

ASYMPTOTIC BEHAVIOR OF THE ECKHOFF APPROXIMATION IN BIVARIATE CASE

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Received Apr. 3, 2011

Abstract. The paper considers the Krylov-Lanczos and the Eckhoff approximations for recovering a bivariate function using limited number of its Fourier coefficients. These approximations are based on certain corrections associated with jumps in the partial derivatives of the approximated function. Approximation of the exact jumps is accomplished by solution of systems of linear equations along the idea of Eckhoff. Asymptotic behaviors of the approximate jumps and the Eckhoff approximation are studied. Exact constants of the asymptotic errors are computed. Numerical experiments validate theoretical investigations.

Key words: *Krylov-Lanczos approximation, Eckhoff approximation, Bernoulli polynomials, convergence acceleration*

AMS (2010) subject classification: 42A10, 65T40

1 Introduction

It is well known that approximation of a 2-periodic and smooth function on the real line by the truncated Fourier series

$$S_N(f; x) := \sum_{n=-N}^N f_n e^{i\pi n x}, \quad f_n := \frac{1}{2} \int_{-1}^1 f(x) e^{-i\pi n x} dx$$

is highly effective. If the 2-periodic extensions of f and its derivatives up to the order p are continuous on the real line, but $f^{(p)}$ is discontinuous, the uniform convergence rate is $O(N^{-p})$ (see[48]). The approximation is accompanied by the Gibbs phenomenon^[48] when the approximated function is discontinuous or non-periodic. The oscillations caused by this phenomenon typically propagate into regions away from singularities and degrade the quality of approximation. Even if the approximated function is analytic but non-periodic the error falls only as fast as

$O(1/N)$ over the entire interval except in zones of width $O(1/N)$ near $x = \pm 1$ where the error is always $O(1)$.

Much efforts were devoted to overcoming the convergence deficiency (see, for example, [3], [11], and [45] with references therein). An efficient approach of convergence acceleration by subtracting a polynomial representing the discontinuities (jumps) in the function and its derivatives was suggested by Krylov in 1906^[23] (see also [40] (pp. 88–99), and [46] (pp. 144–147)). Lanczos, in 1956^[24], in 1964^[25] and in 1966^[26] independently developed the same approach with more formality. He introduced a basic system of polynomials $B(k;x)$ that played a central role in the method and pointed out a close connection between the $B(k;x)$ and Bernoulli polynomials. That was why polynomial subtraction method was called also as Bernoulli method. Jones and Hardy in 1970^[22] and Lyness in 1974^[29] considered convergence acceleration of trigonometric interpolation by polynomial subtraction. They showed the relation of the Krylov-Lanczos method with the theory of Lidstone interpolation^[27]. Since then, it widely considered in the context of Fourier series^{[3],[8]–[11],[19],[31],[32]}, and trigonometric interpolation^{[32],[38]}.

The key problem in the Krylov-Lanczos method is approximation of the exact jump values. Ordinarily, such values are unknown and in general only Fourier coefficients or discrete Fourier coefficients of a given function may be specified. If arbitrary pointwise values of the function can be calculated then the finite difference formulas can be set up for approximation of these quantities. Approaches resembling this approach have been attempted under various names and apparently with varying success for the special case where the approximated function is smooth with only singularity at the end points of the interval: Gottlieb and Orszag in 1977 (polynomial subtractions for nonperiodic problems)^[20], Lanczos in 1966 (increasing the convergence of Fourier series by adding properly chosen boundary terms)^[26] Lyness in 1974 (Lanczos representation)^[29], and Roache in 1978 (reduction to periodicity)^[41]. Whereas, even if the arbitrary pointwise values of the function can be calculated, approximation of jump values via finite differences is not recommended for this purpose^[29]. Even in the case of a uniform grid, finite difference approximations are notoriously unreliable. Moreover, in many applications the Fourier coefficients can be calculated but pointwise values and derivatives are not explicitly available.

As noted in [15], the previous lack of robust methods for the approximation of jump values was the central reason why the polynomial subtraction technique has not been utilized more extensively. The first attempt towards more robust approach was initiated by Gottlieb et al^[21]

and which has been further developed in [1], [2], [12], and [14] by utilizing step functions in the reconstruction of discontinuous functions. Similar idea was applied by Orszag^[35] and Wengle et al^[47] to solve problems with nonperiodic boundary conditions. The general approach was established by Eckhoff in a series of papers^{[15]–[18]}. It was based on the observation that the Fourier coefficients themselves contained sufficient information to reconstruct the jump values. Hence, such values could be approximated to sufficient accuracy using only coefficients. The fundamental aspect of Eckhoff's method was approximation of the jumps by solution of linear system of equations and was rather familiar. A similar idea was used in the Richardson extrapolation process^[42]. Further investigation of the Eckhoff approximation and interpolation were organized in a series of papers [3], [7], [11], [36], [37], and [39]. Application of the polynomial subtraction method for convergence acceleration of the modified Fourier expansions was investigated in [3] (see also references therein).

Polynomial subtraction method for multivariate functions was investigated in [3]-[6], [30], [33], and [34]. In this paper we continue these investigations and focus our attention to bivariate Fourier series

$$S_N(f; x, y) := \sum_{n=-N}^N \sum_{m=-N}^N f_{n,m} e^{i\pi(nx+my)}, \quad f_{n,m} := \frac{1}{4} \int_{-1}^1 \int_{-1}^1 f(x, y) e^{-i\pi(nx+my)} dx dy.$$

Our aim is extraction of exact constants of the asymptotic errors both for jumps and corresponding Eckhoff approximation. Actually we are generalizing the results of [7] for bivariate functions. Our analysis shows how the exact asymptotic constants can be derived also for multidimensional Eckhoff approximation that can complement the results obtained in [5]. This general case will be considered elsewhere. Moreover, our approach shows that exact asymptotic constants can be derived also for multivariate Eckhoff interpolation along the ideas described in [37]. This also will be considered elsewhere. Numerical experiments pointed out the existence of the autocorrection phenomenon, described in [36] and [39], in multivariate approximations and interpolations. These investigations also will be considered elsewhere.

The paper is organized as follows. Section 2 presents the main ideas around the univariate Krylov-Lanczos and Eckhoff approximations. Subsection 2.1 describes the Krylov-Lanczos approximation, subsection 2.2 considers the problem of jump approximation and subsection 2.3 explores the accuracy of the Eckhoff approximation. The results of this section are not new and are coming mainly from the paper [7]. Section 3 considers the Krylov-Lanczos approximation in bivariate case. Section 4 solves the problem of jumps approximation along the idea of Eck-

hoff. Section 5 presents the Eckhoff approximation in bivariate case. In the last three sections exact constants of the asymptotic errors of approximations are calculated. Throughout the paper parallel to theoretical investigations the results of numerical experiments are discussed which validate theoretical conclusions and reveal the practical efficiency of the approximations.

2 The Krylov-Lanczos and Eckhoff Approximations in Univariate case, Approximation of the Jumps

In this section we describe the Krylov-Lanczos and the Eckhoff approximations in univariate case^[7].

2.1 The Krylov-Lanczos Approximation

Suppose $f \in C^q[-1, 1]$ and denote by $A_k(f)$ the exact value of the jump in the k -th derivative of f

$$A_k(f) := f^{(k)}(1) - f^{(k)}(-1), \quad k = 0, \dots, q.$$

The following lemma is crucial for the Krylov-Lanczos approach.

Lemma 1. *Let $f \in C^{q-1}[-1, 1]$ and $f^{(q-1)}$ is absolutely continuous on $[-1, 1]$ for some $q \geq 1$. Then the following expansion is valid*

$$f_n = \frac{(-1)^{n+1}}{2} \sum_{k=0}^{q-1} \frac{A_k(f)}{(i\pi n)^{k+1}} + \frac{1}{2(i\pi n)^q} \int_{-1}^1 f^{(q)}(x) e^{-i\pi n x} dx, \quad n \neq 0.$$

Proof. The proof is trivial due to integration by parts.

Lemma 1 implies the representation

$$f(x) = \sum_{k=0}^{q-1} A_k(f) B(k; x) + F(x), \quad (1)$$

where $B(k; x)$ are 2-periodic extensions of the Bernoulli-like polynomials with the Fourier coefficients

$$B_n(k) := \begin{cases} 0, & n = 0 \\ \frac{(-1)^{n+1}}{2(i\pi n)^{k+1}}, & n \neq 0 \end{cases}$$

and F is a 2-periodic and relatively smooth function on the real line ($F \in C^{q-1}(\mathbf{R})$) with the Fourier coefficients

$$F_n = f_n - \sum_{k=0}^{q-1} A_k(f) B_n(k). \quad (2)$$

The Bernoulli-like polynomials can be calculated recursively

$$B(0;x) = \frac{x}{2}, B(k;x) = \int B(k-1;x)dx, \quad x \in [-1, 1],$$

where the constant of integration is defined by the relation

$$\int_{-1}^1 B(k;x)dx = 0.$$

Approximation of F in (1) by the truncated Fourier series leads to the Krylov-Lanczos (KL) approximation

$$S_{N,q}(f;x) := \sum_{n=-N}^N \left(f_n - \sum_{k=0}^{q-1} A_k(f)B_n(k) \right) e^{i\pi n x} + \sum_{k=0}^{q-1} A_k(f)B(k;x) \tag{3}$$

with the error

$$R_{N,q}(f;x) := f(x) - S_{N,q}(f;x). \tag{4}$$

Denote by $\|\cdot\|$ the standard norm in the space $L_2(-1, 1)$

$$\|f\| := \left(\int_{-1}^1 |f(x)|^2 dx \right)^{1/2}.$$

The following theorem illustrates the basic idea behind the Krylov-Lanczos approach.

Theorem 2.1^[7]. *Suppose $f \in C^q[-1, 1]$ and $f^{(q)}$ is absolutely continuous on $[-1, 1]$. Then the following estimate holds:*

$$\lim_{N \rightarrow \infty} N^{q+\frac{1}{2}} \|R_{N,q}(f)\| = \frac{|A_q(f)|}{\pi^{q+1} \sqrt{2q+1}}.$$

Proof. Lemma 1, Equations (1), (3), and (4) imply

$$R_{N,q}(f) = \sum_{|n|>N} F_n e^{i\pi n x},$$

where

$$\begin{aligned} F_n &= \frac{1}{2(i\pi n)^q} \int_{-1}^1 f^{(q)}(x) e^{-i\pi n x} dx \\ &= \frac{(-1)^{n+1}}{2} \frac{A_q(f)}{(i\pi n)^{q+1}} + \frac{1}{2(i\pi n)^{q+1}} \int_{-1}^1 f^{(q+1)}(x) e^{-i\pi n x} dx. \end{aligned}$$

Taking into account that the second term is $o(n^{-q-1})$ as $n \rightarrow \infty$ according to the Riemann-Lebesgue theorem, we get

$$\|R_{N,q}\|^2 = 2 \sum_{|n|>N} |F_n|^2 = \frac{|A_q|^2}{\pi^{2q+2}} \sum_{n=N+1}^{\infty} \frac{1}{n^{2q+2}} + o(N^{-2q-1}), \quad N \rightarrow \infty.$$

This concludes the proof.

2.2 Approximation of the Jumps

In order to determine the approximate values $\tilde{A}_k(f; N)$ of $A_k(f)$, the fact is used, that the coefficients F_n in (2) asymptotically ($n \rightarrow \infty$) decay faster than the coefficients $B_n(k)$ for $k = 0, \dots, q-1$ and can therefore be discarded for large $|n|$. Hence, we can write

$$f_n = \sum_{k=0}^{q-1} \tilde{A}_k(f; N) B_n(k), \quad n = n_1, n_2, \dots, n_q. \quad (5)$$

The next theorem shows how well the values $\tilde{A}_k(f; N)$ approximate the actual jumps $A_k(f)$ depending on the choice of the indices n_s .

Definition 2.2. By the multiplicity of some number x in a sequence x_1, \dots, x_m we mean the number of indices i for which $x_i = x$.

Theorem 2.3^[7]. Suppose the indices $n_s = n_s(N)$ in (5) are chosen such that

$$\lim_{N \rightarrow \infty} \frac{n_s}{N} = d_s \neq 0, \quad s = 1, \dots, q.$$

Let α be the greatest multiplicity of a number in the sequence d_1, d_2, \dots, d_q . Then, for $f \in C^{q+\alpha-1}[-1, 1]$ such that $f^{(q+\alpha-1)}$ is absolutely continuous on $[-1, 1]$, the following estimate holds:

$$\tilde{A}_j(f; N) = A_j(f) - A_q(f) \frac{\chi_j}{(i\pi N)^{q-j}} + o(N^{-q+j}), \quad N \rightarrow \infty, \quad j = 0, \dots, q-1,$$

where the constants χ_j are coefficients of polynomial

$$\prod_{s=1}^q \left(x - \frac{1}{d_s} \right) = \sum_{s=0}^q \chi_s x^s.$$

2.3 The Eckhoff Approximation

Approximation of $A_k(f)$ in (3) by $\tilde{A}_k(f; N)$ leads to the Eckhoff approximation

$$\tilde{S}_{N,q}(f; x) := \sum_{n=-N}^N \tilde{F}_n e^{i\pi n x} + \sum_{k=0}^{q-1} \tilde{A}_k(f; N) B(k; x) \quad (6)$$

with the error

$$\tilde{R}_{N,q}(f; x) := f(x) - \tilde{S}_{N,q}(f; x), \quad (7)$$

where

$$\tilde{F}_n = f_n - \sum_{k=0}^{q-1} \tilde{A}_k(f; N) B_n(k). \tag{8}$$

The next theorem formulates the analog of Theorem 2.1 for $\tilde{S}_{N,q}(f)$.

Theorem 2.4^[7]. *Let the conditions of Theorem 2.3 be valid. Then the following estimate holds:*

$$\lim_{N \rightarrow \infty} N^{q+\frac{1}{2}} \|\tilde{R}_{N,q}(f)\| = \frac{|A_q(f)|}{\sqrt{2} \pi^{q+1}} \left(\int_{-1}^1 \prod_{s=1}^q \left(x - \frac{1}{d_s}\right)^2 dx \right)^{1/2}.$$

Comparison of Theorems 2.1 and 2.4 shows that approximate calculation of the jumps does not degrade the convergence rate of the Eckhoff approximation.

Theorem 2.4 encompasses the optimal choice of the indices n_s ($d_s = \pm 1$) for the best approximation:

$$n_s = N - s + 1, s = 1, \dots, m; n_s = -(N - s + m + 1), s = m + 1, \dots, 2m \tag{9}$$

for even values of q , $q = 2m$, $m = 1, 2, \dots$, and

$$n_s = N - s + 1, s = 1, \dots, m + 1; n_s = -(N - s + m + 2), s = m + 2, \dots, 2m + 1 \tag{10}$$

for odd values of q , $q = 2m + 1$, $m = 0, 1, \dots$.

As is shown in [36] we have

$$\chi_{2s} = \binom{m}{s} (-1)^{m+s}, s = 0, \dots, m; \chi_{2s+1} = 0, s = 0, \dots, m - 1 \tag{11}$$

for (9), and

$$\chi_{2s} = \binom{m}{s} (-1)^{m+s+1}, \chi_{2s+1} = \binom{m}{s} (-1)^{m+s}, s = 0, \dots, m \tag{12}$$

for (10).

Henceforth, in numerical experiments we will use these choices for the indices n_s .

3 The Krylov-Lanczos Approximation in Bivariate Case

Throughout the paper, $D = [-1, 1] \times [-1, 1]$. As usual $L(D)$ stands for the set of Lebesgue integrable functions on D . $AC[-1, 1]$ and $L[-1, 1]$ are the sets of absolutely continuous and Lebesgue integrable, respectively, functions on $[-1, 1]$.

Carathéodory's definition [13] of absolute continuity for functions of two variables may be stated as follows (see [43]):

Let $\sigma(D)$ denote the system of all rectangles $[x_1, x_2] \times [y_1, y_2]$ contained in D . For any rectangle $P \in \sigma(D)$, $|P|$ denotes the volume of P . We say that rectangles $P_1, P_2 \in \sigma(D)$ do not overlap if they have no interior points in common. Furthermore, rectangles $P_1, P_2 \in \sigma(D)$ are referred to be adjoining if they do not overlap and $P_1 \cup P_2 \in \sigma(D)$.

Definition 3.1^[28]. A finite function $F : \sigma(D) \rightarrow \mathbb{C}$ is said to be additive function of rectangles if, for any adjoining rectangles $P_1, P_2 \in \sigma(D)$, the relation

$$F(P_1 \cup P_2) = F(P_1) + F(P_2)$$

holds.

Definition 3.2^[28]. We say that an additive function of rectangles $F : \sigma(D) \rightarrow R$ is absolutely continuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $P_1, \dots, P_k \in \sigma(D)$ are mutually non-overlapping rectangles with the property

$$\sum_{j=1}^k |P_j| \leq \delta,$$

then the relation

$$\sum_{j=1}^k |F(P_j)| \leq \varepsilon$$

is satisfied.

Having the function $f : \sigma(D) \rightarrow \mathbf{C}$, we put

$$F_f([x_1, x_2] \times [y_1, y_2]) := f(x_1, y_1) - f(x_1, y_2) - f(x_2, y_1) + f(x_2, y_2). \quad (13)$$

The function of rectangles $F_f : \sigma(D) \rightarrow \mathbf{C}$ defined by (13) is said to be a function of rectangles associated with f .

Definition 3.3^{[43],[44]}. We say that a function $f : \sigma(D) \rightarrow \mathbf{C}$ is absolutely continuous on D ($f \in AC(D)$) in the sense of Carathéodory if the following two conditions hold:

- the function of rectangles F_f associated with f is absolutely continuous;
- the functions $f(-1, \cdot) : [-1, 1] \rightarrow \mathbf{C}$ and $f(\cdot, -1) : [-1, 1] \rightarrow \mathbf{C}$ are absolutely continuous.

We will frequently use the following result.

Theorem 3.4^{[43],[44]}. Let $f \in AC(D)$. Then

- $f(\cdot, y) \in AC[-1, 1]$ for every $y \in [-1, 1]$;
- $f(x, \cdot) \in AC[-1, 1]$ for every $x \in [-1, 1]$;
- $f_y(\cdot, y) \in AC[-1, 1]$ for almost every $y \in [-1, 1]$;

- $f_x(x, \cdot) \in AC[-1, 1]$ for almost every $x \in [-1, 1]$;
- $f_{xy} \in L(D)$, $f_{yx} \in L(D)$, and $f_{yx}(x, y) = f_{xy}(x, y)$ for almost every $(x, y) \in D$.

Denote

$$\begin{aligned}
 f^{(k,s)}(x,y) &:= \frac{\partial^{k+s} f(x,y)}{\partial^k x \partial^s y}, \\
 a(k;y) &:= f^{(k,0)}(1,y) - f^{(k,0)}(-1,y), \quad b(s;x) := f^{(0,s)}(x,1) - f^{(0,s)}(x,-1), \\
 a_m(k) &:= \frac{1}{2} \int_{-1}^1 a(k;y) e^{-i\pi m y} dy, \quad b_n(s) := \frac{1}{2} \int_{-1}^1 b(s;x) e^{-i\pi n x} dx, \\
 a^{(s)}(k;y) &:= f^{(k,s)}(1,y) - f^{(k,s)}(-1,y), \quad b^{(k)}(s;x) := f^{(k,s)}(x,1) - f^{(k,s)}(x,-1), \\
 a_m^{(s)}(k) &:= \frac{1}{2} \int_{-1}^1 a^{(s)}(k;y) e^{-i\pi m y} dy, \quad b_n^{(k)}(s) := \frac{1}{2} \int_{-1}^1 b^{(k)}(s;x) e^{-i\pi n x} dx, \\
 c(k,s) &:= f^{(k,s)}(1,1) - f^{(k,s)}(1,-1) - f^{(k,s)}(-1,1) + f^{(k,s)}(-1,-1).
 \end{aligned}$$

The following lemma is crucial for bivariate approximations.

Lemma 2. *Let $f^{(k,s)} \in C(D)$, $k, s = 0, \dots, q-1$ and $f^{(q-1,q-1)} \in AC(D)$. Then the following relations hold:*

$$\begin{aligned}
 f_{n,0} &= \sum_{k=0}^{q-1} B_n(k) a_0(k) + \frac{1}{4(i\pi n)^q} \int_{-1}^1 \int_{-1}^1 f^{(q,0)}(x,y) e^{-i\pi n x} dx dy, \quad n \neq 0, \\
 f_{0,m} &= \sum_{s=0}^{q-1} B_m(s) b_0(s) + \frac{1}{4(i\pi m)^q} \int_{-1}^1 \int_{-1}^1 f^{(0,q)}(x,y) e^{-i\pi m y} dx dy, \quad m \neq 0, \\
 f_{n,m} &= \sum_{k=0}^{q-1} B_n(k) a_m(k) + \sum_{s=0}^{q-1} B_m(s) b_n(s) - \sum_{k,s=0}^{q-1} B_n(k) B_m(s) c(k,s) \\
 &\quad + \frac{1}{4(i\pi n)^q (i\pi m)^q} \int_{-1}^1 \int_{-1}^1 f^{(q,q)}(x,y) e^{-i\pi (nx+my)} dx dy, \quad n, m \neq 0.
 \end{aligned}$$

Proof. The proof is trivial by means of integration by parts in view of Theorem 3.4.

Taking into account that

$$f(x,y) = \sum_{n=-\infty}' \sum_{m=-\infty}' f_{n,m} e^{i\pi (nx+my)} + \sum_{n=-\infty}' f_{n,0} e^{i\pi n x} + \sum_{m=-\infty}' f_{0,m} e^{i\pi m y} + f_{0,0}$$

in view of Lemma 2 we get the main representation

$$f(x,y) = G(x,y) + F(x,y), \tag{14}$$

where

$$G(x,y) := \sum_{k=0}^{q-1} B(k;x) a(k;y) + \sum_{s=0}^{q-1} B(s;y) b(s;x) - \sum_{k=0}^{q-1} \sum_{s=0}^{q-1} B(k;x) B(s;y) c(k,s), \tag{15}$$

and F is a relatively smooth function $F^{(k,s)} \in C(R^2)$, $k, s = 0, \dots, q - 1$.

Henceforth, theoretical investigations will be accompanied by the results of numerical experiments. As a typical example we consider the following function which is smooth on D but has discontinuous 2-periodic extension on R^2

$$f(x) = \sin(3x + 2y^2 + 2). \tag{16}$$

The graph of (16) is shown in Figure 1.

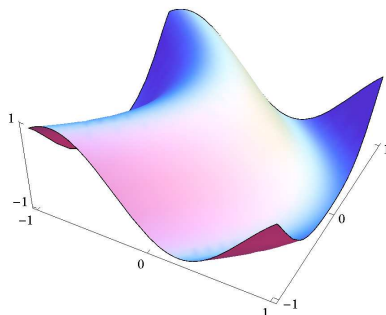


Figure 1. Graph of function (16)

Figure ?? presents graphs of $G(x, y)$ (left) and $F(x, y)$ (right) for $q = 3$. Periodic extension of F onto R^2 is smooth $F \in C^2(R^2)$.

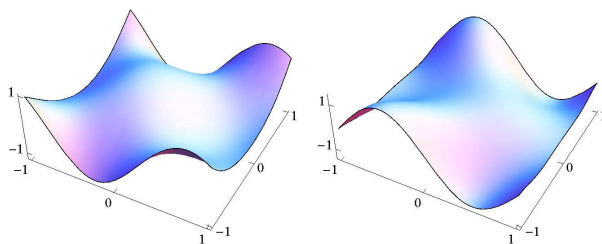


Figure 2. Graphs of $G(x, y)$ (left) and $F(x, y)$ (right) for $q = 3$

The Fourier coefficients of G and F are known explicitly in view of Lemma 2, and Equations (14) and (15)

$$G_{n,m} = \sum_{k=0}^{q-1} B_n(k)a_m(k) + \sum_{s=0}^{q-1} B_m(s)b_n(s) - \sum_{k,s=0}^{q-1} B_n(k)B_m(s)c(k, s), \quad n, m \neq 0,$$

$$G_{0,m} = \sum_{s=0}^{q-1} B_m(s)b_0(s), m \neq 0, G_{n,0} = \sum_{k=0}^{q-1} B_n(k)a_0(k), n \neq 0, G_{0,0} = 0,$$

and

$$F_{n,m} = \frac{1}{4(i\pi n)^q(i\pi m)^q} \int_{-1}^1 \int_{-1}^1 f^{(q,q)}(x,y)e^{-i\pi(nx+my)} dx dy, n, m \neq 0, F_{0,0} = f_{0,0},$$

$$F_{n,0} = \frac{1}{4(i\pi n)^q} \int_{-1}^1 \int_{-1}^1 f^{(q,0)}(x,y)e^{-i\pi nx} dx dy, n \neq 0,$$

$$F_{0,m} = \frac{1}{4(i\pi m)^q} \int_{-1}^1 \int_{-1}^1 f^{(0,q)}(x,y)e^{-i\pi my} dx dy, m \neq 0.$$

Approximation of F in (14) by the truncated Fourier series leads to the *Krylov-Lanczos (KL) approximation* in bivariate case

$$S_{N,q}(f;x,y) := G(x,y) + \sum_{n=-N}^N \sum_{m=-N}^N (f_{n,m} - G_{n,m})e^{i\pi(nx+my)} \tag{17}$$

with the error

$$R_{N,q}(f;x,y) := f(x,y) - S_{N,q}(f;x,y). \tag{18}$$

Note that the KL-approximation uses the exact values of the jump functions $a(k;y)$, $b(s;x)$ and jumps $c(k,s)$ while constructing the correction function G . Later we will discuss the problem of jumps reconstruction.

For further investigations we need the following lemma that reveals the asymptotic behavior of the Fourier coefficients $F_{n,m}$.

Lemma 3. *Let $f^{(k,s)} \in C(D)$, $k, s = 0, \dots, q$ and $f^{(q,q)} \in AC(D)$. Then the following estimates hold:*

$$F_{n,0} = B_n(q)a_0(q) + o(n^{-q-1}), n \rightarrow \infty, \tag{19}$$

$$F_{0,m} = B_m(q)b_0(q) + o(m^{-q-1}), m \rightarrow \infty, \tag{20}$$

$$F_{n,m} = \frac{B_m(q)}{(i\pi n)^q} b_n^{(q)}(q) + \frac{o(m^{-q-1})}{n^q}, m \rightarrow \infty, n \neq 0, \tag{21}$$

$$F_{n,m} = \frac{B_n(q)}{(i\pi m)^q} a_m^{(q)}(q) + \frac{o(n^{-q-1})}{m^q}, n \rightarrow \infty, m \neq 0. \tag{22}$$

Proof. The proof is trivial by means of integration by parts in view of Theorem 3.4 and the Riemann-Lebesgue theorem.

Asymptotic behavior of the KL-approximation depends on the smoothness of F on R^2 and hence on the asymptotic behavior of $F_{n,m}$ as $n, m \rightarrow \infty$ that we established in the last lemma. This encompasses the proof of the following theorem.

Theorem 3.5. *Let $f^{(k,s)} \in C(D)$, $k, s = 0, \dots, q$ and $f^{(q,q)} \in AC(D)$. Then the following estimate holds:*

$$\lim_{N \rightarrow \infty} N^{q+\frac{1}{2}} \|R_{N,q}(f)\| = D_q(f),$$

where

$$D_q(f) := \frac{\sqrt{h_q(f)}}{\pi^{q+1} \sqrt{2q+1}},$$

and

$$h_q(f) := \frac{1}{2} \left| \int_{-1}^1 a(q;x) dx \right|^2 + \frac{1}{2} \left| \int_{-1}^1 b(q;x) dx \right|^2 + \int_{-1}^1 \left| \int_{-1}^1 B(q-1;t) b^{(q)}(q;x-t) dt \right|^2 dx + \int_{-1}^1 \left| \int_{-1}^1 B(q-1;t) a^{(q)}(q;x-t) dt \right|^2 dx.$$

Proof. In view of (14), (17), and (18) we have

$$\begin{aligned} R_{N,q}(f) &= F(x,y) - \sum_{n=-N}^N \sum_{m=-N}^N F_{n,m} e^{i\pi(nx+my)} \\ &= \sum_{|n|>N} F_{n,0} e^{i\pi nx} + \sum_{|m|>N} F_{0,m} e^{i\pi my} + \sum'_{n=-N}^N \sum_{|m|>N} F_{n,m} e^{i\pi(nx+my)} \\ &\quad + \sum'_{m=-N}^N \sum_{|n|>N} F_{n,m} e^{i\pi(nx+my)} + \sum_{|n|>N} \sum_{|m|>N} F_{n,m} e^{i\pi(nx+my)}. \end{aligned}$$

Therefore

$$\begin{aligned} \|R_{N,q}(f)\|^2 &= 4 \sum_{|n|>N} |F_{n,0}|^2 + 4 \sum_{|m|>N} |F_{0,m}|^2 \\ &\quad + 4 \sum'_{n=-N}^N \sum_{|m|>N} |F_{n,m}|^2 + 4 \sum'_{m=-N}^N \sum_{|n|>N} |F_{n,m}|^2 \\ &\quad + 4 \sum_{|n|>N} \sum_{|m|>N} |F_{n,m}|^2. \end{aligned} \tag{23}$$

The first term in (23) we estimate in accordance with (19)

$$\lim_{N \rightarrow \infty} N^{2q+1} \left(4 \sum_{|n|>N} |F_{n,0}|^2 \right) = \frac{|a_0(q)|^2}{\pi^{2q+2}} \lim_{N \rightarrow \infty} N^{2q+1} \sum_{|n|>N} \frac{1}{n^{2q+2}} = \frac{2|a_0(q)|^2}{\pi^{2q+2}(2q+1)}.$$

Similarly, (20) implies

$$\lim_{N \rightarrow \infty} N^{2q+1} \left(4 \sum_{|m| > N} |F_{0,m}|^2 \right) = \frac{2|b_0(q)|^2}{\pi^{2q+2}(2q+1)}.$$

The third term we treat by (21)

$$\begin{aligned} & \lim_{N \rightarrow \infty} N^{2q+1} \left(4 \sum'_{n=-N}^{N'} \sum_{|m| > N} |F_{n,m}|^2 \right) \\ &= \lim_{N \rightarrow \infty} N^{2q+1} \left(\frac{4}{\pi^{2q+2}} \sum_{|m| > N} \frac{1}{m^{2q+2}} \sum'_{n=-N}^{N'} |B_n(q-1)b_n^{(q)}(q)|^2 \right) \\ &= \frac{8}{\pi^{2q+2}(2q+1)} \sum'_{n=-\infty}^{\infty} |B_n(q-1)b_n^{(q)}(q)|^2 \\ &= \frac{1}{\pi^{2q+2}(2q+1)} \int_{-1}^1 \left| \int_{-1}^1 B(q-1;t)b^{(q)}(q;x-t)dt \right|^2 dx. \end{aligned}$$

Similarly, (22) yields

$$\lim_{N \rightarrow \infty} N^{2q+1} \left(4 \sum'_{m=-N}^{N'} \sum_{|n| > N} |F_{n,m}|^2 \right) = \frac{1}{\pi^{2q+2}(2q+1)} \int_{-1}^1 \left| \int_{-1}^1 B(q-1;t)a^{(q)}(q;x-t)dt \right|^2 dx.$$

In view of (21) or (22), the last term is $o(N^{-2q-1})$ as $N \rightarrow \infty$. Substituting all these into (23) we get the required estimate.

Numerical values of $D_q(f)$ for different values of q while approximating function (16) by the KL-approximation are calculated in Table 1. We see that even values of q provide with more accurate (asymptotically) approximation than odd values.

Table 1 Numerical values of $D_q(f)$

q	1	2	3	4	5
$D_q(f)$	0.4904	0.0580	0.5473	0.0972	1.3032

Denote

$$D_{q,N}(f) := N^{q+\frac{1}{2}} \|R_{N,q}(f)\|.$$

It is interesting to calculate the constants $D_{q,N}(f)$ for different values of N and compare them with the theoretical estimate $D_q(f)$. Table 2 shows such values for moderate values of N . For more large values of N calculations are useless due to round off errors as we need to get integrals

from oscillatory functions with high precision. Our calculations are carried out by the MATHEMATICA 8.0 package with standard precision. Comparison of the tables shows that the values of $D_{q,N}(f)$ are close to the values of $D_q(f)$.

Table 2 Numerical values of $D_{q,N}(f)$

	$q = 1$	$q = 2$	$q = 3$	$q = 4$	$q = 5$
$N = 4$	0.3881	0.0998	0.3703	0.1677	0.6317
$N = 8$	0.4124	0.0706	0.4488	0.1215	0.9042
$N = 16$	0.4280	0.0555	0.4963	0.0780	1.0779

Figures 3-6 compare accuracies of approximations by the truncated Fourier series and KL-approximation.

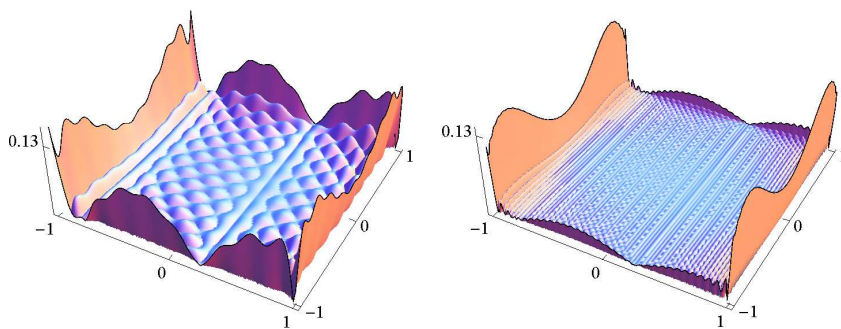


Figure 3. Graphs of the absolute errors while approximating (16) by the truncated Fourier series for $N = 8$ (left) and $N = 32$ (right)

Figure 3 shows non-uniform convergence while approximating non-smooth on R^2 function by the truncated Fourier series. Convergence exists only inside the domain of approximation away from the singularities (see Figure 4). Figures 5 and 6 present the higher accuracy of the KL-approximation on the whole domain of approximation and away from the singularities, respectively. Figure 4 shows increase in accuracy by 46.6 times while theoretical estimate, well-known from the Fourier analysis, is 64 times ($O(N^3)$) while changing N from 8 to 32. Not a bad coincidence for moderate values of N .

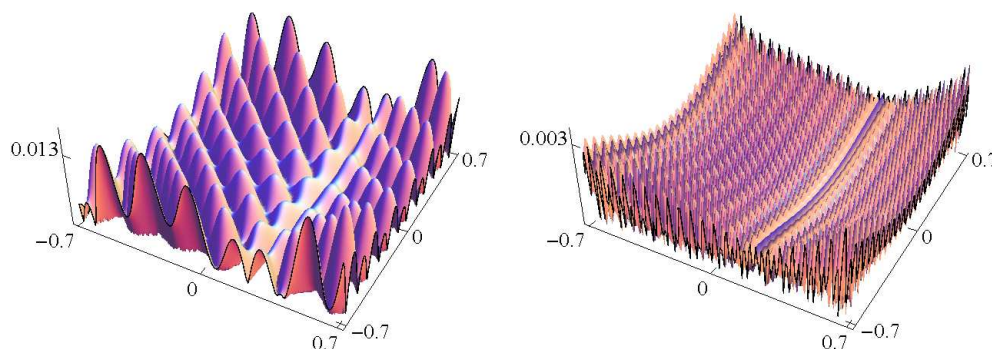


Figure 4. Graphs of the absolute errors while approximating (16) by the truncated Fourier series for $N = 8$ (left) and $N = 32$ (right) away from the singularities

4 Approximation of the Jump Functions

In this section we discuss the problem of reconstruction of the jump functions $a(k; y)$, $b(s; x)$ and the numbers $c(k, s)$ directly from the Fourier coefficients $f_{n,m}$. The procedure of reconstruction includes the following steps: first, we will recover the approximate values of the Fourier coefficients $\tilde{a}_m(k; N)$ of $a_m(k)$ and $\tilde{b}_n(s; N)$ of $b_n(s)$, second, based on these coefficients approximation of $\tilde{a}(k; y; N)$ of $a(k; y)$ and $\tilde{b}(s; x; N)$ of $b(s; x)$ will be performed according to the univariate Eckhoff approximation. Note that the numbers $c(k, s)$ are exact jumps of $a(k; y)$ or $b(s; x)$.

The next lemma reveals the asymptotic behavior of $f_{n,m}$ that outlines the procedure of the Fourier coefficients approximation.

Lemma 4. *Let $f^{(k,s)} \in C(D)$, $k, s = 0, \dots, q-1$ and $f^{(q-1,q-1)} \in AC(D)$. Then the following relations hold:*

$$f_{n,m} = \sum_{k=0}^{q-1} B_n(k) a_m(k) + \frac{1}{4(i\pi n)^q} \int_{-1}^1 \int_{-1}^1 f^{(q,0)}(x, y) e^{-i\pi(nx+my)} dx dy, \quad n \neq 0, \quad (24)$$

$$f_{n,m} = \sum_{s=0}^{q-1} B_m(s) b_n(s) + \frac{1}{4(i\pi m)^q} \int_{-1}^1 \int_{-1}^1 f^{(0,q)}(x, y) e^{-i\pi(nx+my)} dx dy, \quad m \neq 0, \quad (25)$$

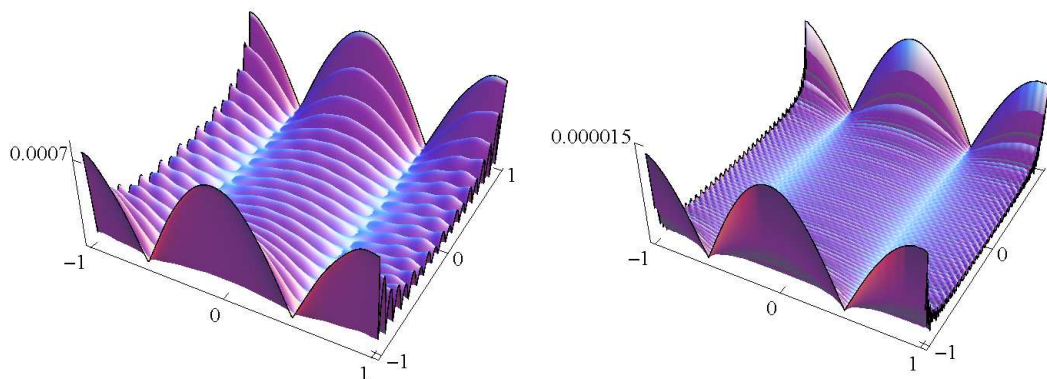


Figure 5. Graphs of the absolute errors while approximating (16) by $S_{N,3}$ for $N = 8$ (left) and $N = 32$ (right)

$$f_{n,m} = \sum_{k,s=0}^{q-1} B_n(k)B_m(s)c(k,s) + \frac{1}{(i\pi m)^q} \sum_{k=0}^{q-1} B_n(k)a_m^{(q)}(k) + \frac{1}{(i\pi n)^q} \sum_{s=0}^{q-1} B_m(s)b_n^{(q)}(s) + \frac{1}{4(i\pi n)^q(i\pi m)^q} \int_{-1}^1 \int_{-1}^1 f^{(q,q)}(x,y)e^{-i\pi(nx+my)} dx dy, \quad n,m \neq 0. \tag{26}$$

Proof. The proof is trivial by means of integration by parts.

As the second term in the right hand side of (24) asymptotically ($n \rightarrow \infty$) decay faster than the first term then it can be discarded for large $|n|$. Hence, from (24) we derive the following systems of linear equations

$$f_{n,m} = \sum_{k=0}^{q-1} B_n(k)\tilde{a}_m(k;N), \quad |m| \leq N, \quad n = n_1, n_2, \dots, n_q. \tag{27}$$

Thus, for any given N we assume to have chosen q different integer indices

$$n_1 = n_1(N), \quad n_2 = n_2(N), \dots, n_q = n_q(N)$$

for evaluating systems (27). Solving these we get the values $\tilde{a}_m(k;N)$, which, as we later prove, approximate the Fourier coefficients $a_m(k)$.

Similarly, from (25) we get the systems

$$f_{n,m} = \sum_{s=0}^{q-1} B_m(s)\tilde{b}_n(s;N), \quad |n| \leq N, \quad m = n_1, n_2, \dots, n_q \tag{28}$$

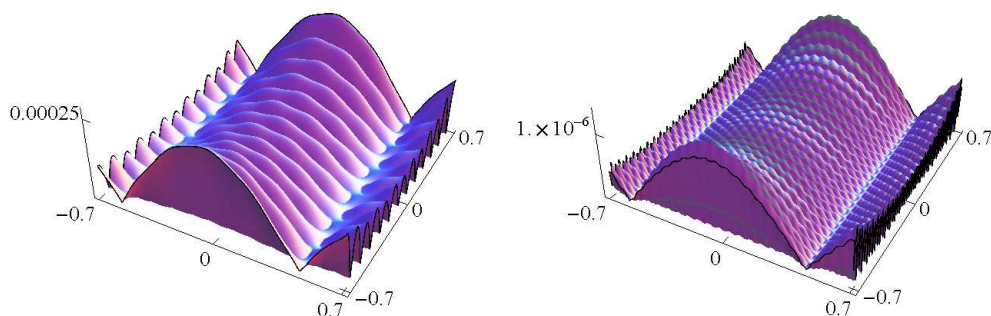


Figure 6. Graphs of the absolute errors while approximating (16) by $S_{N,3}$ for $N = 8$ (left) and $N = 32$ (right) away from the singularities

for approximating the Fourier coefficients $b_n(s)$.

For approximation of the numbers $c(k, s)$ we obtain from (26) the system of equations

$$f_{n,m} = \sum_{k,s=0}^{q-1} B_n(k)B_m(s)\tilde{c}(k,s;N), \quad m, n = n_1, n_2, \dots, n_q. \tag{29}$$

Throughout the paper we will suppose that

$$\alpha N \leq |n_s| \leq N, \quad s = 1, \dots, q \tag{30}$$

for some $0 < \alpha \leq 1$. Note that choices (9) and (10) also satisfy to this estimate.

We will often use the following results.

Lemma 5^[7]. *Let $x_s = (i\pi n_s)^{-1}$ and $\alpha N \leq |n_s| \leq N, s = 1, \dots, q$ for some $0 < \alpha \leq 1$. Denote*

$$\omega_j(q) = \sum_{k=1}^q \frac{x_k^j}{\prod_{\substack{n=1 \\ n \neq k}}^q (x_k - x_n)}, \quad j \geq 0.$$

Then

- (a) $\omega_j(q) = 0$ for $j = 0, \dots, q - 2$;
- (b) $\omega_{q-1}(q) = 1$;
- (c) $\omega_j(q) = O(N^{q-j-1})$ when $N \rightarrow \infty$ for every $j \geq 0$.

Lemma 6^[7]. Suppose the indices n_s satisfy the condition (30) and suppose γ_j is the j -th coefficient of the polynomial $\prod_{s=1}^q (x - x_s) = \sum_{j=0}^q \gamma_j x^j$. Then

$$\gamma_j = O(N^{-q+j}), \quad j = 0, \dots, q-1, \quad N \rightarrow \infty.$$

The next theorem investigates the accuracy of the jumps approximation.

Theorem 4.1. Suppose the indices $n_s = n_s(N)$ are chosen such that

$$\lim_{N \rightarrow \infty} \frac{n_s}{N} = d_s \neq 0, \quad s = 1, \dots, q. \tag{31}$$

Let α be the greatest multiplicity of a number in the sequence d_1, d_2, \dots, d_q . Now, for $f^{(k,s)} \in C(D)$, $k, s = 0, \dots, q + \alpha - 1$ such that $f^{(q+\alpha-1, q+\alpha-1)} \in AC(D)$, the following estimates hold as $N \rightarrow \infty$

$$\tilde{a}_m(j; N) = a_m(j) - a_m(q) \frac{\chi_j}{(i\pi N)^{q-j}} + \varepsilon_m o(N^{-q+j}), \quad j = 0, \dots, q-1, \tag{32}$$

$$\tilde{b}_n(j; N) = b_n(j) - b_n(q) \frac{\chi_j}{(i\pi N)^{q-j}} + \delta_n o(N^{-q+j}), \quad j = 0, \dots, q-1, \tag{33}$$

$$\begin{aligned} \tilde{c}(j, k; N) &= c(j, k) - c(j, q) \frac{\chi_k}{(i\pi N)^{q-k}} \\ &\quad - c(q, k) \frac{\chi_j}{(i\pi N)^{q-j}} + o(N^{-q+j}) + o(N^{-q+k}), \quad j, k = 0, \dots, q-1, \end{aligned} \tag{34}$$

where the constants χ_j are the coefficients of the polynomial

$$\prod_{s=1}^q \left(x - \frac{1}{d_s} \right) = \sum_{s=0}^q \chi_s x^s,$$

and the series $\sum_{n=-\infty}^{\infty} |\delta_n|^2, \sum_{m=-\infty}^{\infty} |\varepsilon_m|^2$ are convergent.

Proof. First we are proving (33). In view of (25) and (28) we get

$$\begin{aligned} \sum_{s=0}^{q-1} (b_n(s) - \tilde{b}_n(s)) x_k^s &= - \sum_{s=q}^{q+\alpha-1} b_n(s) x_k^s \\ &\quad - \frac{(-1)^{n_k+1}}{2} x_k^{q+\alpha-1} \int_{-1}^1 \int_{-1}^1 f^{(0, q+\alpha)}(x, y) e^{-i\pi(nx+n_k y)} dx dy, \end{aligned} \tag{35}$$

$$k = 1, \dots, q,$$

where $x_k = \frac{1}{i\pi n_k}$.

We calculate the inverse of matrix $\{x_k^s\}$ explicitly. Let $P_j(x)$ be the polynomial of degree $q-1$ defined by

$$P_j(x) := \prod_{\substack{n=1 \\ n \neq j}}^q \frac{x - x_n}{x_j - x_n} = \sum_{k=1}^q m_{jk} x^{k-1}, \quad j = 1, \dots, q,$$

where by m_{jk} we denote the coefficients of the polynomial $P_j(x)$. From the equations

$$P_j(x_i) = \sum_{k=1}^q m_{jk} x_i^{k-1} = \delta_{ij}, \quad i, j = 1, \dots, q,$$

(where δ_{ij} is Kronecker's symbol) we see that the transpose of (m_{jk}) is the inverse of the Vandermonde matrix (x_i^{k-1}) . Taking into account that (see [7])

$$m_{jk} = -\frac{1}{x_k^{j+1} \prod_{\substack{n=1 \\ n \neq k}}^q (x_k - x_n)} \sum_{\ell=0}^j \gamma_\ell x_k^\ell, \quad k = 1, \dots, q; \quad j = 0, \dots, q-1 \tag{36}$$

Equation (35) can be written as

$$\begin{aligned} b_n(j) - \tilde{b}_n(j) &= b_n(q) \sum_{\ell=0}^j \gamma_\ell \omega_{\ell-j+q-1} + \sum_{s=q+1}^{q+\alpha-1} b_n(s) \sum_{\ell=0}^j \gamma_\ell \omega_{\ell-j+s-1} \\ &+ \sum_{\ell=0}^j \gamma_\ell \sum_{k=1}^q \frac{x_k^{\ell-j+q+\alpha-2}}{2 \prod_{\substack{r=1 \\ r \neq k}}^q (x_k - x_r)} \int_{-1}^1 \int_{-1}^1 f^{(0,q+\alpha)}(x,y) e^{-i\pi(nx+n_k y)} dx dy. \end{aligned} \tag{37}$$

According to claims (a) and (b) of Lemma 5 we have

$$b_n(q) \sum_{\ell=0}^j \gamma_\ell \omega_{q-j+\ell-1} = b_n(q) \gamma_j, \quad j = 0, \dots, q-1.$$

In view of claim (c) of Lemma 5 and Lemma 6 the second term in the right hand side of (37) is $O(N^{-q+j-1})$ as $N \rightarrow \infty$. For the third term note that

$$\left| \frac{1}{\prod_{\substack{s=1 \\ s \neq k}}^q (x_k - x_s)} \right| \leq \frac{\pi^{q-1} N^{2q-2}}{\prod_{\substack{s=1 \\ s \neq k}}^q |n_k - n_s|} \leq \text{const} \frac{N^{2q-2}}{N^{q-\alpha}} = O(N^{q+\alpha-2}), \quad N \rightarrow \infty$$

as $|n_k - n_s| \geq 1$ whenever k and s differ and $|n_k - n_s| \geq CN$ whenever c_s differs from c_k , which happens at least for $q - \alpha$ indices s . Also, from Lemma 6, we have $\gamma_\ell x_k^\ell = O(N^{-q})$. Therefore, the third term is $\delta_n o(N^{-q+j})$ as $N \rightarrow \infty$. Collecting all of the above estimates we obtain from (37)

$$\tilde{b}_n(j; N) = b_n(j) - b_n(q) \gamma_j + \delta_n o(N^{-q+j}), \quad j = 0, \dots, q-1.$$

Now (33) follows from the Viet formula.

Estimate (32) can be proved similarly.

Then, in view of (26) and (29) we obtain for $m, n = n_1, n_2, \dots, n_q$

$$\begin{aligned} \sum_{k,s=0}^{q-1} \frac{\tilde{c}(k,s) - c(k,s)}{(i\pi n)^k (i\pi m)^s} &= \sum_{k=0}^{q-1} \sum_{s=q}^{q+\alpha-1} \frac{c(k,s)}{(i\pi n)^k (i\pi m)^s} + \sum_{k=q}^{q+\alpha-1} \sum_{s=0}^{q+\alpha-1} \frac{c(k,s)}{(i\pi n)^k (i\pi m)^s} \\ &+ \frac{2(-1)^{m+1}}{(i\pi m)^{q+\alpha-1}} \sum_{k=0}^{q+\alpha-1} \frac{a_m^{(q+\alpha)}(k)}{(i\pi n)^k} + \frac{2(-1)^{n+1}}{(i\pi n)^{q+\alpha-1}} \sum_{s=0}^{q-1} \frac{b_n^{(q+\alpha)}(s)}{(i\pi m)^s} \\ &+ \frac{(-1)^{n+m}}{(i\pi n)^{q+\alpha-1} (i\pi m)^{q+\alpha-1}} \int_{-1}^1 \int_{-1}^1 f^{(q+\alpha, q+\alpha)}(x, y) e^{-i\pi(nx+my)} dx dy. \end{aligned} \quad (38)$$

The remaining can be carried out as above.

Tables 3 and 4 show numerical values of the error $|\tilde{c}(k, s; N) - c(k, s)|$ for $q = 3$ and $N = 8$, $N = 16$, respectively. These values are more sensitive to round-off errors as for they calculation we are inverting the same matrix twice (see (29)). Table 5 shows numerical values of the exact jumps $c(k, s)$ for understanding the relative values while looking at Tables 3 and 4.

Table 3 Absolute values of the error $\tilde{c}(k, s; N) - c(k, s)$ for $q = 3$ and $N = 8$.

q	$s = 0$	$s = 1$	$s = 2$
$k = 0$	0.00041	0.013	0.26
$k = 1$	0.011	0.28	6.65
$k = 2$	0.0058	2.06	3.66

Table 4 Absolute values of the error $\tilde{c}(k, s; N) - c(k, s)$ for $q = 3$ and $N = 16$

q	$s = 0$	$s = 1$	$s = 2$
$k = 0$	0.000074	0.0037	0.19
$k = 1$	0.0011	0.069	2.88
$k = 2$	0.00071	0.86	1.78

Table 5 Numerical values of $c(k, s)$.

q	$s = 0$	$s = 1$	$s = 2$
$k = 0$	0	1.7	0
$k = 1$	0	4.4	0
$k = 2$	0	-15.4	0

Figures 7-10 explore pointwise accuracy of the Fourier coefficients approximation. Figures 7 and 8 present numerical values of the absolute error $|\tilde{a}_m(j; N) - a_m(j)|$ ($j = 0, 1, 2$ from left to right) for $q = 3$ and $N = 8$, $N = 16$, respectively. Theorem 4.1 states that the rate of convergence of $|\tilde{a}_m(j; N) - a_m(j)|$ as $N \rightarrow \infty$ is $O(1/N^3)$, $O(1/N^2)$, and $O(1/N)$ for $j = 0, 1, 2$, respectively. Comparison of Figures 7 and 8 confirms the theoretical estimate. We have increase in accuracy

almost 10, 5, and 3 times (theoretical estimate gives 8, 4, and 2 times) when changing N from $N = 8$ to $N = 16$.

Similar estimates we have in Figures 9 and 10 for the absolute error $|\tilde{b}_n(j;N) - b_n(j)|$.

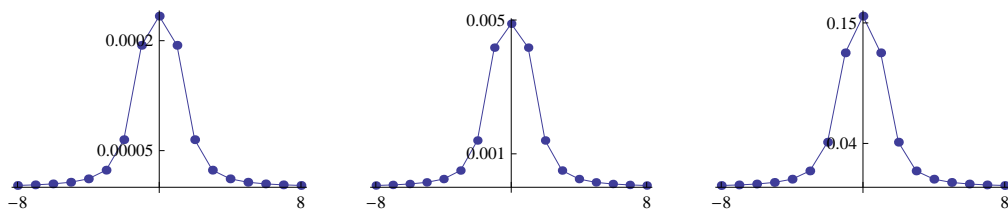


Figure 7. Absolute errors $|\tilde{a}_m(j;N) - a_m(j)|$, $|m| \leq N$ for $N = 8$, $q = 3$ and $j = 0, 1, 2$ (from left to right).

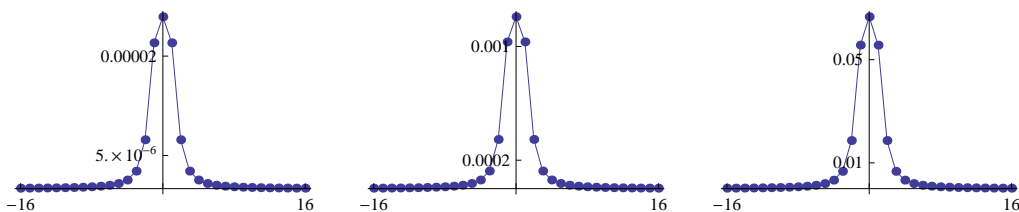


Figure 8. Absolute errors $|\tilde{a}_m(j;N) - a_m(j)|$, $|m| \leq N$ for $N = 16$, $q = 3$ and $j = 0, 1, 2$ (from left to right)

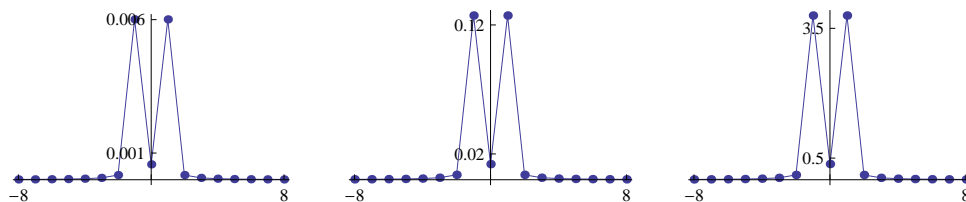


Figure 9. Absolute errors $|\tilde{b}_n(j;N) - b_n(j)|$, $|n| \leq N$ for $N = 8$, $q = 3$ and $j = 0, 1, 2$ (from left to right)

The next theorem immediately follows from the previous one.

Theorem 4.2. *Let the conditions of Theorem 4.1 be valid. Then the following is true*

$$\lim_{N \rightarrow \infty} N^{q-j} \left(\sum_{n=-N}^N |\tilde{b}_n(j) - b_n(j)|^2 \right)^{1/2} = \frac{|\chi_j|}{\sqrt{2\pi}^{q-j}} \|b(q;x)\|, \quad j = 0, \dots, q-1,$$

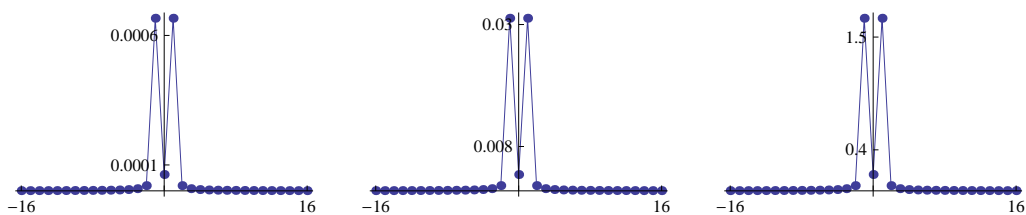


Figure 10. Absolute errors $|\tilde{b}_n(j;N) - b_n(j)|$, $|n| \leq N$ for $N = 16$, $q = 3$ and $j = 0, 1, 2$ (from left to right)

$$\lim_{N \rightarrow \infty} N^{q-j} \left(\sum_{m=-N}^N |\tilde{a}_m(j) - a_m(j)|^2 \right)^{1/2} = \frac{|\chi_j|}{\sqrt{2\pi^{q-j}}} \|a(q; y)\|, \quad j = 0, \dots, q-1.$$

Numerical values of $\frac{|\chi_k|}{\sqrt{2\pi^{q-k}}} \|a(q; y)\|$ and $\frac{|\chi_k|}{\sqrt{2\pi^{q-k}}} \|b(q; x)\|$ are presented in Tables 6 and 7, respectively. By $r_{N,j}(a)$ and $r_{N,j}(b)$ denote the actual errors

$$r_{N,j}(a) := N^{q-j} \left(\sum_{m=-N}^N |\tilde{a}_m(j) - a_m(j)|^2 \right)^{1/2}, \quad r_{N,j}(b) := N^{q-j} \left(\sum_{n=-N}^N |\tilde{b}_n(j) - b_n(j)|^2 \right)^{1/2}.$$

Table 8 shows the values of $r_{N,j}(a)$ and $r_{N,j}(b)$ for different values of N . We see that these values are close to theoretical estimates even for moderate numbers of N .

Table 6 Numerical values of $\frac{|\chi_k|}{\sqrt{2\pi^{q-k}}} \|a(q; y)\|$ for $k = 0, \dots, q-1$.

	$k = 0$	$k = 1$	$k = 2$	$k = 3$
$q = 1$	0.1753	—	—	—
$q = 2$	0.1955	0	—	—
$q = 3$	0.1599	0.5023	1.5779	—
$q = 4$	0.1783	0	3.5186	0

Table 7 Numerical values of $\frac{|\chi_k|}{\sqrt{2\pi^{q-k}}} \|b(q; x)\|$ for $k = 0, \dots, q-1$

	$k = 0$	$k = 1$	$k = 2$	$k = 3$
$q = 1$	1.8067	—	—	—
$q = 2$	0	0	—	—
$q = 3$	3.5708	11.2179	35.2421	—
$q = 4$	0	0	0	0

Table 8 Values of $r_{N,k}(a)$ and $r_{N,k}(b)$ for $q = 3$.

		$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$	$N = 256$
$k = 0$	$r_{N,k}(a)$	0.2682	0.1912	0.1720	0.1654	0.1625	0.1612	0.1605
	$r_{N,k}(b)$	5.8947	4.3638	3.8710	3.700	3.6310	3.6000	3.5850
$k = 1$	$r_{N,k}(a)$	0.5322	0.5095	0.5040	0.5027	0.5024	0.5023	0.5023
	$r_{N,k}(b)$	12.3945	11.5548	11.3033	11.2393	11.2233	11.2192	11.2182
$k = 2$	$r_{N,k}(a)$	2.8428	1.9350	1.7123	1.6357	1.6046	1.5907	1.5842
	$r_{N,k}(b)$	58.1779	43.0687	38.2050	36.5224	35.8362	35.5281	35.3824

Having approximate Fourier coefficients $\tilde{a}_m(k;N)$ and $\tilde{b}_n(s;N)$ the functions $a(k;y)$ and $b(s;x)$, respectively, can be recovered by univariate Eckhoff approximation

$$\tilde{a}(k;y;N) := \sum_{m=-N}^N \left(\tilde{a}_m(k) - \sum_{s=0}^{q-1} \tilde{c}(k,s)B_m(s) \right) e^{i\pi m y} + \sum_{s=0}^{q-1} \tilde{c}(k,s)B(s;y), \quad k = 0, \dots, q-1, \quad (39)$$

and

$$\tilde{b}(s;x;N) := \sum_{n=-N}^N \left(\tilde{b}_n(s) - \sum_{k=0}^{q-1} \tilde{c}(k,s)B_n(k) \right) e^{i\pi n x} + \sum_{k=0}^{q-1} \tilde{c}(k,s)B(k;x), \quad s = 0, \dots, q-1. \quad (40)$$

In the next theorem we explore the accuracy of the approximations $a(k;y)$ and $b(s;x)$ by $\tilde{a}(k;y;N)$ and $\tilde{b}(s;x;N)$, respectively.

Theorem 4.3. *Let indices $n_s = n_s(N)$ be chosen such that*

$$\lim_{N \rightarrow \infty} \frac{n_s}{N} = d_s \neq 0, \quad s = 1, \dots, q.$$

If $f^{(k,s)} \in C(D)$, $k, s = 0, \dots, 2q-1$ such that $f^{(2q-1, 2q-1)} \in AC(D)$ then the estimates hold:

$$\lim_{N \rightarrow \infty} N^{q-k} \|\tilde{a}(k;y;N) - a(k;y)\| = \frac{|\chi_k|}{\pi^{q-k}} \|a(q;y)\|, \quad k = 0, \dots, q-1,$$

and

$$\lim_{N \rightarrow \infty} N^{q-s} \|\tilde{b}(s;x;N) - b(s;x)\| = \frac{|\chi_s|}{\pi^{q-s}} \|b(q;x)\|, \quad s = 0, \dots, q-1.$$

Proof. In view of (39) we write

$$\begin{aligned} \|\tilde{a}(k;y;N) - a(k;y)\|^2 &= 2 \sum_{m=-N}^N |\tilde{a}_m(k) - a_m(k)|^2 \\ &\quad + 2 \sum_{|m|>N} \left| a_m(k) - \sum_{s=0}^{q-1} \tilde{c}(k,s)B_m(s) \right|^2. \end{aligned} \quad (41)$$

The first term we estimate based on Theorem 4.2. For the second term we write

$$a_m(k) = \sum_{s=0}^{q-1} c(k,s)B_m(s) + c(k,q)B_m(q) + o(m^{-q-1}), \quad |m| > N, \quad N \rightarrow \infty.$$

Hence

$$a_m(k) - \sum_{s=0}^{q-1} \tilde{c}(k,s)B_m(s) = \sum_{s=0}^{q-1} (c(k,s) - \tilde{c}(k,s))B_m(s) + c(k,q)B_m(q) + o(m^{-q-1}).$$

Now Equation (4.11) implies

$$a_m(k) - \sum_{s=0}^{q-1} \tilde{c}(k,s)B_m(s) = \frac{1}{N^{q-k}} O\left(\frac{1}{m}\right), \quad |m| > N, \quad N \rightarrow \infty.$$

Therefore the second term in the right hand side of (41) is $o(N^{-2q+2k})$ as $N \rightarrow \infty$. This concludes the proof of the first estimate. The second can be proved similarly.

Figures 11-14 explore the pointwise accuracy of the jumps approximation for $q = 3$ and $N = 8, 16$. Jumps with greater values of k are recovered with less accuracy as they correspond to higher order of derivatives. It is interesting to mention that the order of accuracy of approximations in Theorem 4.3 coincides with the accuracy of coefficients approximation in Theorems 4.1 and 4.2. This differs from the situation in univariate case when the Fourier coefficients are known exactly and the error of approximation depends on the accuracy of jumps approximation.

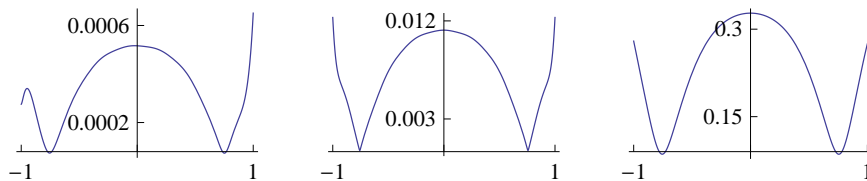


Figure 11. Absolute errors $|\tilde{a}(k; y; N) - a(k; y)|$ for $N = 8, q = 3$ and $k = 0, 1, 2$ (from left to right).

5 The Eckhoff Approximation in Multivariate Case

Replacing $a(k; y)$, $b(s; x)$, and $c(k, s)$ in Equation (15) by their approximated ones $\tilde{a}(k; y; N)$, $\tilde{b}(s; x; N)$, and $\tilde{c}(k, s)$, respectively, we get the following correction function

$$\tilde{G}(x, y; N) = \sum_{k=0}^{q-1} B(k; x) \tilde{a}(k; y; N) + \sum_{s=0}^{q-1} B(s; y) \tilde{b}(s; x; N) - \sum_{k, s=0}^{q-1} B(k; x) B(s; y) \tilde{c}(k, s). \quad (42)$$

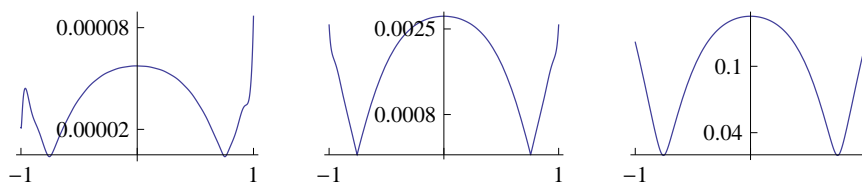


Figure 12. Absolute errors $|\tilde{a}(k; y; N) - a(k; y)|$ for $N = 16, q = 3$ and $k = 0, 1, 2$ (from left to right)

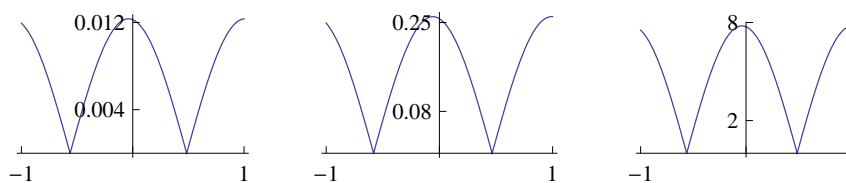


Figure 13. Absolute errors $|\tilde{b}(s; x; N) - b(s; x)|$ for $N = 8, q = 3$ and $k = 0, 1, 2$ (from left to right)

Therefore, the approximated function has the representation

$$f(x, y) = \tilde{F}(x, y; N) + \tilde{G}(x, y; N). \tag{43}$$

Approximation of \tilde{F} by the truncated Fourier series leads to the *Eckhoff approximation* in bivariate case

$$\tilde{S}_{N,q}(f; x, y) := \tilde{G}(x, y) + \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (f_{n,m} - \tilde{G}_{n,m}) e^{i\pi(nx+my)} \tag{44}$$

with the error

$$\tilde{R}_{N,q}(f; x, y) := f(x, y) - \tilde{S}_{N,q}(f; x, y). \tag{45}$$

From (42) the Fourier coefficients $\tilde{G}_{n,m}$ can be calculated explicitly

$$\tilde{G}_{n,m} = \sum_{k=0}^{q-1} B_n(k) \tilde{a}_m(k) + \sum_{s=0}^{q-1} B_m(s) \tilde{b}_n(s) - \sum_{k,s=0}^{q-1} B_n(k) B_m(s) \tilde{c}_{k,s}, \quad n, m \neq 0, \quad \tilde{G}_{0,0} = 0,$$

$$\tilde{G}_{0,m} = \sum_{s=0}^{q-1} B_m(s) \tilde{b}_0(s), \quad m \neq 0, \quad \tilde{G}_{n,0} = \sum_{k=0}^{q-1} B_n(k) \tilde{a}_0(k), \quad n \neq 0.$$

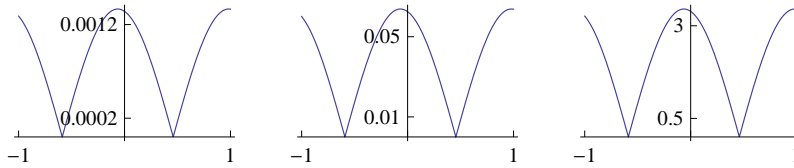


Figure 14. Absolute errors $|\tilde{b}(s;x;N) - b(s;x)|$ for $N = 16$, $q = 3$ and $k = 0, 1, 2$ (from left to right)

In the next theorem we investigate the accuracy of the Eckhoff approximation.

Theorem 5.1. *Let the indices $n_s = n_s(N)$ be chosen such that*

$$\lim_{N \rightarrow \infty} \frac{n_s}{N} = d_s \neq 0, \quad s = 1, \dots, q. \tag{46}$$

If $f^{(k,s)} \in C(D)$, $k, s = 0, \dots, 2q - 1$ such that $f^{(2q-1, 2q-1)} \in AC(D)$ then the estimate holds:

$$\lim_{N \rightarrow \infty} N^{q+\frac{1}{2}} \|\tilde{R}_{N,q}(f)\| = \tilde{D}_q(f), \tag{47}$$

where

$$\tilde{D}_q(f) := \frac{\sqrt{h_q(f)}}{\sqrt{2} \pi^{q+1}} \left(\int_{|x|>1} \left| \sum_{s=0}^q \frac{\chi_s}{x^{s+1}} \right|^2 dx \right)^{1/2}, \tag{48}$$

and $\chi_q = 1$, function $h_q(f)$ is defined in Theorem 3.5 and χ_s are defined in Theorem 2.3.

Proof. In view of (43), (44), and (45) we get

$$\begin{aligned} \|\tilde{R}_{N,q}(f)\|^2 &= \|\tilde{F}(x,y) - \sum_{n=-N}^N \sum_{m=-N}^N \tilde{F}_{n,m} e^{i\pi(nx+my)}\|^2 \\ &= 4 \sum_{|n|>N} |\tilde{F}_{n,0}|^2 + 4 \sum_{|m|>N} |\tilde{F}_{0,m}|^2 \\ &\quad + 4 \sum'_{n=-N} \sum_{|m|>N} |\tilde{F}_{n,m}|^2 + 4 \sum'_{m=-N} \sum_{|n|>N} |\tilde{F}_{n,m}|^2 + 4 \sum_{|n|>N} \sum_{|m|>N} |\tilde{F}_{n,m}|^2, \end{aligned} \tag{49}$$

where $\tilde{F}_{n,m} = f_{n,m} - \tilde{G}_{n,m}$.

We start with the first term in the right hand side of (49)

$$\begin{aligned} \tilde{F}_{n,0} &= f_{n,0} - \tilde{G}_{n,0} = \sum_{k=0}^{q-1} B_n(k)(a_0(k) - \tilde{a}_0(k)) + \frac{1}{4(i\pi n)^q} \int_{-1}^1 \int_{-1}^1 f^{(q+1,0)}(x,y) e^{-i\pi n x} dx dy \\ &= \sum_{k=0}^{q-1} B_n(k)(a_0(k) - \tilde{a}_0(k)) + a_0(q)B_n(q) + o(n^{-q-1}), \quad n \rightarrow \infty. \end{aligned}$$

Hence, in view of Equation (32), we have

$$\begin{aligned} \lim_{N \rightarrow \infty} N^{2q+1} \left(4 \sum_{|n|>N} |F_{n,0}|^2 \right) &= \frac{|a_0(q)|^2}{\pi^{2q+2}} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{|n|>N} \left| \sum_{k=0}^q \frac{\chi_k}{(n/N)^{k+1}} \right|^2 \\ &= \frac{|a_0(q)|^2}{\pi^{2q+2}} \int_{|x|>1} \left| \sum_{k=0}^q \frac{\chi_k}{x^{k+1}} \right|^2 dx. \end{aligned}$$

Similarly

$$\lim_{N \rightarrow \infty} N^{2q+1} \left(4 \sum_{|m|>N} |F_{0,m}|^2 \right) = \frac{|b_0(q)|^2}{\pi^{2q+2}} \int_{|x|>1} \left| \sum_{k=0}^q \frac{\chi_k}{x^{k+1}} \right|^2 dx.$$

For the third term in the right hand side of (49) we have $(n, m \neq 0)$

$$\begin{aligned} \tilde{F}_{n,m} = f_{n,m} - \tilde{G}_{n,m} &= \sum_{s=0}^{q-1} B_m(s) \left(b_n(s) - \tilde{b}_n(s) \right) \\ &\quad - \sum_{k=0}^{q-1} B_n(k) \left(\tilde{a}_m(k) - \sum_{s=0}^{q-1} B_m(s) \tilde{c}(k,s) \right) \\ &\quad + \frac{1}{4(i\pi m)^q} \int_{-1}^1 \int_{-1}^1 f^{(0,q)}(x,y) e^{-i\pi(nx+my)} dx dy. \end{aligned}$$

We need asymptotic behavior of $\tilde{a}_m(k) - \sum_{s=0}^{q-1} B_m(s) \tilde{c}(k,s)$ as $N \rightarrow \infty$ and $|m| > N$. In view of (24) and (27) we get

$$\begin{aligned} \sum_{k=0}^{q-1} \frac{\tilde{a}_m(k) - a_m(k)}{(i\pi n_\ell)^k} &= \sum_{k=q}^{2q-1} \frac{a_m(k)}{(i\pi n_\ell)^k} \\ &\quad + \frac{(-1)^{n_\ell+1}}{2(i\pi n_\ell)^{2q-1}} \int_{-1}^1 \int_{-1}^1 f^{(2q,0)} e^{-i\pi(n_\ell x + my)} dx dy, \quad \ell = 1, \dots, q. \end{aligned}$$

Recalling (36) we get

$$\begin{aligned} \tilde{a}_m(j) - a_m(j) &= - \sum_{\mu=0}^j \gamma_\mu \sum_{k=q}^{2q-1} \omega_{\mu+k-j-1} \sum_{s=0}^{2q-1} B_m(s) c(k,s) \\ &\quad - \sum_{\ell=1}^q \frac{(-1)^{n_\ell+1}}{(i\pi n_\ell)^{2q-1} \prod_{\substack{r=1 \\ r \neq \ell}}^q (x_\ell - x_r)} \sum_{\mu=0}^j \gamma_\mu x_\ell^{\mu-j-1} \sum_{s=0}^{2q-1} B_m(s) b_{n_\ell}^{(2q)}(s) \\ &\quad + o(m^{-2q}), \quad |m| > N, N \rightarrow \infty. \end{aligned}$$

Similarly from (38), with $\alpha = q$, we derive

$$\begin{aligned} \tilde{c}(j, h) - c(j, h) = & - \sum_{v=0}^h \gamma_v \sum_{s=q}^{2q-1} c(j, s) \omega_{v+s-h-1} - \sum_{\mu=0}^j \gamma_\mu \sum_{k=q}^{2q-1} c(k, h) \omega_{\mu+k-j-1} \\ & + \sum_{\mu=0}^j \gamma_\mu \sum_{v=0}^h \gamma_v \sum_{k,s=q}^{2q-1} c(k, s) \omega_{\mu+k-j-1} \omega_{v+s-h-1} \\ & - 2 \sum_{r=1}^q \frac{(-1)^{m_r+1}}{(i\pi m_r)^{2q-1} \prod_{\substack{\ell=1 \\ \ell \neq r}}^q (x_r - x_\ell)} \sum_{v=0}^h \gamma_v x_r^{v-h-1} a_{m_r}^{(2q)}(j) \\ & - 2 \sum_{\ell=1}^q \frac{(-1)^{n_\ell+1}}{(i\pi n_\ell)^{2q-1} \prod_{\substack{r=1 \\ r \neq \ell}}^q (x_\ell - x_r)} \sum_{\mu=0}^j \gamma_\mu x_\ell^{\mu-j-1} b_{n_\ell}^{(2q)}(h) \\ & + \frac{o(1)}{N^{2q-j-h}}, N \rightarrow \infty. \end{aligned}$$

These imply

$$\begin{aligned} \tilde{a}_m(j) - \sum_{s=0}^{q-1} B_m(s) \tilde{c}(j, s) = & a_m(j) - \sum_{s=0}^{q-1} B_m(s) c(j, s) \\ & + c(j, q) \sum_{s=0}^{q-1} B_m(s) \frac{\chi_s}{(i\pi N)^{q-s}} + O\left(\frac{1}{m}\right) o(N^{-q}), \quad |m| > N, N \rightarrow \infty. \end{aligned}$$

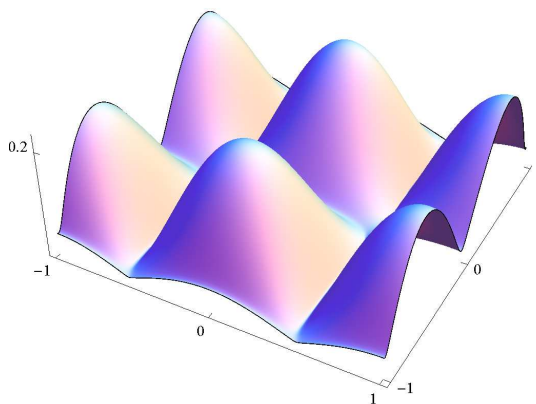


Figure 15. Absolute error $|G(x, y) - \tilde{G}(x, y; N)|$ for $q = 3$ and $N = 8$

Taking into account that

$$a_m(j) - \sum_{s=0}^{q-1} B_m(s) c(j, s) = c(j, q) B_m(q) + o(m^{-q-1}), \quad m \rightarrow \infty$$

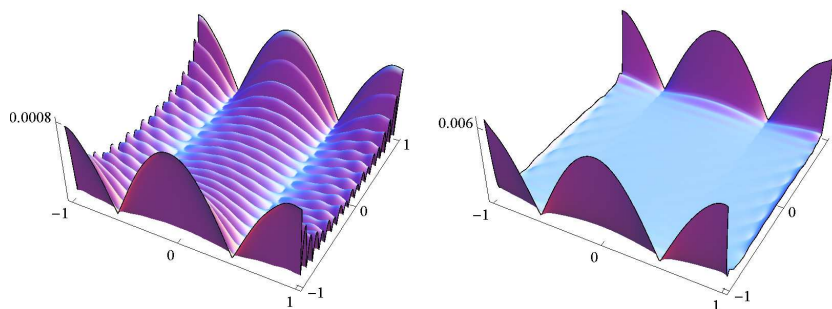


Figure 16. Absolute errors while approximating (16) by the KL-approximation (left) and the Eckhoff approximation (right) for $q = 3$ and $N = 8$

we get the estimate

$$\tilde{a}_m(j) - \sum_{s=0}^{q-1} B_m(s)\tilde{c}(j,s) = c(j,q) \sum_{s=0}^q B_m(s) \frac{\chi_s}{(i\pi N)^{q-s}} + O\left(\frac{1}{m}\right) o(N^{-q}), |m| > N, N \rightarrow \infty.$$

Collecting all these into (??) together with (37) we write

$$\begin{aligned} & \lim_{N \rightarrow \infty} N^{2q+1} \left(4 \sum_{n=-N}^{N'} \sum_{|m|>N} |\tilde{F}_{n,m}|^2 \right) \\ &= \frac{1}{\pi^{2q+2}} \sum_{n=-\infty}^{\infty} \left| b_n(q) - \sum_{k=0}^{q-1} B_n(k)c(k,q) \right|^2 \int_{|x|>1} \left| \sum_{s=0}^q \frac{\chi_s}{x^{s+1}} \right|^2 dx \\ &= \frac{1}{2\pi^{2q+2}} \int_{|x|>1} \left| \sum_{s=0}^q \frac{\chi_s}{x^{s+1}} \right|^2 dx \int_{-1}^1 \left| \int_{-1}^1 B(q-1;t)b^{(q)}(q;x-t)dt \right|^2 dx. \end{aligned}$$

Similarly

$$\begin{aligned} & \lim_{N \rightarrow \infty} N^{2q+1} \left(4 \sum_{m=-N}^{N'} \sum_{|n|>N} |\tilde{F}_{n,m}|^2 \right) \\ &= \frac{1}{2\pi^{2q+2}} \int_{|x|>1} \left| \sum_{s=0}^q \frac{\chi_s}{x^{s+1}} \right|^2 dx \int_{-1}^1 \left| \int_{-1}^1 B(q-1;t)a^{(q)}(q;x-t)dt \right|^2 dx. \end{aligned}$$

Finally, taking into account that the last term in the right hand side of (49) is $o(N^{-2q-1})$ as $N \rightarrow \infty$ we get the required.

Numerical values of $\tilde{D}_q(f)$ while approximating (16) are calculated in Table 9 and the values of $\tilde{D}_{q,N}(f)$

$$\tilde{D}_{q,N}(f) := N^{q+\frac{1}{2}} \|\tilde{R}_{N,q}(f)\|$$

for different values of N are presented in Table 10. Comparison shows how close are the actual constants to their theoretical counterparts.

Table 9 Numerical values of $\tilde{D}_q(f)$.

q	1	2	3	4
$\tilde{D}_q(f)$	0.9808	0.0947	1.1304	0.1859

Table 10 Numerical values of $\tilde{D}_{q,N}(f)$.

	$q = 1$	$q = 2$	$q = 3$	$q = 4$
$N = 4$	0.9354	0.1438	2.0038	0.6711
$N = 8$	0.9208	0.0872	1.5084	0.2111
$N = 16$	0.9269	0.0704	1.3505	0.0980

Figure 15 shows the graph of the absolute error $|G(x,y) - \tilde{G}(x,y;N)|$ for $q = 3$ and $N = 8$. Approximation of $G(x,y)$ by $\tilde{G}(x,y;N)$ in the Eckhoff approximation leads to decrease in accuracy that is shown in Figure 16 (right) compared with the left one where the exact G is applied (KL-approximation).

Figure 17 presents more graphs for the Eckhoff approximation.

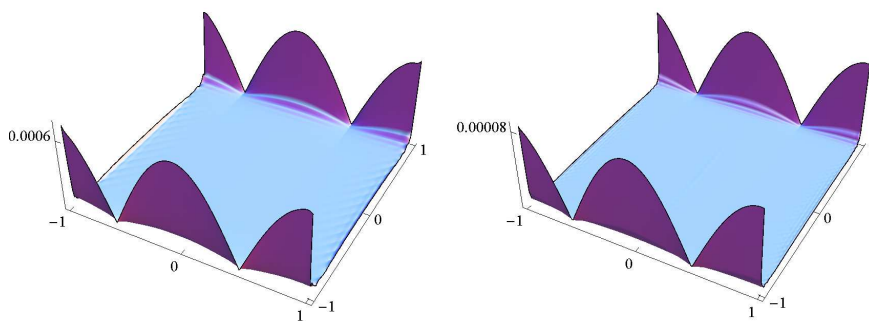


Figure 17. Absolute errors while approximating (16) by the Eckhoff approximation for $N = 16$ (left) and $N = 32$ (right) when $q = 3$.

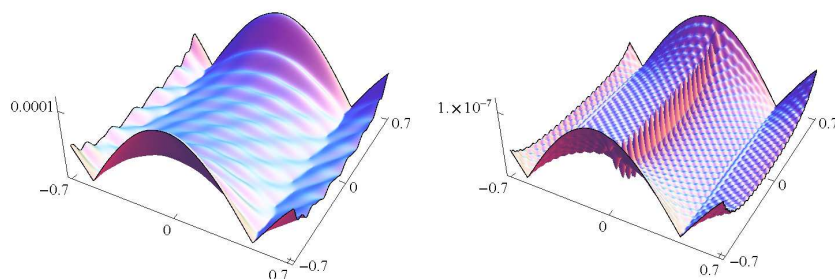


Figure 18. Absolute errors away from the singularities while approximating (16) by the Eckhoff approximation for $N = 8$ (left) and $N = 32$ (right) when $q = 3$

Figure 18 presents the absolute errors away from the singularities. Comparison of Figures 18 and 6 shows that in comparison with the KL-approximation, where the exact values of the jumps are used, in the Eckhoff approximation we have an improvement in convergence. This convergence acceleration phenomenon (away from the singularities), which is quite contrary to the slow convergence that might be expected due to approximate calculation of the jumps, we have called (see [36]) the autocorrection phenomenon of the Eckhoff method. Theoretical background of this phenomenon for multivariate functions will be carried out elsewhere.

Acknowledgement. This work was supported in part by the ANSEF grant PS 1867.

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