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NON-ORTHOGONAL P-WAVELET PACKETS ON THE HALF-LINE

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Abstract. In this paper, the notion of p-wavelet packets on the positive half-line $\mathbb{R}^+$ is introduced. A new method for constructing non-orthogonal wavelet packets related to Walsh functions is developed by splitting the wavelet subspaces directly instead of using the low-pass and high-pass filters associated with the multiresolution analysis as used in the classical theory of wavelet packets. Further, the method overcomes the difficulty of constructing non-orthogonal wavelet packets of the dilation factor $p > 2$.

Key words: *p-Multiresolution analysis, p-wavelet packets, Riesz basis, Walsh function, Walsh-Fourier transform*

AMS (2010) subject classification: 42C40, 42C10, 42C15

1 Introduction

In the early nineties a general scheme for the construction of wavelets was defined. This scheme is based on the notion of multiresolution analysis (MRA) introduced by Mallat[16]. Immediately specialists started to implement new wavelet systems and in recent years, the concept MRA of $\mathbf{R}^n$ has been extended to many different setups, for example, Dahlke introduced multiresolution analysis and wavelets on locally compact Abelian groups[5], Lang[14] constructed compactly supported orthogonal wavelets on the locally compact Cantor dyadic group $\mathcal{C}$ by following the procedure of Daubechies [6] via scaling filters and these wavelets turn out to be certain lacunary Walsh series on the real line. On the otherhand, Jiang et al.[13] pointed out a method for constructing orthogonal wavelets on local field $\mathbf{K}$ with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of $L^2(\mathbf{K})$. Subsequently, the tight wavelet frames on local fields were constructed by Li and Jiang in [15]. Farkov[7] extended the results of Lang[14] on the wavelet analysis on the Cantor dyadic group $\mathcal{C}$ to the locally compact Abelian group G_p which is defined for an integer $p \geq 2$ and coincides with $\mathcal{C}$ when $p = 2$. Concerning the construction of wavelets on a half-line, Farkov [8] has given the general construction of all compactly supported orthogonal p-wavelets in $L^2(\mathbf{R}^+)$ and proved necessary and sufficient conditions for scaling filters with p^n many terms $(p,n \geq 2)$ to generate a p-MRA analysis in $L^2(\mathbf{R}^+)$. These studies were continued by Farkov and his colleagues in [9,10], where they have given some new algorithms for constructing the corresponding biorthogonal and nonstationary wavelets related to the Walsh functions on the positive half-line $\mathbf{R}^+$. On the otherhand, Shah and Debnath[21] have constructed dyadic wavelet frames on the positive half-line $\mathbf{R}^+$ using the Walsh-Fourier transform and have established a necessary condition and a sufficient condition for the system $\{2^{j/2}\psi(2^j x \ominus k) : j \in \mathbf{Z}, k \in \mathbf{Z}^+\}$ to be a frame for $L^2(\mathbf{R}^+)$.

It is well-known that the classical orthonormal wavelet bases have poor frequency localization. For example, if the wavelet ψ is band limited, then the measure of the supp of $(\psi_{j,k})^\wedge$ is 2^j-times that of supp $\hat{\psi}$. To overcome this disadvantage, Coifman et al.[4] constructed univariate orthogonal wavelet packets. The fundamental idea of wavelet packet analysis is to construct a library of orthonormal bases for $L^2(\mathbf{R})$, which can be searched in real time for the best expansion with respect to a given application.

Let $\varphi(x)$ and $\psi(x)$ be the scaling function and the wavelet function associated with a multiresolution analysis $\{V_j\}_{j\in\mathbf{Z}}$. Let W_j be the corresponding wavelet subspaces:

$$W_j = \overline{\text{span}}\left\{\psi_{j,k} : k \in \mathbf{Z}\right\}.$$

Using the low-pass and high-pass filters associated with the MRA, the space W_j can be split into two orthogonal subspaces, each of them can further be split into two parts. Repeating this process j times, W_j is decomposed into 2^j subspaces each generated by integer translates of a single function. If we apply this to each W_j, then the resulting basis of $L^2(\mathbf{R})$ which will consist of integer translates of a countable number of functions, will give a better frequency localization. This basis is called the *wavelet packet basis*. To describe this more formally, we introduce a parameter n to denote the frequency. Set $\omega_0 = \varphi$ and define recursively

$$\omega_{2n}(x) = \sum_{k\in\mathbf{Z}} h_k \omega_n(2x-k), \qquad \omega_{2n+1}(x) = \sum_{k\in\mathbf{Z}} g_k \omega_n(2x-k),$$

where $\{h_k\}_{k\in\mathbf{Z}}$ and $\{g_k\}_{k\in\mathbf{Z}}$ are the low-pass filter and high-pass filter corresponding to $\varphi(x)$ and $\psi(x)$, respectively. Chui and Li[2] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be applied to the spline wavelets and so on. The introduction of biorthogonal wavelet packets attributes to Cohen and Daubechies[3]. Shen[24] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets for the dilation factor $p=2$, however this construction does not work for $p>2$. Other notable generalizations are the orthogonal version of vector-valued wavelet packets[1], non-orthogonal wavelet packets with r-scaling functions[11] and the M-band framelet packets[23].

Recently, Shah[20] has constructed p-wavelet packets related to the Walsh functions on the positive half-line $\mathbf{R}^+$. He proved lemmas on the so-called splitting trick and several theorems concerning the Walsh-Fourier transform of the p-wavelet packets and the construction of p-wavelet packets to show that their translates form an orthonormal basis of $L^2(\mathbf{R}^+)$. Subsequently, the corresponding biorthogonal p-wavelet packets and p-wavelet frame packets on $\mathbf{R}^+$ were studied by the author and Debnath in [18,19,22].

As one of a series of works on the positive half-line $\mathbf{R}^+$, the objective of this paper is to construct non-orthogonal p-wavelet packets related to Walsh functions on $\mathbf{R}^+$ using the splitting trick of wavelets. The splitting trick in our method decomposes the wavelet subspaces directly instead of using the low-pass and high-pass filters as used in the classic theory of wavelet packets, and thus gives the Riesz basis of the wavelet subspaces.

We have organized the article as follows. In Section 2, we state some basic preliminaries, notation and definitions including Walsh functions, the Walsh-Fourier transform and p-MRA. In Section 3, we prove a crucial lemma called the *splitting lemma* which decomposes the wavelet subspaces directly instead of using the low-pass and high-pass filters. By virtue of this lemma, we construct the p-wavelet packets and prove that they generate Reisz basis for $L^2(\mathbf{R}^+)$.

2 Preliminaries and p-Wavelet Packets on $\mathbf{R}^+$

Let p be a fixed natural number greater than 1. As usual, let $\mathbf{R}^+=[0,+\infty)$, $\mathbf{Z}^+=\{0,1,2,\cdots\}$ and $\mathbf{N}=\mathbf{Z}^+-\{0\}$. Set $\Omega_0=\{0,1,2,\cdots,p-1\}$ and $\Omega=\Omega_0-\{0\}$. Denote by $[x]$ the integer part of x. For $x\in\mathbf{R}^+$ and any positive integer j, we set

$$x_j=[p^jx](\mathrm{mod}\,p),\qquad x_{-j}=[p^{1-j}x](\mathrm{mod}\,p). \tag{2.1}$$

We consider on $\mathbf{R}^+$ the addition defined as follows:

$$z=\sum_{j<0}\zeta_j p^{-j-1}+\sum_{j>0}\zeta_j p^{-j}$$

with $\zeta_j = x_j + y_j (\text{mod } p)$ $(j \in \mathbf{Z} \setminus \{0\})$, where $\zeta_j \in \Omega_0$ and x_j, y_j are calculated by (2.1). Note that $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo p in $\mathbf{R}^+$.

For $x \in [0,1)$, let $r_0(x)$ be given by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/p) \\ \\ \varepsilon_p^{\ell}, & \text{if } x \in [\ell p^{-1}, (\ell+1)p^{-1}), \ \ \ell \in \Omega, \end{cases}$$

where $\varepsilon_p = \exp(2\pi i/p)$. The extension of the function r_0 to $\mathbf{R}^+$ is given by the equality $r_0(x+1) = r_0(x), \ x \in \mathbf{R}^+$. Then, the *generalized Walsh functions* $\{w_m(x) : m \in \mathbf{Z}^+\}$ are defined by

$$w_0(x) \equiv 1, \quad w_m(x) = \prod_{j=0}^{k} \left(r_0(p^j x)\right)^{\mu_j}$$

where $m = \sum_{j=0}^{k} \mu_j p^j$, $\mu_j \in \Omega_0, \ \mu_k \neq 0$. They have many properties similar to those of the Haar functions and trigonometric series, and form a complete orthogonal system. Further, by a Walsh polynomial we shall mean a finite linear combination of Walsh functions.

For $x, y \in \mathbf{R}^+$, let

$$\chi(x,y) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j y_{-j} + x_{-j} y_j)\right), \tag{2.2}$$

where x_j, y_j are given by (2.1). Note that $\chi(x, m/p^{n-1}) = \chi(x/p^n, m) = w_m(x/p^n)$ for all $x \in [0, p^n), \ m, n \in \mathbf{Z}^+$. Also, if $x, y, \xi \in \mathbf{R}^+$ and $x \oplus y$ is p-adic irrational, then

$$\chi(x \oplus y, \xi) = \chi(x, \xi)\, \chi(y, \xi), \ \text{ and } \ \chi(x \ominus y, \xi) = \chi(x, \xi)\, \overline{\chi(y, \xi)}.$$

It is shown by Golubov et al[12] that both the systems $\{\chi(\alpha, .)\}_{\alpha=0}^{\infty}$ and $\{\chi(., \alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2[0,1]$.

By p-adic interval $I \subset \mathbf{R}^+$ of range n we mean the intervals of the form

$$I = I_n^k = \left[kp^{-n}, (k+1)p^{-n}\right), \ \ k \in \mathbf{Z}^+.$$

The p-adic topology is generated by the collection of p-adic intervals and each p-adic interval is both open and closed under the p-adic topology (see [17]). The family $\left\{[0, p^{-j}) : j \in \mathbf{Z}\right\}$ forms a fundamental system of the p-adic topology on $\mathbf{R}^+$. Therefore, for each $0 \leq j, k < p^n$, the Walsh function $w_j(x)$ is piecewise constant and hence continuous. Thus $w_j(x) = 1$ for $x \in I_n^0$.

The Walsh-Fourier transform of a function $f \in L^1(\mathbf{R}^+) \cap L^2(\mathbf{R}^+)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbf{R}^+} f(x) \overline{\chi(x, \xi)}\, dx, \tag{2.3}$$

where $\chi(x,\xi)$ is given by (2.2). The Walsh-Fourier operator $\mathcal{F}: L^1(\mathbf{R}^+)\cap L^2(\mathbf{R}^+)\to L^2(\mathbf{R}^+)$, $\mathcal{F}f=\hat{f}$, extends uniquely to the whole space $L^2(\mathbf{R}^+)$. The properties of the Walsh-Fourier transform are quite similar to those of the classic Fourier transform (see[12,17]). In particular, if $f\in L^2(\mathbf{R}^+)$, then $\hat{f}\in L^2(\mathbf{R}^+)$ and

$$\|\hat{f}\|_{L^2(\mathbf{R}^+)}=\|f\|_{L^2(\mathbf{R}^+)}.$$

Definition 2.1. Let $\mathbf{H}$ be a Hilbert space. A sequence $\{f_k\}_{k=1}^{\infty}$ of $\mathbf{H}$ is said to be a Riesz basis for $\mathbf{H}$ if there exist constants A and B, $0<A\le B<\infty$ such that any $f\in\mathbf{H}$ can be represented as a series $f=\sum_{k=1}^{\infty}c_kf_k$ converging in $\mathbf{H}$ with

$$A\|f\|^2\le\sum_{k=1}^{\infty}|c_k|^2\le B\|f\|^2, \tag{2.4}$$

where $\|.\|$ is the norm of $L^2(\mathbf{R}^+)$.

A function $f\in L^2(\mathbf{R}^+)$ is said to be stable if there exist positive constants c_1 and c_2 such that

$$c_1\left(\sum_{k\in\mathbf{Z}^+}|a_k|^2\right)^{1/2}\le\left\|\sum_{k\in\mathbf{Z}^+}a_kf(x\ominus k)\right\|\le c_2\left(\sum_{k\in\mathbf{Z}^+}|a_k|^2\right)^{1/2},$$

for each sequence $\{a_k\}_{k\in\mathbf{Z}^+}\in l^2(\mathbf{Z}^+)$. In other words, f is stable if the system of functions $\{f(x\ominus k):k\in\mathbf{Z}^+\}$ form a Riesz system in $L^2(\mathbf{R}^+)$. Moreover, we recall that (see [8]), f is stable in $L^2(\mathbf{R}^+)$ with constants c_1 and c_2 if and only if

$$c_1\le\sum_{k\in\mathbf{Z}^+}|\hat{f}(\xi\oplus k)|^2\le c_2,\ \text{ for a.e. }\xi\in\mathbf{R}^+.$$

In the following subsection, we introduce the notion of p-multiresolution analysis on $\mathbf{R}^+$ and give the formal definition of p-wavelets of space $L^2(\mathbf{R}^+)$.

*Definition*2.2. A p-multiresolution analysis of $L^2(\mathbf{R}^+)$ is a nested sequence of closed subspaces $\{V_j\}_{j\in\mathbf{Z}}$ such that

(i) $V_j\subset V_{j+1}$ for all $j\in\mathbf{Z}$,

(ii) $\bigcup_{j\in\mathbf{Z}}V_j$ is dense in $L^2(\mathbf{R}^+)$ and $\bigcap_{j\in\mathbf{Z}}V_j=\{0\}$,

(iii) $f\in V_j$ if and only if $f(p.)\in V_{j+1}$ for all $j\in\mathbf{Z}$,

(iv) there exists a function φ in V_0, called the scaling function, such that the system of functions $\{\varphi(.\ominus k):k\in\mathbf{Z}^+\}$ form a Riesz basis for subspace V_0.

Since $\varphi(x)\in V_0\subset V_1$, by Definition 2.2, there exists a finitely supported sequence $\{a_k\}_{k\in\mathbf{Z}^+}\in l^2(\mathbf{Z}^+)$ such that

$$\varphi(x)=\sum_{k\in\mathbf{Z}^+}a_k\varphi(px\ominus k). \tag{2.5}$$

The Walsh-Fourier transform of (2.5) is given by

$$\hat{\varphi}(\xi) = m_0(p^{-1}\xi)\,\hat{\varphi}(p^{-1}\xi), \tag{2.6}$$

where $m_0(\xi) = \sum_{k\in\mathbf{Z}^+} a_k\,\overline{\chi(k,\xi)}$, is a generalized Walsh polynomial, called the mask or symbol of the refinable function φ and is a p-adic step function.

Let W be the wavelet subspace, the complement of V_0 in V_1. If $\psi_1, \psi_2, \ldots, \psi_{p-1}$ are in W such that $\{\psi_\ell(x\ominus k) : k \in \mathbf{Z}^+, \ell \in \Omega\}$ form a Riesz basis for W, then, we call $\psi_1, \psi_2, \ldots, \psi_{p-1}$ the basic p-wavelets associated with the scaling function $\varphi(x)$. Since $\psi_\ell \in W \subset V_1$, for each $\ell \in \Omega$, there exists a sequence $\left\{a_k^\ell\right\}_{k\in\mathbf{Z}^+}$ with $\sum_{k\in\mathbf{Z}^+} |a_k^\ell|^2 < \infty$ such that

$$\psi_\ell(x) = \sum_{k\in\mathbf{Z}^+} a_k^\ell\,\varphi(px\ominus k). \tag{2.7}$$

Implementing the Walsh- Fourier transform for both sides of (2.7) yields

$$\hat{\psi}_\ell(\xi) = m_\ell(p^{-1}\xi)\,\hat{\varphi}(p^{-1}\xi), \tag{2.8}$$

where

$$m_\ell(\xi) = \sum_{k\in\mathbf{Z}^+} a_k^\ell\,\overline{\chi(k,\xi)}, \quad \ell \in \Omega, \tag{2.9}$$

are the integral-periodic functions in $L^2[0,1]$ and are called the *wavelet masks*.

By virtue of the property (ii) of Definition 2.2, we have

$$L^2(\mathbf{R}^+) = \bigoplus_{j\in\mathbf{Z}} \mathcal{D}^j W = V_0 \oplus \Big(\bigoplus_{j\geq 0} \mathcal{D}^j W\Big) \tag{2.10}$$

where $\mathcal{D}$ is the dilation operator such that $\mathcal{D}f(x) = f(px)$, for any $f \in L^2(\mathbf{R}^+)$.

Setting

$$\omega_1 = \psi_1(x), \omega_2 = \psi_2(x), \ldots, \omega_{p-1} = \psi_{p-1}(x).$$

For each integer $n \in \mathbf{Z}^+$, we define ω_n as follows:

$$\omega_n(x) = \omega_{pr+s}(x) = \omega_r(px\ominus\mu_s), \tag{2.11}$$

where r and s are the unique numbers such that $n = pr+s, r \in \mathbf{Z}^+, s \in \Omega_0$.

The functions $\{\omega_n : n \in \mathbf{Z}^+\}$ will be called the basic p-wavelet packets related to the Walsh functions on the positive half-line $\mathbf{R}^+$ (see [20]).

Definition 2.3. Let $\{\omega_n : n \in \mathbf{Z}^+\}$ be the basic p-wavelet packets associated with the p-MRA $\left\{V_j\right\}_{j\in\mathbf{Z}}$ of $L^2(\mathbf{R}^+)$. The collection of functions

$$\mathcal{P} = \left\{\omega_n(p^j x\ominus k) : n \in \mathbf{Z}^+, j \in \mathbf{Z}, k \in \mathbf{Z}^+\right\} \tag{2.12}$$

is called the general p-wavelet packets corresponding to the p-MRA $\left\{V_j\right\}_{j\in\mathbf{Z}}$.

3 Splitting Trick and the Non-orthogonal p-Wavelet Packets

For any $n \in \mathbf{Z}^+$, define

$$U_n = \left\{ f(x) : f(x) = \sum_{k\in\mathbf{Z}^+} b_k \omega_n(x \ominus k),\ \{b_k\}_{k\in\mathbf{Z}^+} \in l^2(\mathbf{Z}^+) \right\}. \tag{3.1}$$

Lemma 3.1. *Let $\{\omega_n : n \in \mathbf{Z}^+\}$ be the basic p-wavelet packets associated with the p-MRA $\{V_j\}_{j\in\mathbf{Z}}$. Then, for all $n \in \mathbf{Z}^+$, $\{\omega_n(x\ominus k) : k \in \mathbf{Z}^+\}$ form a Riesz basis of U_n.*

Proof. We prove this result by induction on n. Since $\{\psi_1, \psi_2, \cdots, \psi_{p-1}\}$ is the basic set of p-wavelets in W and $\{\psi_1(x\ominus k), \psi_2(x\ominus k), \cdots, \psi_{p-1}(x\ominus k) : k \in \mathbf{Z}^+\}$ constitutes Riesz basis of W, therefore,

$$\left\{\psi_1(px\ominus k), \psi_2(px\ominus k), ..., \psi_{p-1}(px\ominus k) : k \in \mathbf{Z}^+\right\}$$

form a Riesz basis of $\mathcal{D}W$. By virtue of Riesz basis and Definition 2.3, the system

$$\left\{\omega_1(x\ominus k), \omega_2(x\ominus k), ..., \omega_{p-1}(x\ominus k) : k \in \mathbf{Z}^+\right\}$$

constitutes Riesz basis of W, and thus $\{\omega_n(x\ominus k) : 1 \le n \le p-1, k \in \mathbf{Z}^+\}$ form a Riesz basis of U_n. Therefore, the claim is true when $1 \le n \le p-1$. Assume that the result is true for $n < \ell, (\ell \ge p)$.

Now, for $n = \ell$, there exist two unique numbers r and s such that $n = pr+s, s \in \Omega_0, r \in \mathbf{Z}^+$ and $r < n = \ell$. Since $r < n = \ell$, the family of functions $\{\omega_r(x\ominus k) : k \in \mathbf{Z}^+\}$ form a Riesz basis of U_n. Therefore, there exist constants c_1 and c_2, $0 < c_1 \le c_2 < \infty$ such that

$$c_1 \le \sum_{k\in\mathbf{Z}^+} \left|\hat{\omega}_r(\xi \oplus k)\right|^2 \le c_2.$$

Further, we have

$$\begin{aligned}\sum_{k\in\mathbf{Z}^+} \left|\hat{\omega}_n(\xi \oplus k)\right|^2 &= \frac{1}{p}\sum_{k\in\mathbf{Z}^+} \left|\hat{\omega}_r(p^{-1}(\xi\oplus k))\right|^2 \left|\chi(p^{-1}\xi, k)\right|^2 \\ &= \frac{1}{p}\sum_{k\in\mathbf{Z}^+} \left|\hat{\omega}_r(p^{-1}\xi\oplus p^{-1}k)\right|^2 \\ &= \frac{1}{p}\sum_{k'\in\mathbf{Z}^+} \left|\hat{\omega}_r(p^{-1}\xi\oplus k')\right|^2 + \frac{1}{p}\sum_{k'\in\mathbf{Z}^+} \left|\hat{\omega}_r(p^{-1}\xi\oplus k'\oplus p^{-1}s_1)\right|^2 + \cdots + \\ &\quad \times \frac{1}{p}\sum_{k'\in\mathbf{Z}^+} \left|\hat{\omega}_r(p^{-1}\xi\oplus k'\oplus p^{-1}s_{p-1})\right|^2\end{aligned}$$

and hence

$$c_1 = p\frac{1}{p} \le \sum_{k\in\mathbf{Z}^+} \left|\hat{\omega}_n(\xi\oplus k)\right|^2 \le p\frac{1}{p} = c_2.$$

Thus, $\{\omega_n(x \ominus k) : k \in \mathbf{Z}^+\}$ forms a Riesz basis of U_n. This completes the proof of the lemma.

Now we establish the splitting trick of our p-wavelet packets.

Lemma 3.2 (Splitting lemma). *For every $n \in \mathbf{Z}^+$, the family of functions*

$$\left\{\omega_{pn+s}(x \ominus k) : s \in \Omega_0, k \in \mathbf{Z}^+\right\}$$

constitutes Riesz basis of $\mathcal{D}U_n$.

Proof. First, we claim that

$$\mathcal{D}U_n = \left\{ f(x) : f(x) = \sum_{s \in \Omega_0} \sum_{k \in \mathbf{Z}^+} b_k^s \omega_{pn+s}(x \ominus k),\ \{b_k^s\}_{k \in \mathbf{Z}^+} \in l^2(\mathbf{Z}^+) \right\}. \tag{3.2}$$

As for any $s \in \Omega_0$, by (2.11) and (3.1), $\omega_{pn+s}(x \ominus k) \in \mathcal{D}U_n$. Assume that $f(x) \in \mathcal{D}U_n$, then there exists a sequence $\{c_k\}_{k \in \mathbf{Z}^+} \in l^2(\mathbf{Z}^+)$ such that

$$f(x) = \sum_{k \in \mathbf{Z}^+} c_k \omega_n(px \ominus k). \tag{3.3}$$

Further, if there exist sequences $\left\{b_k^s\right\}_{k \in \mathbf{Z}^+} \in l^2(\mathbf{Z}^+), s \in \Omega_0$, as for $f(x) \in \mathcal{D}U_n$, such that

$$f(x) = \sum_{s \in \Omega_0} \sum_{k \in \mathbf{Z}^+} b_k^s \omega_{pn+s}(x \ominus k). \tag{3.4}$$

Since $k \in \mathbf{Z}^+$, there exist unique numbers r and s such that $k = pr + s, r \in \mathbf{Z}^+, s \in \Omega_0$. Then, for this choice of $k \in \mathbf{Z}^+$, we obtain

$$\begin{aligned}
f(x) &= \sum_{k \in \mathbf{Z}^+} c_k \omega_n(px \ominus k) \\
&= \sum_{r \in \mathbf{Z}^+} c_{pr} \omega_n(px \ominus pr) + \sum_{r \in \mathbf{Z}^+} c_{pr+s_1} \omega_n(px \ominus pr \ominus s_1) \\
&\quad + \cdots + \sum_{r \in \mathbf{Z}^+} c_{pr+s_{p-1}} \omega_n(px \ominus pr \ominus s_{p-1}) \\
&= \sum_{r \in \mathbf{Z}^+} c_{pr} \omega_{pn}(x \ominus r) + \sum_{r \in \mathbf{Z}^+} c_{pr+s_1} \omega_{pn+1}(x \ominus r) + \cdots + \sum_{r \in \mathbf{Z}^+} c_{pr+s_{p-1}} \omega_{pn+p-1}(x \ominus r) \\
&= \sum_{s \in \Omega_0} \sum_{k \in \mathbf{Z}^+} b_k^s \omega_{pn+s}(x \ominus k),
\end{aligned}$$

where $b_k^s = c_{pr+s_\ell}, \ell \in \Omega_0$ and hence the equality (3.2) follows.

We now show that the set of functions

$$\left\{\omega_{pn+s}(x \ominus k) : s \in \Omega_0, k \in \mathbf{Z}^+\right\}$$

form a Riesz basis of $\mathcal{D}U_n$. In the light of Lemma 3.1, the family $\{\omega_n(x\ominus k):k\in\mathbf{Z}^+\}$ is a Riesz basis of U_n. Therefore, $\{\omega_n(px\ominus k):k\in\mathbf{Z}^+\}$ constitutes a Riesz basis of $\mathcal{D}U_n$. However, $\{\omega_n(px\ominus k):k\in\mathbf{Z}^+\}$ can be splitted into p-disjoint subsets as:

$$\Big\{\omega_n(px\ominus pk):k\in\mathbf{Z}^+\Big\},\Big\{\omega_n(px\ominus pk\ominus s_1):k\in\mathbf{Z}^+\Big\},\cdots,\Big\{\omega_n(px\ominus pk\ominus s_{p-1}):k\in\mathbf{Z}^+\Big\},$$

which can be written as

$$\Big\{\omega_{pn}(x\ominus k):k\in\mathbf{Z}^+\Big\},\Big\{\omega_{pn+1}(x\ominus k):k\in\mathbf{Z}^+\Big\},\ \cdots\ ,\Big\{\omega_{pn+p-1}(x\ominus k):k\in\mathbf{Z}^+\Big\}.$$

Hence, $\Big\{\omega_{pn+s}(x\ominus k):s\in\Omega_0,k\in\mathbf{Z}^+\Big\}$ form a Riesz basis of $\mathcal{D}U_n$.

This is the splitting trick of our method. This splitting trick decomposes the wavelet subspaces directly instead of using the low-pass filter $m_0(\xi)$ and the high-pass filters $m_\ell(\xi),\ell\in\Omega$ by the theory of p-wavelet packets (see [20,22]), and thus gives the Riesz basis of the wavelet subspaces. Applying the splitting trick to the wavelet space W, we can divide W into p-subspaces as follows:

Theorem 3.3. *Let $\{\omega_n:n\in\mathbf{Z}^+\}$ be the p-wavelet packets associated with the scaling function $\varphi(x)$. Then the set of functions*

$$\Big\{\omega_n(x\ominus k):p^{j-1}\le n\le p^j-1,k\in\mathbf{Z}^+\Big\}$$

forms a Riesz basis of $\mathcal{D}^jW$.

Proof. We prove the theorem by induction on j. Since $\{\omega_n:1\le n\le p-1\}$ are the basic p-wavelets related to the Walsh functions and the family $\{\omega_n(x\ominus k):1\le n\le p-1,k\in\mathbf{Z}^+\}$ form a Riesz basis of W. Therefore, the claim is true for $j=1$. Assume that it holds for $j(j\ge 1)$, then

$$\Big\{\omega_n(x\ominus k):p^{j-1}\le n\le p^j-1,k\in\mathbf{Z}^+\Big\}$$

constitutes a Riesz basis of $\mathcal{D}^jW$.

Using Splitting Lemma 3.2 for the case $j+1$, we get

$$\Big\{\omega_{pn}(x\ominus k),\omega_{pn+1}(x\ominus k),...,\omega_{pn+p-1}(x\ominus k):p^{j-1}\le n\le p^j-1,k\in\mathbf{Z}^+\Big\}$$

form a Riesz basis of $\mathcal{D}^{j+1}W$. So $\Big\{\omega_n(x\ominus k):p^j\le n\le p^{j+1}-1,k\in\mathbf{Z}^+\Big\}$ form a Riesz basis of $\mathcal{D}^{j+1}W$.

In the next two theorems, we provide various ways to construct Riesz basis of $L^2(\mathbf{R}^+)$ which is extracted out from the p-wavelet packets $\mathcal{P}$ (see Eq.(2.12)).

Theorem 3.4. *For each fixed* $j > 0, k \in \mathbf{Z}^+$, *the family of functions*

$$\left\{\omega_n(p^\ell x \ominus k) : p^{j-1} \le n \le p^j - 1, \ell \in \mathbf{Z}, k \in \mathbf{Z}^+\right\} \tag{3.5}$$

forms a Riesz basis of $L^2(\mathbf{R}^+)$.

Proof. Since $\{\omega_n(x \ominus k) : p^{j-1} \le n \le p^j - 1, k \in \mathbf{Z}^+\}$ forms a Riesz basis of $\mathcal{D}^j W$. By Theorem 3.3, for each $\ell \in \mathbf{Z}$ the set of functions

$$\left\{\omega_n(p^\ell x \ominus k) : p^{j-1} \le n \le p^j - 1, k \in \mathbf{Z}^+\right\} \tag{3.6}$$

constitutes Riesz basis of $\mathcal{D}^{j+\ell} W$. Also, for each fixed $j > 0$, $L^2(\mathbf{R}^+) = \bigoplus_{\ell \in \mathbf{Z}} \mathcal{D}^{j+\ell} W$, therefore, the set of functions $\{\omega_n(p^\ell x \ominus k) : p^{j-1} \le n \le p^j - 1, \ell \in \mathbf{Z}, k \in \mathbf{Z}^+\}$ forms a Riesz basis of $L^2(\mathbf{R}^+)$.

It is clear from the above construction that the Riesz basis for $L^2(\mathbf{R}^+)$ varies with respect to the integer $j > 0$. Thus, for the case $j = 1$, the sub-collection of

$$\mathcal{P} = \left\{\omega_n(p^\ell x \ominus k) : n \in \mathbf{Z}^+, \ell \in \mathbf{Z}, k \in \mathbf{Z}^+\right\}$$

gives us the known basis $\{\psi_\ell(p^\ell x \ominus k) : k \in \mathbf{Z}^+, \ell \in \mathbf{Z}\}$. Furthermore, in the above construction, the integer j is fixed and the dilation parameter ℓ varies over all integers. In order to construct the Riesz basis from $\mathcal{P}$, we allow j and ℓ in $\mathcal{P}$ to vary simultaneously.

Let $S = \{(j, \ell) : j \in \mathbf{N}, \ell \in \mathbf{Z}\}$ be a disjoint covering of $\mathbf{Z}$, then for each $r \in \mathbf{Z}$, there exist a unique pair $(j, \ell) \in S$ such that $r = j + \ell$. Moreover, this collection S is called an n-finite covering of $\mathbf{Z}$ if there exists a positive integer $J < \infty$ such that for all $(j, \ell) \in S$, $j \le J$.

Theorem 3.5. *Suppose* $\{\omega_n : n \in \mathbf{Z}^+\}$ *are the* p*-wavelet packets associated with the scaling function* $\varphi(x)$. *Then, the family of functions*

$$\left\{\omega_n(p^\ell x \ominus k) : p^{j-1} \le n \le p^j - 1, k \in \mathbf{Z}^+, (j, \ell) \in S\right\} \tag{3.7}$$

constitutes Riesz basis of $L^2(\mathbf{R}^+)$ *if* S *is an* n*-finite cover of* $\mathbf{Z}$.

Proof. Since for each fixed j, family of the functions

$$\left\{\omega_n(x \ominus k) : p^{j-1} \le n \le p^j - 1, k \in \mathbf{Z}^+\right\}$$

constitutes Riesz basis of $\mathcal{D}^j W$ and hence $\{\omega_n(p^\ell x \ominus k) : p^{j-1} \le n \le p^j - 1, k \in \mathbf{Z}^+\}$ form a Riesz basis of $\mathcal{D}^{j+\ell} W$.

For each fixed j, let $S_j = \{(j,\ell) : (j,\ell) \in S\}$. Since S is n-finite, so S can be written as a finite disjoint union of S_j. Therefore, by this property of S, we have

$$L^2(\mathbf{R}^+) = \bigoplus_{j \in \mathbf{N}} \bigoplus_{(j,\ell)\in S_j} \mathcal{D}^{j+\ell} W. \tag{3.8}$$

Thus for each fixed $j > 0$, the family of functions

$$\left\{ \omega_n(p^\ell x \ominus k) : p^{j-1} \le n \le p^j - 1, k \in \mathbf{Z}^+, (j,\ell) \in S_j \right\}$$

form a Riesz basis of $\bigoplus_{(j,\ell)\in S_j} \mathcal{D}^{j+\ell} W$. Using (3.8), it follows that

$$\left\{ \omega_n(p^\ell x \ominus k) : p^{j-1} \le n \le p^j - 1, k \in \mathbf{Z}^+, (j,\ell) \in S \right\}$$

form a Riesz basis of $L^2(\mathbf{R}^+)$.

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