Anal. Theory Appl.
Vol. 28, No. 4 (2012), 377 –384
DOI : 10.3969/j.issn.1672-4070.2012.04.006

THE LOWER DENSITIES OF SYMMETRIC PERFECT SETS

Chengqin Qu

(*South China University of Technology, China*)

Received Aug. 23, 2012; Revised Oct. 30, 2012

Abstract. In this paper, we give the exact lower density of Hausdorff measure of a class of symmetric perfect sets.

Key words: *lower density, Hausdorff measure, symmetric perfect set*

AMS (2010) subject classification: 28A80

1 Introduction

Let $0 \leq s < \infty$ and ν be a measure on $\mathbf{R}^n$. The upper and lower s-densities of ν at $x \in \mathbf{R}^n$ are defined as

$$\Theta^{*s}(\nu,x) = \limsup_{r\to 0} \frac{\nu(B(x,r))}{(2r)^s},$$

and

$$\Theta_*^s(\nu,x) = \liminf_{r\to 0} \frac{\nu(B(x,r))}{(2r)^s},$$

respectively, where $B(x,r)$ denotes the closed ball with diameter $2r$ and center x.

Symmetric perfect sets are nowhere dense perfect subsets of $[0,1]$ constructed in the following manner.Suppose $I=[0,1]$, let $\{c_k\}_{k\geq 1}$ be a real number sequence satisfying $0 < c_k < \frac{1}{2}(k \geq 1)$. For any $k \geq 1$, let

$$D_k = \{(i_1,\cdots,i_k):\ i_j \in \{1,2\},\ D = \bigcup_{k\geq 0} D_k,$$

where $D_0 = \emptyset$. If

$$\sigma = (\sigma_1,\cdots,\sigma_k) \in D_k, \quad \tau = (\tau_1,\cdots,\tau_m) \in D_m,$$

Supported by the NSFC (10771075).

let

$$\sigma * \tau = (\sigma_1, \cdots, \sigma_k, \tau_1, \cdots, \tau_m).$$

Let $\mathcal{F} = \{I_\sigma : \sigma \in D\}$ be the collection of the closed sub-intervals of I satisfying

i) $I_\emptyset = I$;

ii) For any $k \geq 1$ and $\sigma \in D_{k-1}$, $I_{\sigma * i}$ $(i = 1,2)$ are sub-intervals of I_σ. $I_{\sigma*1}, I_{\sigma*2}$ are arranged from the left to the right, $I_{\sigma*1}$ and I_σ have the same left endpoint, $I_{\sigma*2}$ and I_σ have the same right endpoint.

iii) For any $k \geq 1$ and $\sigma \in D_{k-1}$, $j = 1,2$, we have

$$\frac{|I_{\sigma*j}|}{|I_\sigma|} = c_k,$$

where $|A|$ denotes the diameter of A.

Let

$$E_k = \bigcup_{\sigma \in D_k} I_\sigma, \qquad E = \bigcap_{k \geq 0} E_k,$$

we call E the symmetric perfect set and call $\mathcal{F}_k = \{I_\sigma : \sigma \in D_k\}$ the k-order basic intervals of E. The middle-third Cantor set is a well-known example of the symmetric perfect set.

Let x_k be the length of a k-order basic interval, y_k the length of the gap between any two consecutive sub-intervals $I_{\sigma*1}$ and $I_{\sigma*2}$, where $\sigma \in D_{k-1}$. Assume that

(1) There exists $k_0 \in \mathbf{N}$ such that

$$c_k \leq \frac{1}{3}$$

for all $k > k_0$.

(2) $\lim_{k \to \infty} 2^k x_k^s$ exists and is positive finite.

In [8], we gave a formula to calculate the upper s-density of Hausdorff measure for a class of symmetric perfect sets.

Theorem 1. *Let E be a symmetric perfect set, if (1) and (2) hold, then*

$$\Theta^{*s}(\mu_E, x) = \frac{2}{2^s(2^{\frac{1}{s}} - 1)^s} \quad for \quad \mu_E - \text{a. e.} \quad x \in E,$$

where μ_E is the restriction of the Hausdorff measure $\mathcal{H}^s$ over the set E and s is the Hausdorff dimension of the set E.

This paper gives an analogue for the lower s-density of the Hausdorff measure. Our main result is

Theorem 2. *Let E be the symmetric perfect set, if (2) holds, then*

$$\Theta_*^s(\mu_E, x) = \frac{1}{2^s(2^{\frac{1}{s}} - 1)^s} \quad for \quad \mu_E - \text{a. e.} \quad x \in E.$$

Remark 1. From the above theorems we know that there exists non-regular symmetric perfect sets.

2 Proof of Theorem

For any $\sigma=(\sigma_1,\cdots,\sigma_m)\in D_m$, when $0<k\leq m$, we denote

$$\sigma|k=(\sigma_1,\cdots,\sigma_k).$$

By the definition of x_k and y_k, we have

$$x_k=c_1\cdots c_k,\ \ y_k=(1-2c_k)c_1\cdots c_{k-1}.$$

Take

$$B=\lim_{k\to\infty}2^kx_k^s,$$

the assumption (2) implies $0<B<\infty$ and for any $\varepsilon>0$ there exists a positive integer k_0 such that

$$B-\varepsilon<2^sx_k^s<B+\varepsilon, \tag{2.1}$$

for all $k\geq k_0$, and we have

Lemma 2.1. *If the assumption* (2) *holds, then there exists a positive integer* k_0 *such that* $y_{k+1}<y_k$ *for all* $k\geq k_0$, *and* $\mathcal{H}^s(E)=\lim\limits_{k\to\infty}2^kx_k^s$.

Proof. From (2.1) we have

$$y_k=x_{k-1}-2x_k>\frac{2^{\frac{1}{s}}(B-\varepsilon)^{\frac{1}{s}}-2(B+\varepsilon)^{\frac{1}{s}}}{2^{sk}},$$

$$y_{k+1}=x_k-2x_{k+1}<\frac{2^{\frac{1}{s}}(B+\varepsilon)^{\frac{1}{s}}-2(B-\varepsilon)^{\frac{1}{s}}}{2^{\frac{1}{s}}2^{sk}}.$$

Take

$$\varepsilon=\frac{(4^{\frac{1}{s}}+2)^s-2\cdot 3^s}{(4^{\frac{1}{s}}+2)^s+2\cdot 3^s}B,$$

we have

$$2^{-\frac{1}{s}}(2^{\frac{1}{s}}(B+\varepsilon)^{\frac{1}{s}}-2(B-\varepsilon)^{\frac{1}{s}})<2^{\frac{1}{s}}(B-\varepsilon)^{\frac{1}{s}}-2(B+\varepsilon)^{\frac{1}{s}}.$$

Therefore $y_{k+1}<y_k$, and from Themma 1 in [7] we have

$$\mathcal{H}^s(E)=\lim_{k\to\infty}2^kx_k^s,$$

which completes the proof of Lemma 2.1.

Lemma 2.2.[1] *Let E be the symmetric perfect set. If*

$$B=\lim_{k\to\infty}2^k x_k^s$$

exists and is positive finite, then

$$\lim_{k\to\infty}2^k(x_k+y_k)^s=(2^{\frac{1}{s}}-1)^s B.$$

Take

$$\Omega_k=2^k(x_k+y_k)^s,\Omega=(2^{\frac{1}{s}}-1)^s B,$$

then for any $\varepsilon>0$, there exists a positive integer k_0 such that

$$\Omega-\varepsilon<\Omega_k<\Omega+\varepsilon, \tag{2.2}$$

for all $k\geq k_0$.

Let μ be the restriction of the normalized Hausdorff measure $(\mathcal{H}^s(E))^{-1}\mathcal{H}^s$ over the set E, then for any $A\in\mathcal{F}_k$, we have

$$\mu(A)=2^{-k}. \tag{2.3}$$

Let $\sigma\in D_k,\tau\in D_{k+l},(l>0),\tau|k=\sigma$, set

$$I(\sigma,\tau)=I_{\sigma*p_1}\cup I_{\sigma*\sigma(2,p_2)}\cup\cdots\cup I_{\sigma*\sigma(l,p_l)}\cup I_{\sigma*\sigma(l-1,p_{l-1})}\cup I_{\sigma*\sigma(l,1)},$$

where

$$\sigma(m,j)=(p_1+1,p_2+1,\cdots,p_{m-1}+1,j),0\leq p_i\leq 1,j=0,1,$$

and $\sigma*\sigma(l,1)=\tau$, $I_{\sigma*0}=I_{\sigma*\sigma(m,0)}=\emptyset$.

Lemma 2.3. *Let $\sigma\in D_k,\tau\in D_{k+l},(k>k_0)$ and $\tau|k=\sigma$, then*

$$\frac{\mu([a(\sigma),b(\tau)])}{(b(\tau)-a(\sigma))^s}\geq\frac{1}{\Omega+\varepsilon}. \tag{2.4}$$

Proof. By the definition of $I(\sigma,\tau)$, we have

$$\mu(I(\sigma,\tau))=\frac{p_1}{2^{k+1}}+\frac{p_2}{2^{k+2}}+\cdots+\frac{p_l}{2^{k+l}},$$

and

$$\begin{aligned}|I(\sigma,\tau)|^s &\leq (p_1(x_{k+1}+y_{k+1})+p_2(x_{k+2}+y_{k+2})+\cdots+p_l(x_{k+l}+y_{k+l}))^s\\ &\leq p_1(x_{k+1}+y_{k+1})^s+p_2(x_{k+2}+y_{k+2})^s+\cdots+p_l(x_{k+l}+y_{k+l})^s,\end{aligned}$$

therefore

$$\begin{aligned}\frac{\mu(I(\sigma,\tau))}{|I(\sigma,\tau)|^s} &\ge \frac{\frac{p_1}{2^{k+1}}+\frac{p_2}{2^{k+2}}+\cdots+\frac{p_l}{2^{k+l}}}{p_1(x_{k+1}+y_{k+1})^s+p_2(x_{k+2}+y_{k+2})^s+\cdots+p_l(x_{k+l}+y_{k+l})^s}\\ &\ge \min\{\Omega_{k+1},\Omega_{k+2},\cdots,\Omega_{k+l}\}.\end{aligned}$$

From (2.2) we have (2.4). Which completes the proof of Lemma 2.3.

Lemma 2.4. *Let E be the symmetric perfect set. If* (2) *holds, then for all* $x\in E$,

$$\Theta_*^s(\mu,x)\ge 2^{-s}\Omega^{-1}.$$

Proof. Let $x\in E, 0<r<1$ and set $J=[x-r,x+r]$, then there exists a positive integer k, such that J contains at least a $(k+1)$-order basic interval, but it does not contain any k-order basic interval, thus J intersects with at most two k-order basic intervals, and r can be chosen to be sufficient small such that $k>k_0$.

Case 1. J intersects with two k-order basic intervals. Let $I_{\sigma(1)},I_{\sigma(2)}(\sigma(1),\sigma(2)\in D_k)$ be such two basic intervals and set $J=J_1\cup[b(\sigma(1)),a(\sigma(2))]\cup J_2$, where J_1 and J have the same left endpoint, J_2 and J have the same right endpoint. Without loss of generality, let $x\in J_1$, then $a(\sigma(2))-b(\sigma(1))\le|J_1|<|J_1|+|J_2|$, therefore

$$\begin{aligned}\frac{\mu(J)}{|J|^s} &= \frac{\mu(J_1)+\mu(J_2)}{(|J_1|+|J_2|+a(\sigma(2))-b(\sigma(1)))^s}\\ &\ge \frac{\mu(J_1)+\mu(J_2)}{2^s(|J_1|^s+|J_2|^s)}\\ &\ge \frac{1}{2^s}\min\{\frac{\mu(J_1)}{|J_1|^s},\frac{\mu(J_2)}{|J_2|^s}\}.\end{aligned}$$

Let $u=x+r$, i.e. $J_2=[a(\sigma(2)),u]$. If $u=b(\sigma(2))$, in this case, we obviously have

$$\frac{\mu(J_2)}{|J_2|^s}\ge\frac{1}{\Omega+\varepsilon}. \tag{2.5}$$

If

$$u\in E=\bigcap_{k\ge1}\bigcup_{\sigma\in D_k}I_\sigma,$$

but $u\ne b(\sigma(2))$, then there exists $\tau\in D$, such that

$$u=\bigcap_{l\ge1}I_{\tau|l},$$

thus

$$[a(\sigma(2)),u]=\cap_{l\ge1}[a(\sigma(2)),b(\tau|l)],$$

and

$$[a(\sigma(2)),u] \subset \cdots \subset [a(\sigma(2)),b(\tau|(l+1))] \subset [a(\sigma(2)),b(\tau|l)] \subset \cdots.$$

Therefore,

$$\mu(J_2) = \lim_{l\to\infty} \mu([a(\sigma(2)),b(\tau|l)]).$$

On the other hand, we can chose l to be sufficient large such that $I_{\tau|l} \subset I_{\sigma(2)}$, that is $\tau|k = \sigma(2)$, in this case, by Lemma 2.3, we have

$$\frac{\mu(J_2)}{|J_2|^s} = \lim_{l\to\infty} \frac{\mu([a(\sigma(2)),b(\tau|l)])}{|J_2|^s} \geq \lim_{l\to\infty} \frac{\mu([a(\sigma(2)),b(\tau|l)])}{(b(\tau|l)-a(\sigma(2)))^s} \geq \frac{1}{\Omega+\varepsilon}.$$

If $u \notin E$, i.e.

$$u \in I - \bigcap_{k\geq 1} \bigcup_{\sigma\in D_k} I_\sigma = \bigcup_{k\geq 1} (I - \bigcup_{\sigma\in D_k} I_\sigma),$$

then there exists a positive integer $l > k$ such that

$$u \in I - \bigcup_{\sigma\in D_l} I_\sigma,$$

in this case, similar to the proof of Lemma 2.3, we also have

$$\frac{\mu(J_2)}{|J_2|^s} \geq \frac{1}{\Omega+\varepsilon}. \tag{2.6}$$

For the interval J_1, similar to the above argument, we have

$$\frac{\mu(J_1)}{|J_1|^s} \geq \frac{1}{\Omega+\varepsilon}. \tag{2.7}$$

Therefore

$$\frac{\mu(J)}{|J|^s} \geq \frac{1}{2^s(\Omega+\varepsilon)}. \tag{2.8}$$

Case 2. J intersects with only a k-order basic interval, let $I_\sigma(\sigma \in D_k)$ be such a basic interval. If the left endpoint of J lies in the left of $a(\sigma)$, set $J_1 = J \cap I_\sigma$. Since $x \in I_\sigma$, then

$$a(\sigma) - (x-r) < |J_1|,$$

thus

$$\frac{\mu(J)}{|J|^s} \geq \frac{\mu(J_1)}{2^s|J_1|^s}.$$

Similar to the proof in Case 1, we have (2.8).

If the right endpoint of J lies in the right of $b(\sigma)$, or $J \subset I_\sigma$, we also have (2.8), which completes the proof of Lemma 2.4, since ε is arbitary.

Lemma 2.5. *Let E be the symmetric perfect set. If (2) holds, then for almost all $x\in E$,*

$$\Theta^s_*(\mu,x)\le 2^{-s}\Omega^{-1}.$$

Proof. For any $\sigma\in D_k$, let $\tau\in D_k$ and I_τ the first k-order basic interval to the left of I_σ. Since $0<c_k<\frac{1}{2}$, we have

$$a(\sigma)-b(\tau)>0,$$

hence there exists $l>k$ such that

$$r_l=x_l+y_l<a(\sigma)-b(\tau)$$

and

$$\mu([a(\sigma)-r_l,a(\sigma)+r_l])=\frac{1}{2^l}.$$

It follows that

$$\Theta^s_*(\mu,a(\sigma))\le\liminf_{k\to\infty}\frac{\mu([a(\sigma)-r_l,a(\sigma)+r_l])}{(2r_l)^s}=\frac{1}{2^s\Omega}. \tag{2.9}$$

Now, for $k>0$ put

$$\sigma(1)=(1,\cdots,1)\in D_k,$$

and

$$A^k_p=\bigcup_{l=p}^{\infty}\bigcup_{\sigma\in D_l}I_{\sigma*\sigma(1)},A^k=\bigcap_{p=1}^{\infty}A^k_p,A=\bigcap_{k=1}^{\infty}A^k.$$

Similar to the proof of Lemma 2.6 in [7], we know that the measure μ defined in (2.2) is the same as the $\{\frac{1}{2},\frac{1}{2}\}^{\mathbf{N}}$ Bernoulli measure on the symbolic space $\Sigma=\{1,2\}^{\mathbf{N}}$. On the other hand, since the Bernoulli measure is ergodic, we know that the set corresponding to A on the symbolic space is a set of full measure, so A is a set of full measure.

For any $x\in A$ there are infinitely many n such that there exists $\sigma\in D_n$ with

$$|x-a(\sigma)|<\frac{1}{2^k}x_n.$$

Taking

$$r=r_n-\frac{1}{2^k}x_n$$

gives

$$[x-r,x+r]\subset[a(\sigma)-r_n,a(\sigma)+r_n],$$

which implies

$$\Theta^s_*(\mu,x)\le(1-\frac{1}{2^k})^{-s}2^{-s}\Omega^{-1},$$

$\mu-$a.e. on E. Taking $k\to\infty$ we obtain

$$\Theta_*^s(\mu,x)\le 2^{-s}\Omega^{-1}\mu-$$

a.e. on E. This completes the proof of Lemma 2.5.

Proof of Theorem 2. By Lemma 2.4-2.5, we immediately obtain Theorem 2.

Example 1. Let E be the middle-third Cantor set, it is well known that $\dim_H(E)=s=\dfrac{\log 2}{\log 3}$, and $\mathcal{H}^s(E)=1$, where $\dim_H(E)$ is the Hausdorff dimension of the set E, and $\mathcal{H}^s(E)$ is the Hausdorff measure of the set E. By Theorem 1 and Theorem 2 we obtain

$$\Theta^{*s}(\mu_E,x)=\frac{2}{4^s},\ \Theta_*^s(\mu_E,x)=\frac{1}{4^s}\quad for\quad \mu_E-\text{a. e. } x\in E.$$

References

[1] Ayer. E. and Strichartz, R. S., Exact Hausdorff Measure and Intervals of Maximum Density for Cantor Measures, Trans. Amer. Soc., 351(1999), 3725-3741.

[2] Baek, K. K., Packing Dimension and Measure of Homogeneous Cantor Sets, Bulletin of the Australian Mathematical Society 74(2006), 443-448.

[3] Cutler, C. D., The Density Theorem and Hausdorff Inequality for Packing Measure in General Metric Spaces, Illinois J.Math. 39(1995), 676-694.

[4] Feng, D. J., Wen, Z. Y. and Wu, J., Dimension of the Homogeneous Moran Sets, Science in China (Series A), 40(1997), 475-482.

[5] Marion, D. J., Measure de Hausdorff et Theorie de Perron-Frobinius des Matrice Non-negatives, Ann. Inst. Fourier, Grenoble 35(1985), 99-125.

[6] Mattila, P., Geometry of Sets and Measures in Euclidean Spaces, Cambridge University Press, 1995.

[7] Qu, C. Q., Rao, H. and Su, W. Y., Hausdorff Measure of Homogeneous Cantor Set, Acta Math. Sinica, English Series, 17(2001), 1-11.

[8] Qu, C. Q., Zhou, Z. L. and Jia, B. G., The Upper Densities of Symmetric Perfect Sets, J. Math. Anal. Appl., 292(2004), 23-32.

Department of Applied Mathematics
South China University of Technology
Guangzhou, 510640
P. R. China

E-mail: chengqingqu@tom.com