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Order of Magnitude of Multiple Fourier Coefficients

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Abstract. The order of magnitude of multiple Fourier coefficients of complex valued functions of generalized bounded variations like $(\Lambda^1,\cdots,\Lambda^N)BV^{(p)}$ and $r-BV$, over $[0,2\pi]^N$, are estimated.

Key Words: Order of magnitude of multiple Fourier coefficients, function of $(\Lambda^1,\cdots,\Lambda^N)BV^{(p)}$, $r-BV$ and $\mathrm{Lip}(p;\alpha_1,\cdots,\alpha_N)$.

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1 Introduction

Recently, V. Fülöp and F. Móricz [3] studied the order of magnitude of multiple Fourier coefficients of functions in BV ($\overline{\mathbf{T}}^N$), where $\mathbf{T}=[0,2\pi)$, in the sense of Vitali and Hardy. Here, we have generalized these results by estimating the order of magnitude of multiple Fourier coefficients of complex valued functions in $(\Lambda^1,\cdots,\Lambda^N)BV^{(p)}$, $r-BV$ and $\mathrm{Lip}(p;\alpha_1,\cdots,\alpha_N)$ over $\overline{\mathbf{T}}^N$.

Definition 1.1. For a given $f\in L^p(\overline{\mathbf{T}}^2)$, $1\le p<\infty$, the p-integral modulus of continuity of f is defined as

$$\omega^{(p)}(f;\delta_1,\delta_2)=\sup\left\{\left(\frac{1}{4\pi^2}\iint_{\overline{\mathbf{T}}^2}|\Delta f(x,y;h,k)|^p dxdy\right)^{1/p}:\ 0<h\le\delta_1,\ 0<k\le\delta_2\right\},$$

where

$$\Delta f(x,y;h,k)=f(x+h,y+k)-f(x,y+k)-f(x+h,y)+f(x,y).$$

For every $f\in L^p(\overline{\mathbf{T}}^2)$, $\omega^{(p)}(f;\delta_1,\delta_2)\to 0$ as $\max\{\delta_1,\delta_2\}\to 0$.

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For $p \geq 1$ and $\alpha_1, \alpha_2 \in (0,1]$, we say that $f \in \text{Lip}(p;\alpha_1,\alpha_2)$ if

$$\omega^{(p)}(f;\delta_1,\delta_2) = \mathcal{O}(\delta_1^{\alpha_1}\delta_2^{\alpha_2}) \ \text{ as } \ \delta_1 \text{and } \delta_2 \to 0.$$

For $p=\infty$, we write $\omega(f;\delta_1,\delta_2)$ for $\omega^{(\infty)}(f;\delta_1,\delta_2)$, Definition 1.1 gives the modulus of continuity of f and in that case the class $\text{Lip}(p;\alpha_1,\alpha_2)$ reduces to Lipschitz class $\text{Lip}(\alpha_1,\alpha_2)$.

Definition 1.2. Let $\mathbf{L}$ be the class of all non-decreasing sequences $\Lambda' = \{\lambda_n'\}$ $(n=1,2,\cdots)$ of positive numbers such that $\sum_n (\lambda_n')^{-1}$ diverges. For given $\bigwedge = (\Lambda^1,\Lambda^2)$, where $\Lambda^k = \{\lambda_n^k\} \in \mathbf{L}$ for $k=1,2$ and $p \geq 1$. A complex valued measurable function f defined on a rectangle $R := [a,b]\times[c,d]$ is said to be of p-(Λ^1,Λ^2)-bounded variation (that is, $f \in (\Lambda^1,\Lambda^2)BV^{(p)}(R)$), if

$$V_{\bigwedge_p}(f,R) = \sup_{P=P_1\times P_2} \Big(\sum_{i=1}^{m}\sum_{j=1}^{l} \frac{|\Delta f(x_i,y_j)|^p}{\lambda_i^1 \lambda_j^2} \Big)^{1/p} < \infty,$$

where

$$\Delta f(x_i,y_j) = \Delta f(x_i,y_j;\Delta x_i,\Delta y_j), \qquad \Delta x_i = x_{i+1} - x_i,$$
$$\Delta y_j = y_{j+1} - y_j, \qquad P_1 : a = x_0 < x_1 < x_2 < \cdots < x_m = b$$

and

$$P_2 : c = y_0 < y_1 < y_2 < \cdots < y_l = d.$$

If $f \in (\Lambda^1,\Lambda^2)BV^{(p)}(R)$ is such that the marginal functions $f(a,\cdot) \in \Lambda^2 BV^{(p)}([c,d])$ and $f(\cdot,c) \in \Lambda^1 BV^{(p)}([a,b])$ (refer [6]) for the definition of $\Lambda BV^{(p)}([a,b])$), then f is said to be of p-$(\Lambda^1,\Lambda^2)^*$-bounded variation over R (that is, $f \in (\Lambda^1,\Lambda^2)^* BV^{(p)}(R)$).

If $f \in (\Lambda^1,\Lambda^2)^* BV^{(p)}(R)$ then f is bounded and each of the marginal function $f(\cdot,t) \in \Lambda^1 BV^{(p)}([a,b])$ and $f(s,\cdot) \in \Lambda^2 BV^{(p)}([c,d])$, where $t \in [c,d]$ and $s \in [a,b]$ are fixed.

Note that, for $\Lambda^1 = \Lambda$ and $\Lambda^2 = \{1\}$ (that is, $\lambda_n^1 = \lambda_n$ and $\lambda_n^2 = 1$, $\forall n$) the class $(\Lambda^1,\Lambda^2)BV^{(p)}(R)$ and the class $(\Lambda^1,\Lambda^2)^* BV^{(p)}(R)$ reduce to the class $\Lambda BV^{(p)}(R)$ and the class $\Lambda^* BV^{(p)}(R)$ respectively; for $p=1$, we omit writing p, the class $(\Lambda^1,\Lambda^2)BV^{(p)}(R)$ and the class $(\Lambda^1,\Lambda^2)^* BV^{(p)}(R)$ reduce to the class $(\Lambda^1,\Lambda^2)BV(R)$ (Definition 2, [1]) and the class $(\Lambda^1,\Lambda^2)^* BV(R)$ respectively and for $p=1$ the class $\Lambda BV^{(p)}(R)$ and the class $\Lambda^* BV^{(p)}(R)$ reduce to the class $\Lambda BV(R)$ and the class $\Lambda^* BV(R)$ respectively (Definition 3, [2]). Moreover, for $\Lambda^1 = \Lambda^2 = \{1\}$ and for $p=1$ the class $(\Lambda^1,\Lambda^2)BV^{(p)}(R)$ and the class $(\Lambda^1,\Lambda^2)^* BV^{(p)}(R)$ reduces to the class $BV_V(R)$ (bounded variation in the sense of Vitali) and the class $BV_H(R)$ (bounded variation in the sense of Hardy) respectively.

Observe that the characteristic function of $E = \{(x,y); x \in [0,1] \text{ and } y \in [0,1-x]\}$ is in $\Lambda BV^{(p)}([0,1]^2)$ if

$$\sum_n \Big(\frac{1}{\lambda_n}\Big)^2 < \infty. \tag{1.1}$$

If Λ satisfies (1.1), the requirement of measurability cannot be omitted from Definition 1.2, otherwise the class $\Lambda BV^{(p)}$ would include functions which are not Lebesgue measurable. Even under the assumption of measurability, Dyachenko and Waterman (Proposition 1, [2]) proved that there exists a $f\in\Lambda BV(R)$ which is everywhere discontinuous.

Definition 1.3. For a given positive integer r, a complex valued function f defined on a rectangle $R:=[a,b]\times[c,d]$ is said to be of r-bounded variation (that is, $f\in r-BV(R)$) if the following two conditions are satisfied:

(i)
$$V_r(f,R)=\sup_{P=P_1\times P_2} V_r(f,R,P)<\infty,$$

where
$$V_r(f,R,P)=\Big(\sum_{i=1}^{m-r}\sum_{j=1}^{n-r}|\Delta^r f(x_i,y_j)|\Big),$$

P, P_1, P_2, $\Delta f(x_i,y_j)$ are defined in Definition 1.2 and
$$\Delta^k f(x_i,y_j)=\Delta^{k-1}(\Delta f(x_i,y_j)),\quad k\geq 2,$$

so that
$$\Delta^r f(x_i,y_j)=\sum_{s=1}^{r}\sum_{t=1}^{r}(-1)^{s+t}\tbinom{r}{s}\tbinom{r}{t}f(x_{i+r-s},y_{j+r-t}).$$

(ii) The marginal functions $f(\cdot,c)\in r-BV([a,b])$ and $f(a,\cdot)\in r-BV([c,d])$.

It is easy to prove that $f\in r-BV(R)$ implies f is bounded on R, $BV_H(R)\subset r-BV(R)$ and each of the marginal functions $f(\cdot,y_0)\in r-BV([a,b])$ and $f(x_0,\cdot)\in r-BV([c,d])$ (refer to (Definition 4, pp. 115, [6]) for the definition of $r-BV[a,b]$), where $y_0\in[c,d]$ and $x_0\in[a,b]$ are fixed.

Definition 1.4. A function f defined on the rectangle $R:=[a,b]\times[c,d]$ is said to be absolutely continuous (that is, $f\in AC(R)$) if the following two conditions are satisfied:

(i) Given $\epsilon>0$, there exists $\delta=\delta(\epsilon)>0$ such that
$$\sum_{\{R_k\}\in R}|f(a_k,c_k)-f(b_k,c_k)-f(a_k,d_k)+f(b_k,d_k)|<\epsilon,$$

whenever $\{R_k:=[a_k,b_k]\times[c_k,d_k]\}_{k=1,2,\cdots,}$ is a infinite collection of pairwise non-overlapping sub-rectangles of R with
$$\sum_{\{R_k\}\in R}(b_k-a_k)(d_k-c_k)<\delta.$$

(ii) The marginal functions $f(\cdot,c)\in AC([a,b])$ and $f(a,\cdot)\in AC([c,d])$.

An absolutely continuous function f on R is uniformly continuous and each of the marginal functions $f(\cdot,y_0)\in AC([a,b])$ and $f(x_0,\cdot)\in AC([c,d])$, where $y_0\in[c,d]$ and $x_0\in[a,b]$ are fixed.

2 New results for functions of two variables

For any $\mathbf{x}=(x_1,x_2)\in\overline{\mathbf{T}}^2$ and $\mathbf{k}=(k_1,k_2)\in\mathbf{Z}^2$, denote their scalar product by $\mathbf{k}\cdot\mathbf{x}=k_1x_1+k_2x_2$.

For any $f\in L^1(\overline{\mathbf{T}}^2)$, where f is 2π-periodic in each variable, its Fourier series is defined as

$$f(\mathbf{x})\sim\sum_{\mathbf{k}\in\mathbf{Z}^2}\hat{f}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x})},$$

where

$$\hat{f}(\mathbf{k})=\frac{1}{(2\pi)^2}\int_{\overline{\mathbf{T}}^2}f(\mathbf{x})e^{-i(\mathbf{k}\cdot\mathbf{x})}d\mathbf{x}$$

denotes the $\mathbf{k}^{th}$ Fourier coefficient of f.

We prove the following theorems.

Theorem 2.1. *If $f\in(\Lambda^1,\Lambda^2)BV^{(p)}(\overline{\mathbf{T}}^2)\cap L^p(\overline{\mathbf{T}}^2)$ $(p\geq 1)$ and $\boldsymbol{k}=(k_1,k_2)\in\mathbf{Z}^2$ is such that $k_1\cdot k_2\neq 0$, then*

$$|\hat{f}(\boldsymbol{k})|=\mathcal{O}\bigg(\frac{1}{(\sum_{i=1}^{|k_1|}\sum_{j=1}^{|k_2|}\frac{1}{\lambda_i^1\lambda_j^2})^{1/p}}\bigg).$$

Theorem 2.1 generalizes the result (Theorem 1 (iii), [5]) for functions of two variables.

Corollary 2.1. If $f\in(\Lambda^1,\Lambda^2)^*BV^{(p)}(\overline{\mathbf{T}}^2)$ $(p\geq 1)$ and $\mathbf{k}=(k_1,k_2)\in\mathbf{Z}^2$ is such that $k_1\cdot k_2\neq 0$, then

$$|\hat{f}(\mathbf{k})|=\mathcal{O}\bigg(\frac{1}{(\sum_{i=1}^{|k_1|}\sum_{j=1}^{|k_2|}\frac{1}{\lambda_i^1\lambda_j^2})^{1/p}}\bigg).$$

Theorem 2.2. *If $f\in r-BV(\overline{\mathbf{T}}^2)$ and $\boldsymbol{k}=(k_1,k_2)\in\mathbf{Z}^2$ is such that $k_1\cdot k_2\neq 0$, then*

$$|\hat{f}(\boldsymbol{k})|=\mathcal{O}\Big(\frac{1}{|k_1\cdot k_2|}\Big).$$

Theorem 2.3. *If $f\in\mathrm{Lip}(p;\alpha_1,\alpha_2)$ over $\overline{\mathbf{T}}^2$ $(p\geq 1,\ \alpha_1,\ \alpha_2\in(0,1])$ and $\boldsymbol{k}=(k_1,k_2)\in\mathbf{Z}^2$ is such that $k_1\cdot k_2\neq 0$, then*

$$|\hat{f}(\boldsymbol{k})|=\mathcal{O}\bigg(\frac{1}{|k_1|^{\alpha_1}|k_2|^{\alpha_2}}\bigg).$$

Theorem 2.4. *If $f\in AC(\overline{\mathbf{T}}^2)$ and $\boldsymbol{k}=(k_1,k_2)\in\mathbf{Z}^2$ is such that $k_1\cdot k_2\neq 0$, then*

$$|\hat{f}(\boldsymbol{k})|=o\Big(\frac{1}{|k_1\cdot k_2|}\Big).$$

3 Proof of the results

Proof of Theorem 2.1: Since

$$\hat{f}(k_1,k_2)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x_1,x_2)e^{-ik_1x_1}e^{-ik_2x_2}dx_1dx_2,$$

we have

$$\begin{aligned}4|\hat{f}(k_1,k_2)|=\frac{1}{4\pi^2}\Big|\int_0^{2\pi}\int_0^{2\pi}\Big(f\Big(x_1+\frac{\pi}{k_1},x_2+\frac{\pi}{k_2}\Big)-f\Big(x_1,x_2+\frac{\pi}{k_2}\Big)\\ -f\Big(x_1+\frac{\pi}{k_1},x_2\Big)+f(x_1,x_2)\Big)e^{-ik_1x_1}e^{-ik_2x_2}dx_1dx_2\Big|.\end{aligned}$$

Because of the periodicity of f in each variable, we get

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi}|\Delta f_{r_1r_2}(x_1,x_2)|dx_1dx_2=\int_0^{2\pi}\int_0^{2\pi}\Big|f\Big(x_1+\frac{\pi}{k_1},x_2+\frac{\pi}{k_2}\Big)\\ -f\Big(x_1,x_2+\frac{\pi}{k_2}\Big)-f\Big(x_1+\frac{\pi}{k_1},x_2\Big)+f(x_1,x_2)\Big|dx_1dx_2,\end{aligned}$$

where

$$\begin{aligned}\Delta f_{r_1r_2}(x_1,x_2)=f\Big(x_1+\frac{r_1\pi}{k_1},x_2+\frac{r_2\pi}{k_2}\Big)-f\Big(x_1+\frac{(r_1-1)\pi}{k_1},x_2+\frac{r_2\pi}{k_2}\Big)\\ -f\Big(x_1+\frac{r_1\pi}{k_1},x_2+\frac{(r_2-1)\pi}{k_2}\Big)+f\Big(x_1+\frac{(r_1-1)\pi}{k_1},x_2+\frac{(r_2-1)\pi}{k_2}\Big),\end{aligned}$$

for any $r_1,r_2\in\mathbf{Z}$. Therefore

$$|\hat{f}(k_1,k_2)|\le\frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{2\pi}|\Delta f_{r_1r_2}(x_1,x_2)|dx_1dx_2. \tag{3.1}$$

Dividing both sides by $\lambda^1_{r_1}\lambda^2_{r_2}$ and then summing over $r_1=1$ to $|k_1|$ and $r_2=1$ to $|k_2|$, we get

$$|\hat{f}(k_1,k_2)|\Big(\sum_{r_1=1}^{|k_1|}\sum_{r_2=1}^{|k_2|}\frac{1}{\lambda^1_{r_1}\lambda^2_{r_2}}\Big)\le\frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{2\pi}\Big(\sum_{r_1=1}^{|k_1|}\sum_{r_2=1}^{|k_2|}\frac{|\Delta f_{r_1r_2}(x_1,x_2)|}{(\lambda^1_{r_1}\lambda^2_{r_2})^{\frac{1}{p}+\frac{1}{q}}}\Big)dx_1dx_2,$$

where q is the index conjugate to p.

Applying Hölder's inequality on the right side of the above inequality, we have

$$\begin{aligned}&|\hat{f}(k_1,k_2)|\Big(\sum_{r_1=1}^{|k_1|}\sum_{r_2=1}^{|k_2|}\frac{1}{\lambda^1_{r_1}\lambda^2_{r_2}}\Big)\\ \le&\frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{2\pi}\Big(\sum_{r_1=1}^{|k_1|}\sum_{r_2=1}^{|k_2|}\frac{|\Delta f_{r_1r_2}(x_1,x_2))|^p}{\lambda^1_{r_1}\lambda^2_{r_2}}\Big)^{\frac{1}{p}}\Big(\sum_{r_1=1}^{|k_1|}\sum_{r_2=1}^{|k_2|}\frac{1}{\lambda^1_{r_1}\lambda^2_{r_2}}\Big)^{\frac{1}{q}}dx_1dx_2.\end{aligned}$$

Hence

$$|\hat{f}(k_1,k_2)|\Big(\sum_{r_1=1}^{|k_1|}\sum_{r_2=1}^{|k_2|}\frac{1}{\lambda^1_{r_1}\lambda^2_{r_2}}\Big)^{\frac{1}{p}} \le \frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{2\pi}\Big(\sum_{r_1=1}^{|k_1|}\sum_{r_2=1}^{|k_2|}\frac{|\Delta f_{r_1r_2}(x_1,x_2)|^p}{\lambda^1_{r_1}\lambda^2_{r_2}}\Big)^{\frac{1}{p}}dx_1dx_2$$
$$\le \frac{1}{4}V_{\Lambda_p}(f,\overline{\mathbf{T}}^2).$$

This completes the proof.

Proof of Corollary 2.1: Observe that $f\in(\Lambda^1,\Lambda^2)^*BV^{(p)}(\overline{\mathbf{T}}^2)$ then f is bounded and $(\Lambda^1,\Lambda^2)^*BV^{(p)}(\overline{\mathbf{T}}^2)\subset(\Lambda^1,\Lambda^2)BV^{(p)}(\overline{\mathbf{T}}^2)$.

Hence the corollary follows.

Proof of Theorem 2.2: Proceeding as in the proof of Theorem 2.1, from (3.1) it follows that

$$|\hat{f}(k_1,k_2)|\le\Big(\frac{1}{16\pi^2}\Big)\int_0^{2\pi}\int_0^{2\pi}|\Delta f_{r_1r_2}(x_1,x_2)|dx_1dx_2.$$

Similarly, we get

$$|\hat{f}(k_1,k_2)|\le\Big(\frac{1}{16\pi^2}\Big)^r\int_0^{2\pi}\int_0^{2\pi}|\Delta^r f_{r_1r_2}(x_1,x_2)|dx_1dx_2.$$

Summing the above inequality over $r_1=1$ to $|k_1|-r$ and $r_2=1$ to $|k_2|-r$, we get

$$(|k_1|-r)(|k_2|-r)|\hat{f}(k_1,k_2)|\le\Big(\frac{1}{16\pi^2}\Big)^r\int_0^{2\pi}\int_0^{2\pi}\sum_{r_1=1}^{|k_1|-r}\sum_{r_2=1}^{|k_2|-r}|\Delta^r f_{r_1r_2}(x_1,x_2)|dx_1dx_2.$$

This together with

$$\sum_{r_1=1}^{|k_1|-r}\sum_{r_2=1}^{|k_2|-r}|\Delta^r f_{r_1r_2}(x_1,x_2)|\le V_r(f;[0,2\pi]^2),$$

$|k_1|\approx|k_1|-r$ and $|k_2|\approx|k_2|-r$ implies

$$|\hat{f}(k_1,k_2)|=\mathcal{O}\Big(\frac{1}{|k_1k_2|}\Big).$$

Proof of Theorem 2.3: Proceeding as in the proof of Theorem 2.1, one gets (3.1). By applying Hölder's inequality to the right side of (3.1), we obtain

$$|\hat{f}(k_1,k_2)|=\mathcal{O}(1)\Big(\int_0^{2\pi}\int_0^{2\pi}|\Delta f_{r_1r_2}(x_1,x_2)|^p dx_1dx_2\Big)^{1/p}.$$

Hence the result follows.

Proof of Theorem 2.4: Theorem 2.4 can be proved in a similar way to the proof of Theorem 2.1.

4 Extension of the results for functions of several variables

For any $\mathbf{x}=(x_1,\cdots,x_N)\in\overline{\mathbf{T}}^N$ and $\mathbf{k}=(k_1,\cdots,k_N)\in\mathbf{Z}^N$ denotes their scalar product by

$$\mathbf{k}\cdot\mathbf{x}=k_1x_1+\cdots+k_Nx_N.$$

For $f\in L^1(\overline{\mathbf{T}}^N)$, where f is complex valued function which is 2π-periodic in each variable, its Fourier series is defined as

$$f(\mathbf{x})\sim\sum_{\mathbf{k}\in\mathbf{Z}^N}\hat{f}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x})},$$

where

$$\hat{f}(\mathbf{k})=\frac{1}{(2\pi)^N}\int_{\overline{\mathbf{T}}^N}f(\mathbf{x})e^{-i(\mathbf{k}\cdot\mathbf{x})}d\mathbf{x}$$

denotes the $\mathbf{k}^{th}$ Fourier coefficient of f.

Given $\mathbf{x}=(x_1,\cdots,x_N)\in\mathbf{T}^N$ and $\mathbf{h}=(h_1,\cdots,h_N)\in\mathbf{T}^N$, define

$$\begin{aligned}\Delta f(\mathbf{x};\mathbf{h})&=T_{\mathbf{h}}f(\mathbf{x})-f(\mathbf{x})=\Delta f(x_1,\cdots,x_N;h_1,\cdots,h_N)\\&=\sum_{\eta_1=0}^{1}\cdots\sum_{\eta_N=0}^{1}(-1)^{\eta_1+\cdots+\eta_N}f(x_1+\eta_1h_1,\cdots,x_N+\eta_Nh_N).\end{aligned}$$

For $p\geq1$, the p-integral modulus of continuity of a function $f\in L^p(\overline{\mathbf{T}}^N)$ is defined as

$$\omega^{(p)}(f;\delta_1,\cdots,\delta_N)=\sup\Big\{\Big(\frac{1}{(2\pi)^N}\int_{\overline{\mathbf{T}}^N}|\Delta f(\mathbf{x};\mathbf{h})|^pd\mathbf{x}\Big)^{1/p}:\ 0<h_j\leq\delta_j,\ j=1,\cdots,N\Big\}.$$

Obviously, $\omega^{(p)}(f;\delta_1,\cdots,\delta_N)\to0$ as $\max\{\delta_1,\cdots,\delta_N\}\to0$.

For $p=\infty$, we omit writing p, one gets $\omega(f;\delta_1,\cdots,\delta_N)$, the modulus of continuity of f.

A function $f\in L^p(\overline{\mathbf{T}}^N)$ is said to belongs to $\mathrm{Lip}(p;\alpha_1,\cdots,\alpha_N)$, the Lipschitz class in the mean of order p, if $\omega^{(p)}(f;\delta_1,\cdots,\delta_N)=\mathcal{O}(\delta_1^{\alpha_1},\cdots,\delta_N^{\alpha_N})$ as $\delta_i\to0$, $\forall i=1,\cdots,N$.

For $p=\infty$, the class $\mathrm{Lip}(p;\alpha_1,\cdots,\alpha_N)$ reduces to the Lipschitz class $\mathrm{Lip}(\alpha_1,\cdots,\alpha_N)$. Obviously, $\mathrm{Lip}(\alpha_1,\cdots,\alpha_N)\subset\mathrm{Lip}(p;\alpha_1,\cdots,\alpha_N)$.

For given $\bigwedge=(\Lambda^1,\cdots,\Lambda^N)$, where $\Lambda^1,\cdots,\Lambda^N\in\mathbf{L}$ and $p\geq1$. A function $f:\overline{\mathbf{T}}^N\to\mathbf{C}$ is said to be of $p-(\Lambda^1,\cdots,\Lambda^N)$ bounded variation (that is, $f\in(\Lambda^1,\cdots,\Lambda^N)BV^{(p)}(\overline{\mathbf{T}}^N)$) if

$$V_{\Lambda_p}(f,\overline{\mathbf{T}}^N)=\sup_P\Big(\sum_{r_1=1}^{s_1}\cdots\sum_{r_N=1}^{s_N}\frac{|\Delta f(x_1^{r_1-1},\cdots,x_N^{r_N-1};h_1^{r_1},\cdots,h_N^{r_N})|^p}{\lambda^1_{r_1}\lambda^2_{r_2}\cdots\lambda^N_{r_N}}\Big)^{1/p}<\infty,$$

where the supremum is extended over all partitions $P=P_1\times P_2\times\cdots\times P_N$ of the closed cube $\overline{\mathbf{T}}^N$, $P_j=\{0=x_j^0<x_j^1<\cdots<x_j^{s_j}=2\pi\}$ and $s_j\geq1$; $r_j=1,2,\cdots,s_j$; $h_j^{r_j}=x_j^{r_j}-x_j^{r_j-1}$; $j=1,2,\cdots,N$.

Moreover, a function $f\in(\Lambda^1,\cdots,\Lambda^N)BV^{(p)}(\overline{\mathbf{T}}^N)$ is said to be of $p-(\Lambda^1,\cdots,\Lambda^N)^*$ bounded variation (that is, $f\in(\Lambda^1,\cdots,\Lambda^N)^*BV^{(p)}(\overline{\mathbf{T}}^N)$), if for each of its marginal functions

$$f(x_1,\cdots,x_{i-1},0,x_{i+1},\cdots,x_N)\in(\Lambda^1,\cdots,\Lambda^{i-1},\Lambda^{i+1},\cdots,\Lambda^N)^*BV^{(p)}(\overline{\mathbf{T}}^N(0_i)),$$

$\forall i=1,2,\cdots,N$, where

$$\overline{\mathbf{T}}^N(0_i)=\Big\{(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_N)\in\overline{\mathbf{T}}^{N-1}:\ 0\le x_k\le 2\pi,\ \text{for}\ \ k=1,\cdots,i-1,i+1,\cdots,N\Big\}.$$

It is easy to prove that $f\in(\Lambda^1,\cdots,\Lambda^N)^*BV^{(p)}(\overline{\mathbf{T}}^N)$ implies it is bounded. In particular f is Lebesgue integrable over $\overline{\mathbf{T}}^N$.

Similarly, we say that a function $f\in r-BV(\overline{\mathbf{T}}^N)$ if the following two conditions are satisfied:

(i)
$$V_r(f,\overline{\mathbf{T}}^N)=\sup_P\bigg(\sum_{r_1=1}^{s_1}\cdots\sum_{r_N=1}^{s_N}|\Delta^r f(x_1^{r_1-1},\cdots,x_N^{r_N-1};h_1^{r_1},\cdots,h_N^{r_N})|\bigg)<\infty,$$

where, for $k\ge 2$,

$$\Delta^k f(x_1^{r_1-1},\cdots,x_N^{r_N-1};h_1^{r_1},\cdots,h_N^{r_N})=\Delta^{k-1}\big(\Delta f(x_1^{r_1-1},\cdots,x_N^{r_N-1};h_1^{r_1},\cdots,h_N^{r_N})\big).$$

(ii) Each of its marginal functions

$$f(x_1,x_2,\cdots,x_{i-1},0,x_{i+1},\cdots,x_N)\in r-BV(\overline{\mathbf{T}}^N(0_i)),\quad \forall i=1,\cdots,N.$$

A function $f=f(x_1,\cdots,x_N)$ is said to be absolutely continuous over $\overline{\mathbf{T}}^N$ (that is, $f\in AC(\overline{\mathbf{T}}^N)$) if the following two conditions are satisfied [4]:

(i) Given $\epsilon>0$, there exists $\delta=\delta(\epsilon)>0$ such that

$$\sum_{R_k\in R'}|\Delta f(c_1^k,\cdots,c_N^k;h_1^k,\cdots,h_N^k)|<\epsilon$$

with $h_j^k=d_j^k-c_j^k$, $j=1,2,\cdots,N$; whenever

$$R'=\big\{R_k=[c_1^k,d_1^k]\times[c_2^k,d_2^k]\times\cdots\times[c_N^k,d_N^k]\big\}$$

is a finite collection of pairwise non-overlapping sub-rectangles of $\overline{\mathbf{T}}^N$ with

$$\sum_{R_k\in R'}\prod_{j=1}^N(d_j^k-c_j^k)<\delta.$$

(ii) Each of its marginal functions

$$f(x_1,x_2,\cdots,x_{i-1},0,x_{i+1},\cdots,x_N)\in AC(\overline{\mathbf{T}}^N(0_i)),\quad \forall i=1,\cdots,N.$$

Now, we extend the above mentioned theorems of Section 2 for higher dimensional space as following:

Theorem 4.1. *If $f\in(\Lambda^1,\cdots,\Lambda^N)BV^{(p)}(\overline{\mathbf{T}}^N)\cap L^p(\overline{\mathbf{T}}^N)$ $(p\geq 1)$ and $\boldsymbol{k}=(k_1,\cdots,k_N)\in\mathbf{Z}^N$ is such that $k_1\cdots k_N\neq 0$, then*

$$|\hat{f}(\boldsymbol{k})|=\mathcal{O}\bigg(\frac{1}{\big(\sum_{r_1=1}^{|k_1|}\cdots\sum_{r_N=1}^{|k_N|}\frac{1}{\lambda^1_{r_1}\cdots\lambda^N_{r_N}}\big)^{1/p}}\bigg).$$

Obviously, Theorem 4.1 generalizes the result (Theorem, [3]).

Corollary 4.1. If $f\in(\Lambda^1,\cdots,\Lambda^N)^*BV^{(p)}(\overline{\mathbf{T}}^N)(p\geq 1)$ and $\mathbf{k}=(k_1,\cdots,k_N)\in\mathbf{Z}^N$ is such that $k_1\cdots k_N\neq 0$, then

$$|\hat{f}(\mathbf{k})|=\mathcal{O}\bigg(\frac{1}{\big(\sum_{r_1=1}^{|k_1|}\cdots\sum_{r_N=1}^{|k_N|}\frac{1}{\lambda^1_{r_1}\cdots\lambda^N_{r_N}}\big)^{1/p}}\bigg).$$

Theorem 4.2. *If $f\in r-BV(\overline{\mathbf{T}}^N)(r\geq 1)$ and $\boldsymbol{k}=(k_1,\cdots,k_N)\in\mathbf{Z}^N$ is such that $k_1\cdots k_N\neq 0$, then*

$$|\hat{f}(\boldsymbol{k})|=\mathcal{O}\Big(\frac{1}{|k_1\cdots k_N|}\Big).$$

Theorem 4.3. *If $f\in\mathrm{Lip}(p;\alpha_1,\cdots,\alpha_N)$ over $\overline{\mathbf{T}}^N$ $(p\geq 1,\ \alpha_1,\cdots,\alpha_N\in(0,1])$ and $\boldsymbol{k}=(k_1,\cdots,k_N)\in\mathbf{Z}^N$ is such that $k_1\cdots k_N\neq 0$, then*

$$|\hat{f}(\boldsymbol{k})|=\mathcal{O}\Big(\frac{1}{|k_1|^{\alpha_1}\cdots|k_N|^{\alpha_N}}\Big).$$

Theorem 4.4. *If $f\in AC(\overline{\mathbf{T}}^N)$ and $\boldsymbol{k}=(k_1,\cdots,k_N)\in\mathbf{Z}^N$ is such that $k_1\cdots k_N\neq 0$, then*

$$|\hat{f}(\boldsymbol{k})|=o\Big(\frac{1}{|k_1\cdots k_N|}\Big).$$

The above results of this section can be proved in the same way as we do in Section 2.

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