of at most two points in X ([2], Proposition II.15). By Lemma 2, there is a $u \in IR(\sigma)$. Hence, for every $t > 0$, $\Psi_u^*(\sigma,t) > 0$, and there is an $s > 0$ such that $\Psi_u(\sigma,s) = 0$. There are at most two preimages, u_0 and u_1, of u. Then, by the continuity, $\Psi_{u_i}(f^m,r) = 0$, for some $r > 0$ and $i = 0,1$, and $\Psi_{u_i}^*(f^m,k) > 0$ for at least one $i \in \{0,1\}$ and every $k > 0$. Thus, $u_0 \in IR(f^m)$ or $u_1 \in IR(f^m)$ and, by Theorem 1, $IR(f) \neq \emptyset$.

Recall that $f \in \mathcal{C}(X)$ is *Li-Yorke chaotic*, or *LYC*, if there is an uncountable set $S \subseteq X$ such that for every $x \neq y$ in S, $\liminf_{n\to\infty} \rho(\varphi^n(x),\varphi^n(y)) = 0$ and $\limsup_{n\to\infty} \rho(\varphi^n(x),\varphi^n(y)) > 0$.

Theorem 3. *For $f \in \mathcal{C}(I)$, $W(f) \neq AP(f)$ implies that f is Li-Yorke chaotic, but does not imply $h(f) > 0$.*

Proof. Every continuous map of a compact metric space with positive topological entropy is Li-Yorke chaotic [1]. Hence to prove the theorem it suffices to consider the class $\mathcal{C}_0 \subset \mathcal{C}(I)$ of maps with zero topological entropy and show that

(i) for every $f \in \mathcal{C}_0$, $W(f) \neq AP(f)$ implies *LYC*, and

(ii) there is an $f \in \mathcal{C}_0$ with $W(f) \neq AP(f)$.

For $f \in \mathcal{C}_0$, $R(f) = UR(f)$, see, e.g., [2], Corollary VI.8. Hence, by (4), $W(f) \neq AP(f)$ implies that f has an infinite minimal ω-limit set $\widetilde{\omega}$ possessing a point which is not in $AP(f)$. Recall that for every such $\widetilde{\omega}$ there is an associated system $\{J_n\}_{n\geq 1}$ of compact periodic intervals such that J_n has period 2^n, and $\widetilde{\omega} \subseteq \bigcap_{n\geq 1} \bigcup_{0\leq j<2^n} f^j(J_n)$ [8]. For every $x \in \widetilde{\omega}$ there is a sequence $\iota(x) = \{j_n\}_{n\geq 1}$ of integers, $0 \leq j_n < 2^n$, such that

$$x \in \bigcap_{n\geq 1} f^{j_n}(J_n) =: Q_x.$$

For every $x \in \widetilde{\omega}$, the set $\widetilde{\omega} \cap Q_x$ contains one (i.e., the point x) or two points. In the second case $Q_x = [a,b]$ is a compact wandering interval (i.e., $f^n(Q_x) \cap Q_x = \emptyset$ for every $n \geq 1$) such that $a,b \in \widetilde{\omega}$ and either $x = a$ or $x = b$. Moreover, if, for every $x \in \widetilde{\omega}$, $\widetilde{\omega} \cap Q_x$ is a singleton then f restricted to $\widetilde{\omega}$ is the adding machine, and $\widetilde{\omega} \subseteq AP(f)$, see [3]. Consequently, $W(f) \neq AP(f)$ implies the existence of an infinite ω-limit set $\widetilde{\omega}$ such that

$$\widetilde{\omega} \cap Q_x = \{a,b\},\ a < b,\ \text{for some } x \in \widetilde{\omega}. \tag{11}$$

This condition characterizes *LYC* maps in $\mathcal{C}_0$ (see [8] or subsequent books like[11]) which proves (i).

To prove (ii) note that there are maps $f \in \mathcal{C}_0$ such that both a and b in (11) are non-isolated points of $\widetilde{\omega}$, see [3] or [6]. Then $a,b \in UR(f)$ are minimal points. We show that in this case either $a \notin AP(f)$ or $b \notin AP(f)$ (actually, neither a nor b is in $AP(f)$ but we do not need this stronger property). So assume that $a,b \in AP(f)$ and U_a, U_b are their disjoint open neighborhoods. Then there is an *even* m, $m = (2k+1)2^n$, with $n \geq 1$, such that $f^{jm}(a) \in U_a$ and $f^{jm}(b) \in U_b$, for every $j \geq 0$. Let $\{J_n\}_{n\geq 1}$ be the system of compact periodic intervals associated with $\widetilde{\omega}$. Without loss of generality we may assume that, for some n, $[a,b] \subset J_n$. Since J_n has period 2^n, for arbitrary odd j, $f^{jm}(J_n) \cap J_n = \emptyset$. If $f^{jm}(J_n)$ is to the left of J_n, then $f^{jm}(J_n) \cap U_b = \emptyset$, otherwise $f^{jm}(J_n) \cap U_a = \emptyset$. In any case, $f^{jm}(a) \notin U_a$ or $f^{jm}(b) \notin U_b$, which is a contradiction.

3 Generalization for Maps on More General One-dimensional Spaces

Here we show that the results given in Theorems 2 and 3 concerning maps in $\mathcal{C}(I)$ can be generalized to more general one-dimensional compact metric spaces like topological graphs or trees, but not dendrites. Recall that X is a topological graph if X is a non-empty compact connected metric space which is the union of finitely many arcs (i.e., continuous images of the interval I) such that every two arcs can have only end-points in common. A tree is a topological graph which contains no subset homeomorphic to the circle. A dendrite is a locally connected continuum containing no subset homeomorphic to the circle. The proof of generalized results is based on the same ideas as that of Theorems 2 and 3. We only need some recent, nontrivial results concerning the structure of ω-limit sets of such maps, see [4] and [5]. Therefore we give here only outline of the proof, pointing out only main differences.

Theorem 4. *Let* $f \in \mathcal{C}(X)$.

(i) *If X is a topological graph then $h(f) > 0$ is equivalent to $QW(f) \neq W(f)$.*

(ii) *There is a dendrit X such that $h(f) > 0$ and $QW(f) = W(f) = UR(f)$.*

Proof. To prove (i) note that, for $f \in \mathcal{C}(X)$ where X is a topological graph, $h(f) > 0$ if and only if, for some $n \geq 1$, f^n is turbulent [4]. Hence the proof of Lemma 3 applies also to this case and $h(f) > 0$ implies $IR(f) \neq \emptyset$. On the other hand, if $h(f) = 0$ then every infinite ω-limit set is a solenoid (i.e., it has an associated system of compact periodic intervals $\{J_n\}_{n\geq 1}$, J_n with period 2^n) and consequently, $R(f) = UR(f)$ [4] which gives the other implication.

(ii) In [5] there is an example of a dendrit X with a continuous map f possessing exactly two ω-limit sets: a minimal Cantor-type set Q such that $h(f|_Q) \geq 0$ and a fixed point p such that $\omega(f,x) = \{p\}$ for every $x \in X \setminus Q$.

Theorem 5. *Let* $f \in \mathcal{C}(X)$.

(i) *If X is a compact tree then $W(f) \neq AP(f)$ implies LYC, but does not imply $h(f) > 0$.*

(ii) *If X is a dendrit, or a topological graph containing a circle then $W(f) \neq AP(f)$ implies neither LYC nor $h(f) > 0$.*

Proof. (i) Similarly as in the proof of Theorem 3we may assume $h(f) = 0$. Then every infinite ω-limit set of f is a solenoid and the argument with obvious modifications applies.

(ii) If X is the circle, take f to be an irrational rotation. Then obvioulsy $X = UR(f) \setminus AP(f) = W(f) \setminus AP(f)$ but f is not *LYC*. On the other hand, let $\widetilde{\omega}$ be the ω-limit set used in the proof of part (ii) of Theorem 3. Thus, $\widetilde{\omega}$ is a minimal set intersecting $UR(f) \setminus AP(f)$. A modification of the construction from [5] yields a dendrite with exactly two ω-limit sets, an infinite minimal set $Q = \widetilde{\omega}$ and a fixed point q (see the proof of part (ii) of the preceding theorem). It is easy to see that f is not *LYC*.

Remark 1. By Theorems 4 and 5, for a map $f \in \mathcal{C}(X)$ where X is a compact metric space, the properties $h(f) > 0$ and $W(f) \neq AP(f)$ are independent. Similarly, $h(f) > 0$ and $IR(f) \neq \emptyset$ are independent. Example of a map f with $h(f) = 0$ and $IR(f) \neq \emptyset$ is given in [7] (see also the text at the end of Section 1), and any minimal map f with $h(f) > 0$ yields $IR(f) = \emptyset$.

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