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A CLASS OF HARMONIC STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS ASSOCIATED WITH WRIGHT GENERALIZED HYPERGEOMETRIC FUNCTION

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Abstract. Making use of Wright operator we introduce a new class of complex-valued harmonic functions with respect to symmetric points which are orientation preserving, univalent and starlike. We obtain coefficient conditions, extreme points, distortion bounds, and convex combination.

Key words: *harmonic, univalent, Wright operator, symmetric point*

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1 Introduction

Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g}\,, \tag{1.1}$$

which are analytic univalent and sense-preserving in the unit disc $U = \{z : |z| < 1\}$. So that f is normalized by $f(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, we may express the analytic functions h and g in the forms

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k \qquad |b_1| < 1\,. \tag{1.2}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{H}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{H}$ (see [4]).Hence

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}, \ |b_1| < 1. \tag{1.3}$$

We denote $\overline{\mathcal{H}}$ the subclass of $\mathcal{H}$ consists of harmonic functions $f = h + \overline{g}$ of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}, \ |b_1| < 1. \tag{1.4}$$

Let the Hadamard product (or convolution) of two power series $\Phi(z) = z + \sum_{k=2}^{\infty} \phi_k z^k$ and $\Psi(z) = z + \sum_{k=2}^{\infty} \psi_k z^k$ be defined by

$$(\Phi * \Psi)(z) = z + \sum_{k=2}^{\infty} \phi_k \psi_k z^k = (\Psi * \Phi)(z).$$

Let $\alpha_1, A_1, \cdots, \alpha_q, A_q$ and $\beta_1, B_1, \cdots, \beta_s, B_s$ $(q, s \in \mathbf{N})$ be positive and real parameters such that

$$1 + \sum_{j=1}^{s} B_j - \sum_{j=1}^{q} A_j \geq 0.$$

The Wright generalized hypergeometric function[19] (see also [12)

$${}_q\Psi_s[(\alpha_1, A_1), ..., (\alpha_q, A_q); (\beta_1, B_1), ..., (\beta_s, B_s); z] = {}_q\Psi_s\left[(\alpha_i, A_i)_q; (\beta_i, B_i)_s; z\right]$$

is defined by

$${}_q\Psi_s\left[(\alpha_i, A_i)_q; (\beta_i, B_i)_s; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{q} \Gamma(\alpha_i + nA_i)}{\prod_{i=1}^{s} \Gamma(\beta_i + nB_i)} \frac{z^n}{n!}, \qquad z \in U.$$

If $A_i = 1 (i = 1, \cdots, q)$ and $B_i = 1 (i = 1, \cdots, s)$, we have the relationship:

$$\Omega_q\Psi_s\left[(\alpha_i, 1)_q; (\beta_i, 1)_s; z\right] = {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z),$$

where ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ is the generalized hypergeometric function (see for details [6], [7], [8], [9], [13]) and

$$\Omega = \frac{\prod_{i=1}^{s} \Gamma(\beta_i)}{\prod_{i=1}^{q} \Gamma(\alpha_i)}. \tag{1.5}$$

The Wright generalized hypergeometric functions were invoked in the geometric function theory (see [5], [6], [15], [16] and [17]).

By using the generalized hypergeometric function Dziok and Srivastava[7] introduced a linear operator. In [5] Dziok and Riana and in [3] Aouf and Dziok extended the linear operator by using Wright generalized hypergeometric function.

First we define a function ${}_q\Phi_s\left[(\alpha_i,A_i)_q;(\beta_i,B_i)_s;z\right]$ by

$$
{}_q\Phi_s\left[(\alpha_i,A_i)_q;(\beta_i,B_i)_s;z\right]=\Omega\, z\ {}_q\Psi_s\left[(\alpha_i,A_i)_q;(\beta_i,B_i)_s;z\right]
$$

and consider the following linear operator

$$
\theta_{q,s}\left[(\alpha_i,A_i)_q;(\beta_i,B_i)_s\right]:S_H\to S_H,
$$

defined by the convolution

$$
\theta_{q,s}\left[(\alpha_i,A_i)_q;(\beta_i,B_i)_s\right]f(z)={}_q\Phi_s\left[(\alpha_i,A_i)_q;(\beta_i,B_i)_s;z\right]*f(z).
$$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$
\theta_{q,s}\left[(\alpha_i,A_i)_q;(\beta_i,B_i)_s\right]f(z)=z+\sum_{k=2}^{\infty}\Omega\sigma_k(\alpha_1)a_kz^k, \tag{1.6}
$$

where Ω is given by (1.5) and $\sigma_k(\alpha_1)$ is defined by

$$
\sigma_k(\alpha_1)=\frac{\Gamma(\alpha_1+A_1(k-1))\ldots\Gamma(\alpha_q+A_q(k-1))}{\Gamma(\beta_1+B_1(k-1))\ldots\Gamma(\beta_s+B_s(k-1))(k-1)!}. \tag{1.7}
$$

If, for convenience, we write

$$
\theta_{q,s}[\alpha_1,A_1,B_1]f(z)=\theta_{q,s}[(\alpha_1,A_1),\ldots,(\alpha_q,A_q);(\beta_1,B_1),\ldots,(\beta_s,B_s)]f(z),
$$

then one can easily verify from the definition (1.6) that

$$
\begin{aligned}
&zA_1\left(\theta_{q,s}[\alpha_1,A_1,B_1]f(z)\right)'\\
&\quad=\alpha_1\theta_{q,s}[\alpha_1+1,A_1,B_1]f(z)-(\alpha_1-A_1)\theta_{q,s}[\alpha_1,A_1,B_1]f(z).
\end{aligned} \tag{1.8}
$$

We note that for $A_i=1\,(i=1,2,\cdots,q)$ and $B_i=1\,(i=1,2,\cdots,s)$, we obtain $\theta_{q,s}[\alpha_1,1,1]f(z)=H_{q,s}[\alpha_1]f(z)$, which was introduced and studied by Dziok and Srivastava $[7]$.

Applying the Wright operator to the harmonic functions $f = h + \overline{g}$ given by (1.1) we get

$$\theta_{q,s}[\alpha_1, A_1, B_1] f(z) = \theta_{q,s}[\alpha_1, A_1, B_1] h(z) + \overline{\theta_{q,s}[\alpha_1, A_1, B_1] g(z)}. \tag{1.9}$$

Motivated by Jahangiri et al.[10,11] and Ahuja and Jahangiri [1], we define a new subclass $HS_{s^*}([\alpha_1, A_1, B_1], \gamma)$ of $\mathcal{H}$ that are starlike with respect to symmetric points.

Definition 1. For $0 \leq \gamma < 1$ and $z = re^{i\theta} \in U$, we let $\mathcal{H}S_{s^*}([\alpha_1, A_1, B_1], \gamma)$ a subclass of $\mathcal{H}$ of the form $f = h + \overline{g}$ be given by (1.3) and satisfying the analytic criteria

$$\operatorname{Re}\left\{ \frac{2z\left(\theta_{q,s}[\alpha_1, A_1, B_1] f(z)\right)'}{z'\left[\theta_{q,s}[\alpha_1, A_1, B_1] f(z) - \theta_{q,s}[\alpha_1, A_1, B_1] f(-z)\right]} \right\} > \gamma, \tag{1.10}$$

where $\theta_{q,s}[\alpha_1, A_1, B_1] f(z)$ is defined by (1.9) and $z' = \frac{\partial}{\partial\theta}\left(z = re^{i\theta}\right)$.

We also let $\overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma) = \mathcal{H}S_{s^*}([\alpha_1, A_1, B_1], \gamma) \cap \overline{\mathcal{H}}$.

The family $\mathcal{H}S_{s^*}([\alpha_1, A_1, B_1], \gamma)$ is of special interest because for suitable choices of q, s, $[A_1]$, $[B_1]$ and $[\alpha_1]$, we note that

(i) If $A_i = 1\,(i = 1, \cdots, q)$ and $B_j = 1\,(j = 1, \cdots, s)$, we have $\mathcal{H}S_{s^*}([\alpha_1, 1, 1], \gamma) = \mathcal{H}S_{s^*}([\alpha_1], \gamma)$, which was studied by Murugusundaramoorthy et al. [14];

(ii) If $f(-z) = -f(z)$, $A_i = 1\,(i = 1, ..., q)$ and $B_j = 1\,(j = 1, \cdots, s)$, we have $\mathcal{H}S_{s^*}([\alpha_1, 1, 1], \gamma) = S^*_{\mathcal{H}}(\alpha_1, \gamma)$, which was studied by Al-Kharsani and AL-Khal [2].

Remark 1. If the co-analytic part of $f = h + \overline{g}$ is zero, $\alpha_i = A_i = 1\,(i = 1, ..., q)$ and $\beta_j = B_j = 1\,(j = 1, ..., s)$ then $\mathcal{H}S_{s^*}([(1,1), \gamma])$ turns out to be the class $S^*_s(\gamma)$ of starlike functions with respect to symmetric points which was introduced by Sakaguchi [18].

In this paper, we have obtained the coefficient conditions for the classes $\mathcal{H}S_{s^*}([\alpha_1, A_1, B_1], \gamma)$ and $\overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$. Further a representation theorem, inclusion properties and distortion bounds for the class $\overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$ are also established.

2 Coefficient Characterization

Unless otherwise mentioned, we assume throughout this paper that $q, s \in \mathbf{N}$, $a_1 = 1$, $\alpha_1, A_1, \cdots, \alpha_q, A_q$, $\beta_1, B_1, \cdots, \beta_s, B_s \in \mathbf{R}^+$ and $0 \leq \gamma < 1$. We begin with a sufficient condition for functions in $\mathcal{H}S_{s^*}([\alpha_1, A_1, B_1], \gamma)$.

Theorem 1. *Let* $f = h + \overline{g}$ *be given by* (1.3). *Furthermore, let*

$$\sum_{k=2}^{\infty} \frac{\left[2k - \gamma\left(1 - (-1)^k\right)\right]}{2(1-\gamma)} \Omega\sigma_k(\alpha_1)|a_k| + \sum_{k=1}^{\infty} \frac{\left[2k + \gamma\left(1 - (-1)^k\right)\right]}{2(1-\gamma)} \Omega\sigma_k(\alpha_1)|b_k| \leq 1, \tag{2.1}$$

where Ω *and* $\sigma_k(\alpha_1)$ *are defined by* (1.5) *and* (1.7). Then f *is sense-preserving, harmonic univalent in* U *and* $f \in \mathcal{H}S_{s^*}([\alpha_1, A_1, B_1], \gamma)$.

Proof. According the condition (1.10), we only need to show that if (2.1) holds, then

$$\mathrm{Re}\left\{\frac{2z\,(\theta_{q,s}[\alpha_1,A_1,B_1]\,f(z))'}{z'\,[\theta_{q,s}[\alpha_1,A_1,B_1]\,f(z)-\theta_{q,s}[\alpha_1,A_1,B_1]\,f(-z)]}\right\}=\mathrm{Re}\frac{A(z)}{B(z)}>\gamma,$$

where

$$A(z)=2z\,(\theta_{q,s}[\alpha_1,A_1,B_1]\,f(z))'=2z'\left[z+\sum_{k=2}^{\infty}k\Omega\sigma_k(\alpha_1)\,a_kz^k-\sum_{k=1}^{\infty}k\Omega\sigma_k(\alpha_1)\,\overline{b_k z^k}\right]$$

and

$$\begin{aligned}B(z)&=z'\,[\theta_{q,s}[\alpha_1,A_1,B_1]\,f(z)-\theta_{q,s}[\alpha_1,A_1,B_1]\,f(-z)]\\&=z'\left[2z+\sum_{k=2}^{\infty}\left[1-(-1)^k\right]\Omega\sigma_k(\alpha_1)\,a_kz^k+\sum_{k=1}^{\infty}\left[1-(-1)^k\right]\Omega\sigma_k(\alpha_1)\,\overline{b_k z^k}\right].\end{aligned}$$

Using the fact that $\mathrm{Re}\{w(z)\}>\gamma$ if and only if $|1-\gamma+w|>|1+\gamma-w|$, it suffices to show that

$$|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)|>0. \tag{2.2}$$

Substituting for $A(z)$ and $B(z)$ in (2.2) and by using (2.1), we obtain

$$\begin{aligned}&\Bigg|2(2-\gamma)z+\sum_{k=2}^{\infty}\left[2k+(1-\gamma)(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)\,a_k\,z^k\\&\quad-\sum_{k=1}^{\infty}\left[2k-(1-\gamma)(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)\,\overline{b_k\,z^k}\Bigg|\\&\quad-\Bigg|-2\gamma z+\sum_{k=2}^{\infty}\left[2k-(1+\gamma)(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)\,a_k\,z^k\\&\quad-\sum_{k=1}^{\infty}\left[2k+(1+\gamma)(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)\,\overline{b_k\,z^k}\Bigg|\\&\geq 4(1-\gamma)|z|-2\sum_{k=2}^{\infty}\left[2k-\gamma(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)\,|a_k|\,|z|^k\\&\quad-2\sum_{k=1}^{\infty}\left[2k+\gamma(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)\,|b_k|\,|z|^k\\&\quad=4(1-\gamma)|z|\left[1-\sum_{k=2}^{\infty}\frac{\left[2k-\gamma(1-(-1)^k)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)\,|a_k|\,|z|^{k-1}\right.\end{aligned}$$

$$-\sum_{k=1}^{\infty}\frac{\left[2k+\gamma(1-(-1)^k)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|b_k|\,|z|^{k-1}\Bigg]$$

$$\geq 4(1-\gamma)\Bigg[1-\sum_{k=2}^{\infty}\frac{\left[2k-\gamma(1-(-1)^k)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|a_k|$$

$$-\sum_{k=1}^{\infty}\frac{\left[2k+\gamma(1-(-1)^k)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|b_k|\Bigg]\geq 0.$$

This last expression is non-negative by (2.1).

The harmonic univalent functions

$$f(z)=z+\sum_{k=2}^{\infty}\frac{2(1-\gamma)}{\left[2k-\gamma(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)}X_k z^k+\sum_{k=1}^{\infty}\frac{2(1-\gamma)}{\left[2k-\gamma(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)}\overline{Y_k}\,\overline{z}^k, \tag{2.3}$$

where $\sum_{k=2}^{\infty}|X_k|+\sum_{k=1}^{\infty}|Y_k|=1$, show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.3) are in $\mathcal{H}S_{s^*}([\alpha_1,A_1,B_1],\gamma)$ because

$$\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|a_k|+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|b_k|$$

$$=\sum_{k=2}^{\infty}|X_k|+\sum_{k=1}^{\infty}|Y_k|=1.$$

This completes the proof of Theorem 1.

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f(z)$ of the form (1.4).

Theorem 2. *Let* $f=h+\overline{g}$ *be given by* (1.4). *Then* $f\in\overline{\mathcal{H}S_{s^*}}([\alpha_1,A_1,B_1],\gamma)$ *if and only if*

$$\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|a_k|+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|b_k|\leq 1, \tag{2.4}$$

where Ω *and* $\sigma_k(\alpha_1)$ *are defined by* (1.5) *and* (1.7), respectively.

Proof. Since $\overline{\mathcal{H}S_{s^*}}([\alpha_1,A_1,B_1],\gamma)\subset\mathcal{H}S_{s^*}([\alpha_1,A_1,B_1],\gamma)$, we only need to prove the "only if" part of the theorem. To this end, for functions $f(z)$ of the form (1.4), we notice that the condition

$$\operatorname{Re}\left\{\frac{2z\left(\theta_{q,s}[\alpha_1,A_1,B_1]f(z)\right)'}{z'\left[\theta_{q,s}[\alpha_1,A_1,B_1]f(z)-\theta_{q,s}[\alpha_1,A_1,B_1]f(-z)\right]}\right\}>\gamma$$

is equivalent to

$$\mathrm{Re}\left\{\frac{2(1-\gamma)-\sum\limits_{k=2}^{\infty}\left[2k-\gamma(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)a_kz^{k-1}-\frac{\overline{z}}{z}\sum\limits_{k=1}^{\infty}[2k+\gamma(1-(-1)^k)]\Omega\sigma_k(\alpha_1)b_k\overline{z}^{k-1}}{2-\sum\limits_{k=2}^{\infty}(1-(-1)^k)\Omega\sigma_k(\alpha_1)a_kz^{k-1}+\frac{\overline{z}}{z}\sum\limits_{k=1}^{\infty}(1-(-1)^k)\Omega\sigma_k(\alpha_1)b_k\overline{z}^{k-1}}\right\}>0. \tag{2.5}$$

The above required condition (2.5) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we must have

$$\left\{\frac{2(1-\gamma)-\sum\limits_{k=2}^{\infty}\left[2k-\gamma(1-(-1)^k)\right]\Omega\sigma_k(\alpha_1)a_kr^{k-1}-\sum\limits_{k=1}^{\infty}[2k+\gamma(1-(-1)^k)]\Omega\sigma_k(\alpha_1)b_kr^{k-1}}{2-\sum\limits_{k=2}^{\infty}(1-(-1)^k)\Omega\sigma_k(\alpha_1)a_kr^{k-1}+\sum\limits_{k=1}^{\infty}(1-(-1)^k)\Omega\sigma_k(\alpha_1)b_kr^{k-1}}\right\}>0. \tag{2.6}$$

If the condition (2.4) does not hold, then the numerator in (2.6) is negative for r sufficiently close to 1. Hence there exists $z_0=r_0$ in $(0,1)$ for which the quotient in (2.6) is negative. This contradicts the required condition for $f(z)\in\overline{\mathcal{H}S_{s^*}}([\alpha_1,A_1,B_1],\gamma)$ and so the proof of Theorem 2 is completed.

3 Extreme Points and Distortion Theorem

Our next theorem is on the extreme points of convex hulls of $\overline{\mathcal{H}S_{s^*}}([\alpha_1,A_1,B_1],\gamma)$ denoted by $clco\,\overline{\mathcal{H}S_{s^*}}([\alpha_1,A_1,B_1],\gamma)$.

Theorem 3. *A function $f_k(z)\in clco\,\overline{\mathcal{H}S_{s^*}}([\alpha_1,A_1,B_1],\gamma)$ if and only if $f_k(z)$ can be expressed by the form*

$$f_k(z)=\sum_{k=1}^{\infty}\left[X_k h_k(z)+Y_kg_k(z)\right], \tag{3.1}$$

where $h_1(z)=z$,

$$h_k(z)=z-\frac{2(1-\gamma)}{\left[2k-\gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)}z^k\ (k\geq 2),$$

and

$$g_k(z)=z+\frac{2(1-\gamma)}{\left[2k+\gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)}\overline{z}^k\quad (k\geq 1),$$

$$X_k\geq 0,\ Y_k\geq 0,\ \sum_{k=1}^{\infty}(X_k+Y_k)=1.$$

In particular, the extreme points of $\overline{\mathcal{H}S_{s^*}}([\alpha_1,A_1,B_1],\gamma)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For functions $f_k(z)$ of the form (3.1), we have

$$f_k(z) = z - \sum_{k=2}^{\infty} \frac{2(1-\gamma)}{\left[2k-\gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)} X_k z^k + \sum_{k=1}^{\infty} \frac{2(1-\gamma)}{\left[2k+\gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)} Y_k \overline{z}^k.$$

Then by Theorem 2

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|a_k| + \sum_{k=1}^{\infty} \frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|b_k| \\
&= \sum_{k=2}^{\infty} \frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)\frac{2(1-\gamma)}{\left[2k-\gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)}X_k \\
&\quad + \sum_{k=1}^{\infty} \frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)\left(\frac{2(1-\gamma)}{\left[2k+\gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)}Y_k\right) \\
&= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \le 1
\end{aligned}$$

and so $f_k \in \overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$.

Conversely, if $f_k \in clco\,\overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$. Setting

$$X_k = \frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|a_k|, \qquad k \ge 2,$$

and

$$Y_k = \frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|b_k|, \qquad k \ge 1.$$

We obtain $f_k(z) = \sum_{k=1}^{\infty} [X_k h_k(z) + Y_k g_k(z)]$ as required.

Theorem 4. *Let the functions $f(z)$ defined by (1.4) be in the class $\overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$ Then for $|z| = r < 1$, we have*

$$|f(z)| \le (1+|b_1|)r + \frac{1}{\Omega\sigma_2(\alpha_1)}\left\{\frac{1-\gamma}{2} - \frac{1+\gamma}{2}|b_1|\right\}r^2,$$

and

$$|f(z)| \ge (1-|b_1|)r - \frac{1}{\Omega\sigma_2(\alpha_1)}\left\{\frac{1-\gamma}{2} - \frac{1+\gamma}{2}|b_1|\right\}r^2.$$

The result is sharp.

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Let $f(z) \in \overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$. Taking the absolute value of f we have

$$\begin{aligned}
|f(z)| &\leq (1+|b_1|)\,r+\sum_{k=2}^{\infty}(|a_k|+|b_k|)\,r^k\\
&\leq (1+|b_1|)r+r^2\sum_{k=2}^{\infty}(|a_k|+|b_k|)\\
&\leq (1+|b_1|)r+\tfrac{(1-\gamma)}{\Omega\sigma_2(\alpha_1)}\sum_{k=2}^{\infty}\frac{\Omega\sigma_k(\alpha_1)}{1-\gamma}(|a_k|+|b_k|)r^2\\
&= (1+|b_1|)r+\tfrac{(1-\gamma)r^2}{\Omega\sigma_2(\alpha_1)}\sum_{k=2}^{\infty}\left\{\tfrac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{4(1-\gamma)}|a_k|+\tfrac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{4(1-\gamma)}|b_k|\right\}\Omega\sigma_k(\alpha_1)\\
&= (1+|b_1|)r+\frac{(1-\gamma)r^2}{2\Omega\sigma_2(\alpha_1)}\sum_{k=2}^{\infty}\left\{\tfrac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}|a_k|+\tfrac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}|b_k|\right\}\Omega\sigma_k(\alpha_1)\\
&\leq (1+|b_1|)r+\frac{(1-\gamma)r^2}{2\Omega\sigma_2(\alpha_1)}\left(1-\frac{1+\gamma}{1-\gamma}|b_1|\right)\\
&= (1+|b_1|)r+\frac{1}{\Omega\sigma_2(\alpha_1)}\left[\frac{1-\gamma}{2}-\frac{1+\gamma}{2}|b_1|\right]r^2.
\end{aligned}$$

The bounds given in Theorem 4 for functions $f=h+\overline{g}$ of the form (1.4) also hold for functions of the form (1.2) if the coefficient condition (2.1) is satisfied. The upper bound given for $f\in\overline{\mathcal{H}S_{s^*}}([\alpha_1,A_1,B_1],\gamma)$ is sharp and the equality occurs for the functions

$$f(z)=z+\overline{b_1 z}+\frac{1}{\Omega\sigma_2(\alpha_1)}\left[\frac{1-\gamma}{2}-\frac{1+\gamma}{2}b_1\right]\overline{z}^2,$$

showing that the bounds given in Theorem 4 are sharp. This completes the proof of Theorem 4.

4 Convolution and Convex Combination

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form:

$$f(z)=z-\sum_{k=2}^{\infty}a_k z^k+\sum_{k=1}^{\infty}b_k\overline{z}^k,\ |b_1|<1 \tag{4.1}$$

and

$$G(z)=z-\sum_{k=2}^{\infty}A_k z^k+\sum_{k=1}^{\infty}B_k\overline{z}^k\ (A_k\geq 0;\,B_k\geq 0) \tag{4.2}$$

we define the convolution of f and G as

$$(f*G)(z)=f(z)*G(z)=z-\sum_{k=2}^{\infty}a_kA_kz^k+\sum_{k=1}^{\infty}b_kB_k\overline{z}^k. \tag{4.3}$$

Using this definition, we show that the class $\overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$ is closed under convolution.

Theorem 5. *For* $0 \le \mu \le \gamma < 1$, *let* $f \in \overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$ *and* $G \in \overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \mu)$. *Then* $f * G \in \overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma) \subset \overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \mu)$.

Proof. Let the function $f(z)$ defined by (4.1) be in the class $\overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$ and let the function $G(z)$ defined by (4.2) be in the class $\overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \mu)$. Then the convolution $f * G$ is given by (4.3). We wish to show that the coefficients of $f * G$ satisfy the required condition given in Theorem 2. For $G \in \overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \mu)$ we note that $0 \le A_k \le 1$ and $0 \le B_k \le 1$. Now, for the convolution function $f * G$ we obtain

$$\sum_{k=2}^{\infty}\left[2k-\gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)|a_k|A_k+\sum_{k=1}^{\infty}\left[2k+\gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)|b_k|B_k$$

$$\begin{aligned}
&\le \sum_{k=2}^{\infty}\left[2k-\gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)|a_k|+\sum_{k=1}^{\infty}\left[2k+\gamma\left(1-(-1)^k\right)\right]\Omega\sigma_k(\alpha_1)|b_k| \\
&\le 2(1-\gamma),
\end{aligned}$$

since $0 \le \mu \le \gamma < 1$ and $f \in \overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$. Therefore $f * G \in \overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma) \subset \overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \mu)$, since the above inequality bounded by $2(1-\gamma)$ while $2(1-\gamma) \le 2(1-\mu)$.

Now, we show that the class $\overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$ is closed under convex combinations of its members.

Theorem 6. *The class* $\overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$ *is closed under convex combination.*

Proof. For $i = 1, 2, \cdots$, let $f_i \in \overline{\mathcal{H}S_{s^*}}([\alpha_1, A_1, B_1], \gamma)$, where f_i is given by

$$f_i(z) = z - \sum_{k=2}^{\infty}|a_{k_i}|\,z^k + \sum_{k=1}^{\infty}|b_{k_i}|\,\overline{z}^k,\ (a_{k_i} \ge 0\,;\ b_{k_i} \ge 0\,;\ z \in U).$$

Then by using Theorem 2, we have

$$\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|a_{k_i}|+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)|b_{k_i}| \le 1. \quad (4.4)$$

For $\sum\limits_{i=1}^{\infty} t_i = 1, 0 \le t_i \le 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i\, f_i(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i\,|a_{k_i}|\right)z^k + \sum_{k=1}^{\infty}(\sum_{i=1}^{\infty} t_i\,|b_{k_i}|)\overline{z}^k. \quad (4.5)$$

Then, by using (4.4), we have

$$\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)\left(\sum_{i=1}^{\infty}t_i\,|a_{k_i}|\right)+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)\left(\sum_{i=1}^{\infty}t_i\,|b_{k_i}|\right)$$
$$=\sum_{i=1}^{\infty}t_i\left[\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)\,|a_{k_i}|+\sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k(\alpha_1)\,|b_{k_i}|\right]$$
$$\leq\sum_{i=1}^{\infty}t_i=1,$$

this is the necessary and sufficient condition given by (2.4) and so $\sum_{i=1}^{\infty}t_i f_i(z)\in\overline{\mathcal{H}S_{s^*}}\left([\alpha_1,A_1,B_1],\gamma\right)$. This completes the proof of Theorem 6.

5 Properties of Certain Integral Operator

Finally, we study properties of certain integral operator.

Theorem 7. *Let the functions $f(z)$ defined by (1.4) be in the class $\overline{\mathcal{H}S_{s^*}}\left([\alpha_1,A_1,B_1],\gamma\right)$ and let c be a real number such that $c>-1$. Then the function $F(z)$ defined by*

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \tag{5.1}$$

belongs to the class $\overline{\mathcal{H}S_{s^*}}\left([\alpha_1,A_1,B_1],\gamma\right)$.

Proof. From the representation of $F(z)$, it follows that

$$\begin{aligned}F(z) &= \frac{c+1}{z^c}\int_0^z t^{c-1}\left\{h(t)+\overline{g(t)}\right\}\mathrm{d}t\\ &= \frac{c+1}{z^c}\left(\int_0^z t^{c-1}\left(t-\sum_{k=2}^{\infty}a_k t^k\right)\mathrm{d}t+\overline{\int_0^z t^{c-1}\left(\sum_{k=1}^{\infty}b_k t^k\right)\mathrm{d}t}\right)\\ &= \frac{c+1}{z^c}\left(\int_0^z t^c dt-\sum_{k=2}^{\infty}a_k\int_0^z t^{c+k-1}\mathrm{d}t+\overline{\sum_{k=1}^{\infty}b_k\int_0^z t^{c+k-1}\mathrm{d}t}\right)\end{aligned}$$
$$=z-\sum_{k=2}^{\infty}A_k z^k+\sum_{k=1}^{\infty}B_k\overline{z}^k,$$

where $A_k = \dfrac{c+1}{c+k}a_k,\ B_k = \dfrac{c+1}{c+k}b_k$. Therefore

$$\sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k\left(\alpha_1\right)\frac{c+1}{c+k}|a_k| + \sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k\left(\alpha_1\right)\frac{c+1}{c+k}|b_k|$$
$$\leq \sum_{k=2}^{\infty}\frac{\left[2k-\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k\left(\alpha_1\right)|a_k| + \sum_{k=1}^{\infty}\frac{\left[2k+\gamma\left(1-(-1)^k\right)\right]}{2(1-\gamma)}\Omega\sigma_k\left(\alpha_1\right)|b_k| \leq 1.$$

Since $f(z) \in \overline{HS_{s^*}}\left(\left[\alpha_1, A_1, B_1\right], \gamma\right)$, we have from Theorem 2, $F(z) \in \overline{HS_{s^*}}\left(\left[\alpha_1, A_1, B_1\right], \gamma\right)$.

Remark 2. Putting $A_i = 1\,(i = 1, \ldots, q)$ and $B_j = 1\,(j = 1, \ldots, s)$ in our results we obtain the results obtained by Murugusundaramoorthy et al. [14].

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