A CLASS OF HARMONIC STARLIKE FUNCTIONS
WITH RESPECT TO SYMMETRIC POINTS
ASSOCIATED WITH WRIGHT GENERALIZED
HYPERGEOMETRIC FUNCTION

M. K. Aouf  R. M. El-Ashwah  A. Shamandy  and  S. M. El-Deeb

(Mansoura University, Egypt)

Received Apr. 27, 2012

Abstract. Making use of Wright operator we introduce a new class of complex-valued harmonic functions with respect to symmetric points which are orientation preserving, univalent and starlike. We obtain coefficient conditions, extreme points, distortion bounds, and convex combination.

Key words: harmonic, univalent, Wright operator, symmetric point

AMS (2010) subject classification: 30C45

1 Introduction

Denote by \( \mathcal{H} \) the family of functions

\[
f = h + \overline{g},
\]

which are analytic univalent and sense-preserving in the unit disc \( U = \{ z : |z| < 1 \} \). So that \( f \) is normalized by \( f(0) = f_{\overline{z}}(0) - 1 = 0 \). Thus, for \( f = h + \overline{g} \in \mathcal{H} \), we may express the analytic functions \( h \) and \( g \) in the forms

\[
h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1 .
\]
where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $\mathcal{H}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{H}$ (see [4]). Hence

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \quad (1.3)$$

We denote $\mathcal{H}$ the subclass of $\mathcal{H}$ consists of harmonic functions $f = h + \psi$ of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \quad (1.4)$$

Let the Hadamard product (or convolution) of two power series $\Phi(z) = z + \sum_{k=2}^{\infty} \phi_k z^k$ and $\Psi(z) = z + \sum_{k=2}^{\infty} \psi_k z^k$ be defined by

$$(\Phi \ast \Psi)(z) = z + \sum_{k=2}^{\infty} \phi_k \psi_k z^k = (\Psi \ast \Phi)(z).$$

Let $\alpha_1, A_1, \ldots, \alpha_q, A_q$ and $\beta_1, B_1, \ldots, \beta_s, B_s \ (q, s \in \mathbb{N})$ be positive and real parameters such that

$$1 + \sum_{j=1}^{s} B_j - \sum_{j=1}^{q} A_j \geq 0.$$ 

The Wright generalized hypergeometric function$^{[19]}$ (see also $^{[12]}$

$$q \Psi_s [(\alpha_1, A_1), \ldots, (\alpha_q, A_q); (\beta_1, B_1), \ldots, (\beta_s, B_s); z] = q \Psi_s [(\alpha_i, A_i)_q; (\beta_i, B_i)_s; z]$$

is defined by

$$q \Psi_s [(\alpha_i, A_i)_q; (\beta_i, B_i)_s; z] = \sum_{n=0}^{\infty} \prod_{j=1}^{q} \frac{\Gamma(\alpha_i + nA_j)}{n!} \frac{z^n}{\prod_{j=1}^{s} \Gamma(\beta_i + nB_j)^{n_j}} , \quad z \in U.$$ 

If $A_i = 1 \ (i = 1, \ldots, q)$ and $B_i = 1 \ (i = 1, \ldots, s)$, we have the relationship:

$$\Omega_q \psi_s [(\alpha_i, 1)_q; (\beta_i, 1)_s; z] = q F_s (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),$$

where $q F_s (\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ is the generalized hypergeometric function (see for details $^{[6]}$, $^{[7]}$, $^{[8]}$, $^{[9]}$, $^{[13]}$) and

$$\Omega = \prod_{i=1}^{q} \frac{\Gamma(\beta_i)}{\prod_{i=1}^{q} \Gamma(\alpha_i)}.$$  

(1.5)
The Wright generalized hypergeometric functions were invoked in the geometric function theory (see [5], [6], [15], [16] and [17]).


First we define a function $\Phi_s \left[ (\alpha_i, A_i) ; (\beta_i, B_i)_s ; z \right]$ by

$$
\Phi_s \left[ (\alpha_i, A_i) ; (\beta_i, B_i)_s ; z \right] = \Omega z \quad \Psi_s \left[ (\alpha_i, A_i) ; (\beta_i, B_i)_s ; z \right]
$$

and consider the following linear operator

$$
\theta_{q,s} \left[ (\alpha_i, A_i) ; (\beta_i, B_i)_s \right] : S_H \to S_H,
$$

defined by the convolution

$$
\theta_{q,s} \left[ (\alpha_i, A_i) ; (\beta_i, B_i)_s \right] f(z) = q \Phi_s \left[ (\alpha_i, A_i) ; (\beta_i, B_i)_s ; z \right] * f(z).
$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$
\theta_{q,s} \left[ (\alpha_i, A_i) ; (\beta_i, B_i)_s \right] f(z) = z + \sum_{k=2}^{\infty} \Omega \sigma_k (\alpha_1) a_k z^k, \quad (1.6)
$$

where $\Omega$ is given by (1.5) and $\sigma_k (\alpha_1)$ is defined by

$$
\sigma_k (\alpha_1) = \frac{\Gamma (\alpha_1 + A_1 (k - 1)) \cdots \Gamma (\alpha_q + A_q (k - 1))}{\Gamma (\beta_1 + B_1 (k - 1)) \cdots \Gamma (\beta_s + B_s (k - 1)) (k - 1)!}. \quad (1.7)
$$

If, for convenience, we write

$$
\theta_{q,s} \left[ \alpha_1, A_1, B_1 \right] f(z) = \theta_{q,s} \left[ (\alpha_1, A_1), \ldots, (\alpha_q, A_q) ; (\beta_1, B_1), \ldots, (\beta_s, B_s) \right] f(z),
$$

then one can easily verify from the definition (1.6) that

$$
zA_1 \left( \theta_{q,s} \left[ \alpha_1, A_1, B_1 \right] f(z) \right) = \alpha_1 \theta_{q,s} \left[ \alpha_1 + 1, A_1, B_1 \right] f(z) - (\alpha_1 - A_1) \theta_{q,s} \left[ \alpha_1, A_1, B_1 \right] f(z). \quad (1.8)
$$

We note that for $A_i = 1 \ (i = 1, 2, \ldots, q)$ and $B_i = 1 \ (i = 1, 2, \ldots, s)$, we obtain $\theta_{q,s} \left[ \alpha_1, 1 \right] f(z) = H_{q,s} [\alpha_1] f(z)$, which was introduced and studied by Dzio̧k and Srivastava [7].
Applying the Wright operator to the harmonic functions \( f = h + \overline{g} \) given by (1.1) we get
\[
\theta_{q,s}[\alpha_1,A_1,B_1]f(z) = \theta_{q,s}[\alpha_1,A_1,B_1]h(z) + \theta_{q,s}[\alpha_1,A_1,B_1]g(z).
\] (1.9)

Motivated by Jahangiri et al.\(^{[10,11]}\) and Ahuja and Jahangiri \(^{[1]}\), we define a new subclass \( HS_\gamma^*([\alpha_1,A_1,B_1],\gamma) \) of \( \mathcal{H} \) that are starlike with respect to symmetric points.

**Definition 1.** For \( 0 \leq \gamma < 1 \) and \( z = re^{i\theta} \in U \), we let \( \mathcal{H} S_\gamma^*([\alpha_1,A_1,B_1],\gamma) \) a subclass of \( \mathcal{H} \) of the form \( f = h + \overline{g} \) be given by (1.3) and satisfying the analytic criteria
\[
\Re \left\{ \frac{2z}{z} \left[ \theta_{q,s}[\alpha_1,A_1,B_1]f'(-z) - \theta_{q,s}[\alpha_1,A_1,B_1]f(z) \right] \right\} > \gamma,
\] (1.10)
where \( \theta_{q,s}[\alpha_1,A_1,B_1]f(z) \) is defined by (1.9) and \( z' = \frac{\partial}{\partial \theta}(z = re^{i\theta}) \).

We also let \( \mathcal{H} S_{\gamma}([\alpha_1,A_1,B_1],\gamma) = \mathcal{H} S_{\gamma}([\alpha_1,A_1,B_1],\gamma) \cap \mathcal{H} \).

The family \( \mathcal{H} S_{\gamma}([\alpha_1,A_1,B_1],\gamma) \) is of special interest because for suitable choices of \( q, s, [A_1], [B_1] \) and \([\alpha_1] \), we note that

(i) If \( A_1 = 1 (i = 1,\ldots,q) \) and \( B_j = 1 (j = 1,\ldots,s) \), we have \( \mathcal{H} S_{\gamma}([\alpha_1,1,1],\gamma) = \mathcal{H} S_{\gamma}([\alpha_1],\gamma) \), which was studied by Murugusundaramoorthy et al.\(^{[14]}\);

(ii) If \( f(-z) = -f(z), A_i = 1 (i = 1,\ldots,q) \) and \( B_j = 1 (j = 1,\ldots,s) \), we have \( \mathcal{H} S_{\gamma}([\alpha_1,1,1],\gamma) = S_{\mathcal{H}^*}^p(\alpha_1,\gamma) \), which was studied by Al-Kharsani and AL-Khal\(^{[2]}\).

**Remark 1.** If the co-analytic part of \( f = h + \overline{g} \) is zero, \( \alpha_i = A_i = 1 (i = 1,\ldots,q) \) and \( \beta_j = B_j = 1 (j = 1,\ldots,s) \) then \( \mathcal{H} S_{\gamma}(([1,1],\gamma)) \) turns out to be the class \( S_{\mathcal{H}^*}^p(\gamma) \) of starlike functions with respect to symmetric points which was introduced by Sakaguchi\(^{[18]}\).

In this paper, we have obtained the coefficient conditions for the classes \( \mathcal{H} S_{\gamma}([\alpha_1,A_1,B_1],\gamma) \) and \( \mathcal{H} S_{\gamma}([\alpha_1,A_1,B_1],\gamma) \). Further a representation theorem, inclusion properties and distortion bounds for the class \( \mathcal{H} S_{\gamma}^*([\alpha_1,A_1,B_1],\gamma) \) are also established.

## 2 Coefficient Characterization

Unless otherwise mentioned, we assume throughout this paper that \( q,s \in \mathbb{N}, a_1 = 1, \)
\( \alpha_1,A_1,\ldots,\alpha_q,A_q, \beta_1,B_1,\ldots,\beta_s,B_s \in \mathbb{R}^+ \) and \( 0 \leq \gamma < 1 \). We begin with a sufficient condition for functions in \( \mathcal{H} S_{\gamma}([\alpha_1,A_1,B_1],\gamma) \).

**Theorem 1.** Let \( f = h + \overline{g} \) be given by (1.3). Furthermore, let
\[
\sum_{k=2}^{\infty} \left[ 2k - \gamma \left( 1 - (-1)^k \right) \right] \frac{\Omega \sigma_k(\alpha_1)}{2(1-\gamma)} |a_k| + \sum_{k=1}^{\infty} \left[ 2k + \gamma \left( 1 - (-1)^k \right) \right] \frac{\Omega \sigma_k(\alpha_1)}{2(1-\gamma)} |b_k| \leq 1,
\] (2.1)
where $\Omega$ and $\sigma_k(\alpha_1)$ are defined by (1.5) and (1.7). Then $f$ is sense-preserving, harmonic univalent in $U$ and $f \in \mathcal{H}(S_+ \cup \{\alpha_1, A_1, B_1\}, \gamma)$.

Proof. According the condition (1.10), we only need to show that if (2.1) holds, then

$$
\text{Re} \left\{ \frac{2z \left( \theta_{q,s}[\alpha_1, A_1, B_1] f(z) \right)'}{z' \left( \theta_{q,s}[\alpha_1, A_1, B_1] f(z) - \theta_{q,s}[\alpha_1, A_1, B_1] f(-z) \right)} \right\} = \frac{\text{Re} A(z)}{B(z)} > \gamma,
$$

where

$$
A(z) = 2z \left( \theta_{q,s}[\alpha_1, A_1, B_1] f(z) \right)' = 2z' \left[ z + \sum_{k=2}^{\infty} k \Omega \sigma_k(\alpha_1) a_k z^k - \sum_{k=1}^{\infty} k \Omega \sigma_k(\alpha_1) \overline{b_k z^k} \right]
$$

and

$$
B(z) = z' \left[ \theta_{q,s}[\alpha_1, A_1, B_1] f(z) - \theta_{q,s}[\alpha_1, A_1, B_1] f(-z) \right] = z' \left[ 2z + \sum_{k=2}^{\infty} \left[ 1 - (-1)^k \right] \Omega \sigma_k(\alpha_1) a_k z^k + \sum_{k=1}^{\infty} \left[ 1 - (-1)^k \right] \Omega \sigma_k(\alpha_1) \overline{b_k z^k} \right].
$$

Using the fact that $\text{Re} \{w(z)\} > \gamma$ if and only if $|1 - \gamma + w| > |1 + \gamma - w|$, it suffices to show that

$$
|A(z) + (1 - \gamma)B(z)| - |A(z) - (1 + \gamma)B(z)| > 0.
$$

(2.2)

Substituting for $A(z)$ and $B(z)$ in (2.2) and by using (2.1), we obtain

$$
\left| 2(2 - \gamma)z + \sum_{k=2}^{\infty} \left[ 2k + (1 - \gamma)(1 - (-1)^k) \right] \Omega \sigma_k(\alpha_1) a_k z^k \right.
$$

$$
- \sum_{k=1}^{\infty} \left[ 2k - (1 - \gamma)(1 - (-1)^k) \right] \Omega \sigma_k(\alpha_1) \overline{b_k z^k}
$$

$$
- \left[ 2k - (1 + \gamma)(1 - (-1)^k) \right] \Omega \sigma_k(\alpha_1) a_k z^k
$$

$$
- \sum_{k=1}^{\infty} \left[ 2k + (1 + \gamma)(1 - (-1)^k) \right] \Omega \sigma_k(\alpha_1) \overline{b_k z^k}
$$

$$
\geq 4(1 - \gamma)|z| - 2 \sum_{k=2}^{\infty} \left[ 2k - \gamma(1 - (-1)^k) \right] \Omega \sigma_k(\alpha_1) |a_k| |z|^k
$$

$$
- 2 \sum_{k=1}^{\infty} \left[ 2k + \gamma(1 - (-1)^k) \right] \Omega \sigma_k(\alpha_1) |b_k| |z|^k
$$

$$
= 4(1 - \gamma)|z| \left[ 1 - \sum_{k=2}^{\infty} \frac{2k - \gamma(1 - (-1)^k)}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1) |a_k| |z|^{k-1} \right]
$$
The harmonic univalent functions

\[
f(z) = z + \sum_{k=2}^{\infty} \frac{2(1 - \gamma)}{2(1 - \gamma)} X_k z^k + \sum_{k=1}^{\infty} \frac{2(1 - \gamma)}{2(1 - \gamma)} Y_k z^k,
\]

where \( \sum_{k=2}^{\infty} |X_k| + \sum_{k=1}^{\infty} |Y_k| = 1 \), show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.3) are in \( \mathcal{H}S_\gamma([\alpha_1, A_1, B_1], \gamma) \) because

\[
\sum_{k=2}^{\infty} \frac{2k - \gamma(1 - (-1)^k)}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1) |a_k| + \sum_{k=1}^{\infty} \frac{2k + \gamma(1 - (-1)^k)}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1) |b_k| = \sum_{k=2}^{\infty} |X_k| + \sum_{k=1}^{\infty} |Y_k| = 1.
\]

This completes the proof of Theorem 1.

In the following theorem, it is shown that the condition (2.1) is also necessary for functions \( f(z) \) of the form (1.4).

**Theorem 2.** Let \( f = h + g \) be given by (1.4). Then \( f \in \mathcal{H}S_\gamma([\alpha_1, A_1, B_1], \gamma) \) if and only if

\[
\sum_{k=2}^{\infty} \frac{2k - \gamma(1 - (-1)^k)}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1) |a_k| + \sum_{k=1}^{\infty} \frac{2k + \gamma(1 - (-1)^k)}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1) |b_k| \leq 1, \tag{2.4}
\]

where \( \Omega \) and \( \sigma_k(\alpha_1) \) are defined by (1.5) and (1.7), respectively.

**Proof.** Since \( \mathcal{H}S_\gamma([\alpha_1, A_1, B_1], \gamma) \subset \mathcal{H}S_\gamma([\alpha_1, A_1, B_1], \gamma) \), we only need to prove the "only if" part of the theorem. To this end, for functions \( f(z) \) of the form (1.4), we notice that the condition

\[
\text{Re} \left\{ \frac{2z (\theta_{q,s}[\alpha_1,A_1,B_1] f(z))'}{z', \theta_{q,s}[\alpha_1,A_1,B_1] f(z) - \theta_{q,s}[\alpha_1,A_1,B_1] f(-z)} \right\} > \gamma
\]
is equivalent to
\[
\text{Re}\left\{ \frac{2(1-\gamma) - \sum_{k=2}^{\infty} \frac{2k-\gamma(1-(-1)^k)}{2k-\gamma(1-(-1)^k)} \Omega \sigma_k(\alpha) \left( a_k z^{k-1} + \frac{2k-\gamma(1-(-1)^k)}{2k-\gamma(1-(-1)^k)} \Omega \sigma_k(\alpha) \right) z^k}{2 - \sum_{k=2}^{\infty} (1-(-1)^k) \Omega \sigma_k(\alpha) \left( a_k z^{k-1} + \frac{2k-\gamma(1-(-1)^k)}{2k-\gamma(1-(-1)^k)} \Omega \sigma_k(\alpha) \right) z^k} \right\} > 0. \quad (2.5)
\]

The above required condition (2.5) must hold for all values of \( z \) in \( U \). Upon choosing the values of \( z \) on the positive real axis where \( 0 \leq z = r < 1 \), we must have
\[
\text{Re}\left\{ \frac{2(1-\gamma) - \sum_{k=2}^{\infty} \frac{2k-\gamma(1-(-1)^k)}{2k-\gamma(1-(-1)^k)} \Omega \sigma_k(\alpha) \left( a_k z^{k-1} + \frac{2k-\gamma(1-(-1)^k)}{2k-\gamma(1-(-1)^k)} \Omega \sigma_k(\alpha) \right) z^k}{2 - \sum_{k=2}^{\infty} (1-(-1)^k) \Omega \sigma_k(\alpha) \left( a_k z^{k-1} + \frac{2k-\gamma(1-(-1)^k)}{2k-\gamma(1-(-1)^k)} \Omega \sigma_k(\alpha) \right) z^k} \right\} > 0. \quad (2.6)
\]

If the condition (2.4) does not hold, then the numerator in (2.6) is negative for \( r \) sufficiently close to 1. Hence there exists \( z_0 = r_0 \) in \((0,1)\) for which the quotient in (2.6) is negative. This contradicts the required condition for \( f(z) \in \overline{SH_r}([\alpha_1, A_1, B_1], \gamma) \) and so the proof of Theorem 2 is completed.

### 3 Extreme Points and Distortion Theorem

Our next theorem is on the extreme points of convex hulls of \( \overline{SH_r}([\alpha_1, A_1, B_1], \gamma) \) denoted by \( clco \overline{SH_r}([\alpha_1, A_1, B_1], \gamma) \).

**Theorem 3.** A function \( f_k(z) \in \overline{SH_r}([\alpha_1, A_1, B_1], \gamma) \) if and only if \( f_k(z) \) can be expressed by the form
\[
f_k(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_k(z)), \quad (3.1)
\]
where \( h_1(z) = z \),
\[
h_k(z) = z - \frac{2(1-\gamma)}{2k-\gamma \left( 1 - (-1)^k \right)} \Omega \sigma_k(\alpha) z^k \quad (k \geq 2),
\]
and
\[
g_k(z) = z + \frac{2(1-\gamma)}{2k+\gamma \left( 1 - (-1)^k \right)} \Omega \sigma_k(\alpha) z^k \quad (k \geq 1),
\]
\[
X_k \geq 0, Y_k \geq 0, \sum_{k=1}^{\infty} (X_k + Y_k) = 1.
\]

In particular, the extreme points of \( \overline{SH_r}([\alpha_1, A_1, B_1], \gamma) \) are \( \{ h_k \} \) and \( \{ g_k \} \).
Proof. For functions $f_k(z)$ of the form (3.1), we have

$$f_k(z) = z - \sum_{k=2}^{\infty} \sum_{j \geq k} 2(1-\gamma) \sigma_k(\alpha_1) X_k z^k + \sum_{k=1}^{\infty} \sum_{j > k} 2(1-\gamma) \sigma_k(\alpha_1) Y_k z^k.$$ 

Then by Theorem 2

$$\sum_{k=2}^{\infty} \frac{2k - \gamma(1 - (-1)^k)}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1) |a_k| + \sum_{k=1}^{\infty} \frac{2k + \gamma(1 - (-1)^k)}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1) |b_k|$$

$$= \sum_{k=2}^{\infty} \frac{2k - \gamma(1 - (-1)^k)}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1) X_k + \sum_{k=1}^{\infty} \frac{2k + \gamma(1 - (-1)^k)}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1) Y_k$$

$$= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1$$

and so $f_k \in \mathcal{FS}_\gamma(\{\alpha_1, A_1, B_1\}, \gamma)$. 

Conversely, if $f_k \in clco \mathcal{FS}_\gamma(\{\alpha_1, A_1, B_1\}, \gamma)$. Setting

$$X_k = \frac{2k - \gamma(1 - (-1)^k)}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1) |a_k|, \quad k \geq 2,$$

and

$$Y_k = \frac{2k + \gamma(1 - (-1)^k)}{2(1 - \gamma)} \Omega \sigma_k(\alpha_1) |b_k|, \quad k \geq 1.$$

We obtain $f_k = \sum_{k=1}^{\infty} [X_k h_k(z) + Y_k g_k(z)]$ as required.

**Theorem 4.** Let the functions $f(z)$ defined by (1.4) be in the class $\mathcal{FS}_\gamma(\{\alpha_1, A_1, B_1\}, \gamma)$.

Then for $|z| = r < 1$, we have

$$|f(z)| \leq (1 + |b_1|) r + \frac{1}{\Omega \sigma_2(\alpha_1)} \left\{ \frac{1 - \gamma}{2} - \frac{1 + \gamma}{2} |b_1| \right\} r^2,$$

and

$$|f(z)| \geq (1 - |b_1|) r - \frac{1}{\Omega \sigma_2(\alpha_1)} \left\{ \frac{1 - \gamma}{2} - \frac{1 + \gamma}{2} |b_1| \right\} r^2.$$

The result is sharp.

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Let $f(z) \in \mathcal{FS}_\gamma(\{\alpha_1, A_1, B_1\}, \gamma)$. Taking the absolute value of $f$ we
have

\[
|f(z)| \leq (1 + |b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^k
\]

\[
\leq (1 + |b_1|)r + r^2 \sum_{k=2}^{\infty} (|a_k| + |b_k|)
\]

\[
\leq (1 + |b_1|)r + \frac{(1-\gamma)^r}{\Omega \sigma_1(\alpha_1)} \sum_{k=2}^{\infty} \frac{\Omega \sigma_k(\alpha_1)}{1-\gamma} (|a_k| + |b_k|) r^2
\]

\[
= (1 + |b_1|)r + \frac{(1-\gamma)^r}{\Omega \sigma_2(\alpha_1)} \sum_{k=2}^{\infty} \left\{ \frac{2k-\gamma(1-(-1)^k)}{4(1-\gamma)} |a_k| + \frac{2k+\gamma(1-(-1)^k)}{4(1-\gamma)} |b_k| \right\} \Omega \sigma_1(\alpha_1)
\]

\[
\leq (1 + |b_1|)r + \frac{(1-\gamma)^r}{\Omega \sigma_2(\alpha_1)} \left[ 1 - \frac{1+\gamma}{2} |b_1| \right] r^2.
\]

The bounds given in Theorem 4 for functions \(f = h + \overline{g}\) of the form (1.4) also hold for functions of the form (1.2) if the coefficient condition (2.1) is satisfied. The upper bound given for \(f \in \overline{\mathcal{K}_r([\alpha_1, A_1, B_1], \gamma)}\) is sharp and the equality occurs for the functions

\[
f(z) = z + \overline{b_1}z + \frac{1}{\Omega \sigma_2(\alpha_1)} \left[ \frac{1-\gamma}{2} - \frac{1+\gamma}{2} |b_1| \right] z^2,
\]

showing that the bounds given in Theorem 4 are sharp. This completes the proof of Theorem 4.

4 Convolution and Convex Combination

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form:

\[
f(z) = z - \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k \overline{z}^k, \quad |b_1| < 1
\]

(4.1)

and

\[
G(z) = z - \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k \overline{z}^k (A_k \geq 0; B_k \geq 0)
\]

(4.2)

we define the convolution of \(f\) and \(G\) as

\[
(f \ast G)(z) = f(z) \ast G(z) = z - \sum_{k=2}^{\infty} a_k A_k z^k + \sum_{k=1}^{\infty} b_k B_k \overline{z}^k.
\]

(4.3)
Using this definition, we show that the class $\mathcal{F}_r^e ([\alpha_1,A_1,B_1], \gamma)$ is closed under convolution.

**Theorem 5.** For $0 \leq \mu \leq \gamma < 1$, let $f \in \mathcal{F}_r^e ([\alpha_1,A_1,B_1], \gamma)$ and $G \in \mathcal{F}_r^e ([\alpha_1,A_1,B_1], \mu)$. Then $f * G \in \mathcal{F}_r^e ([\alpha_1,A_1,B_1], \gamma) \subset \mathcal{F}_r^e ([\alpha_1,A_1,B_1], \mu)$.

**Proof.** Let the function $f(z)$ defined by (4.1) be in the class $\mathcal{F}_r^e ([\alpha_1,A_1,B_1], \gamma)$ and let the function $G(z)$ defined by (4.2) be in the class $\mathcal{F}_r^e ([\alpha_1,A_1,B_1], \mu)$. Then the convolution $f * G$ is given by (4.3). We wish to show that the coefficients of $f * G$ satisfy the required condition given in Theorem 2. For $G \in \mathcal{F}_r^e ([\alpha_1,A_1,B_1], \mu)$ we note that $0 \leq A_k \leq 1$ and $0 \leq B_k \leq 1$.

Now, for the convolution function $f * G$ we obtain

$$
\sum_{k=2}^{\infty} \left[ 2k - \gamma \left( 1 - (-1)^k \right) \right] \Omega \sigma_k (\alpha_1) |a_k| A_k + \sum_{k=1}^{\infty} \left[ 2k + \gamma \left( 1 - (-1)^k \right) \right] \Omega \sigma_k (\alpha_1) |b_k| B_k
$$

$$
\leq \sum_{k=2}^{\infty} \left[ 2k - \gamma \left( 1 - (-1)^k \right) \right] \Omega \sigma_k (\alpha_1) |a_k| + \sum_{k=1}^{\infty} \left[ 2k + \gamma \left( 1 - (-1)^k \right) \right] \Omega \sigma_k (\alpha_1) |b_k|
$$

$$
\leq 2 (1 - \gamma),
$$

since $0 \leq \mu \leq \gamma < 1$ and $f \in \mathcal{F}_r^e ([\alpha_1,A_1,B_1], \gamma)$. Therefore $f * G \in \mathcal{F}_r^e ([\alpha_1,A_1,B_1], \gamma) \subset \mathcal{F}_r^e ([\alpha_1,A_1,B_1], \mu)$, since the above inequality bounded by $2 (1 - \gamma)$ while $2 (1 - \gamma) \leq 2 (1 - \mu)$.

Now, we show that the class $\mathcal{F}_r^e ([\alpha_1,A_1,B_1], \gamma)$ is closed under convex combinations of its members.

**Theorem 6.** The class $\mathcal{F}_r^e ([\alpha_1,A_1,B_1], \gamma)$ is closed under convex combination.

**Proof.** For $i = 1, 2, \cdots$, let $f_i \in \mathcal{F}_r^e ([\alpha_1,A_1,B_1], \gamma)$, where $f_i$ is given by

$$
f_i(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| z^k, \quad (a_k \geq 0; b_k \geq 0; \ z \in U).
$$

Then by using Theorem 2, we have

$$
\sum_{k=2}^{\infty} \left[ \frac{2k - \gamma \left( 1 - (-1)^k \right)}{2 (1 - \gamma)} \right] \Omega \sigma_k (\alpha_1) |a_k| + \sum_{k=1}^{\infty} \left[ \frac{2k + \gamma \left( 1 - (-1)^k \right)}{2 (1 - \gamma)} \right] \Omega \sigma_k (\alpha_1) |b_k| \leq 1. \quad (4.4)
$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of $f_i$ may be written as

$$
\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i |a_k| \right) z^k + \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i |b_k| \right) z^k. \quad (4.5)
$$
Then, by using (4.4), we have

\[
\sum_{k=2}^{\infty} \frac{[2k-\gamma(1-(-1)^k)]}{2(1-\gamma)} \Omega \sigma_k(\alpha_1) \left( \sum_{i=1}^{\infty} t_i |a_k| \right) + \sum_{k=1}^{\infty} \frac{[2k+\gamma(1-(-1)^k)]}{2(1-\gamma)} \Omega \sigma_k(\alpha_1) \left( \sum_{i=1}^{\infty} t_i |b_k| \right)
\]

\[
= \sum_{i=1}^{\infty} t_i \left[ \sum_{k=2}^{\infty} \frac{[2k-\gamma(1-(-1)^k)]}{2(1-\gamma)} \Omega \sigma_k(\alpha_1) |a_k| + \sum_{k=1}^{\infty} \frac{[2k+\gamma(1-(-1)^k)]}{2(1-\gamma)} \Omega \sigma_k(\alpha_1) |b_k| \right]
\]

\[
\leq \sum_{i=1}^{\infty} t_i = 1,
\]

this is the necessary and sufficient condition given by (2.4) and so \( \sum_{i=1}^{\infty} t_i f_i(z) \in \mathcal{H}_S(\alpha_1, A_1, B_1, \gamma) \).

This completes the proof of Theorem 6.

## 5 Properties of Certain Integral Operator

Finally, we study properties of certain integral operator.

**Theorem 7.** Let the functions \( f(z) \) defined by (1.4) be in the class \( \mathcal{H}_S(\alpha_1, A_1, B_1, \gamma) \) and let \( c \) be a real number such that \( c > -1 \). Then the function \( F(z) \) defined by

\[
F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt
\]

belong to the class \( \mathcal{H}_S(\alpha_1, A_1, B_1, \gamma) \).

**Proof.** From the representation of \( F(z) \), it follows that

\[
F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} \left\{ h(t) + \frac{g(t)}{t^c} \right\} dt
\]

\[
= \frac{c+1}{z^c} \left( \int_0^z t^{c-1} \left( t - \sum_{k=2}^{\infty} a_k t^k \right) dt + \int_0^z t^{c-1} \left( \sum_{k=1}^{\infty} b_k t^k \right) dt \right)
\]

\[
= \frac{c+1}{z^c} \left( \int_0^z t dt - \sum_{k=2}^{\infty} a_k \int_0^z t^{c+k-1} dt + \sum_{k=1}^{\infty} b_k \int_0^z t^{c+k-1} dt \right)
\]

\[
= z - \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k z^k,
\]
where \( A_k = \frac{c+1}{c+k} a_k \), \( B_k = \frac{c+1}{c+k} b_k \). Therefore

\[
\sum_{k=2}^{\infty} \left[ \frac{2k-\gamma (1-(-1)^k)}{2(1-\gamma)} \Omega \sigma_k (\alpha_1) \frac{c+1}{c+k} a_k \right] + \sum_{k=1}^{\infty} \left[ \frac{2k+\gamma (1-(-1)^k)}{2(1-\gamma)} \Omega \sigma_k (\alpha_1) \frac{c+1}{c+k} b_k \right] \\
\leq \sum_{k=2}^{\infty} \left[ \frac{2k-\gamma (1-(-1)^k)}{2(1-\gamma)} \Omega \sigma_k (\alpha_1) |a_k| \right] + \sum_{k=1}^{\infty} \left[ \frac{2k+\gamma (1-(-1)^k)}{2(1-\gamma)} \Omega \sigma_k (\alpha_1) |b_k| \right] \leq 1.
\]

Since \( f(z) \in HS^s ([\alpha_1, A_1, B_1], \gamma) \), we have from Theorem 2, \( F(z) \in HS^s ([\alpha_1, A_1, B_1], \gamma) \).

**Remark 2.** Putting \( A_i = 1 \) \((i = 1, \ldots, q)\) and \( B_j = 1 \) \((j = 1, \ldots, s)\) in our results we obtain the results obtained by Murugusundaramoorthy et al. [14].

**References**


M. K. Aouf
Department of Mathematics
Faculty of Science, Mansoura University
Mansoura 35516
Egypt
E-mail: mkaouf127@yahoo.com

R. M. El-Ashwah
Department of Mathematics
Faculty of Science at Damietta, Mansoura University
New Damietta 34517
Egypt
E-mail: r_elashwah@yahoo.com
A. Shamandy
Department of Mathematics
Faculty of Science, Mansoura University
Mansoura 35516
Egypt
E-mail: shamandy16@hotmail.com

S. M. El-Deeb
Department of Mathematics
Faculty of Science at Damietta, Mansoura University
New Damietta 34517
Egypt
E-mail: shezaeldeeb@yahoo.com