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Existence of Solution for a Coupled System of Fractional Integro-Differential Equations on an Unbounded Domain

Azizollah Babakhani*

Department of Mathematics, Faculty of Basic Science, Babol University of Technology, Bobol-Iran

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Abstract. We present the existence of solution for a coupled system of fractional integro-differential equations. The differential operator is taken in the Caputo fractional sense. We combine the diagonalization method with Arzela-Ascoli theorem to show a fixed point theorem of Schauder.

Key Words: Fractional derivative/integral, coupled system, Volterra integral equation, diagonalization method.

AMS Subject Classifications: 34LXX, 34GXX

1 Introduction

Fractional differential equations have gained considerable importance due to their various applications in visco-elasticity, electro-analytical chemistry and many physical problems [1–3]. So far there have been several fundamental works on the fractional derivative and fractional differential equations, written by Miller and Ross [4], Podlubny [5] and others in [6–8]. Mathematical aspects of fractional order differential equations have been discussed in details by many authors [9–17].

Consider the Volterra integral equation of the second kind of the form:

$$u(t)=\lambda\int_0^t K(t,s)ds+f(t),$$

where f, K are given functions, λ is a parameter and u is the solution. This equation arises very often in solving various problems of mathematical physics, especially in describing

*Corresponding author. *Email address:* `babakhani@nit.ac.ir` (A. Babakhani)

physical processes after effects [23,24]. Rabha W. Ibrahim and Shaher Momani [25] discussed the upper and lower bounds of solutions for fractional integral equations of the form:

$$u^m(t)=a(t)I^\alpha\{b(t)u(t)\}+f(t), \quad m\geq 1,$$

where $a(t)$, $b(t)$, $f(t)$ are real positive functions in $C([0,T],\mathbf{R})$ and $\alpha\in(0,1)$. Jinhua Wang et al. have investigated the existence and uniqueness of positive solution to nonzero boundary valued problem for a coupled system of nonlinear fractional equation and the reader is referted to [18]. A. Arara et al. [19] have considered a class of boundary valued problems involving Caputo fractional derivative on the half line by using the diagonalization process.

In this paper, we investigate the existence of solution for the coupled system of nonlinear fractional differential equation:

$$^cD^\alpha x(t)=tI^\gamma f(t,y(t))+f(t,y(t)), \quad t\in(0,\infty), \tag{1.1a}$$

$$^cD^\beta y(t)=tI^\eta g(t,x(t))+g(t,x(t)), \quad t\in(0,\infty), \tag{1.1b}$$

$$x(0)=x_0,\ y(0)=y_0,\ x(t) \text{ and } y(t) \text{ are bounded on } [0,\infty), \tag{1.1c}$$

where $1<\alpha,\beta\leq 2$, $^cD^\alpha$ and $^cD^\beta$ are the Caputo fractional derivatives, γ,η are real positive numbers, I^γ and I^η are Riemann-Liouville fractional integral and $f,g:[0,\infty)\times\mathbf{R}\to\mathbf{R}$ are given continuous functions.

2 Basic definitions and preliminaries

We begin in this section to recall some notations, definitions and results for fractional calculus which are used throughout this paper [4,5,7,20].

Let $\mathcal{J}_n=[0,n]$, $L^1(\mathcal{J}_n,\mathbf{R})$ denote the Banach space of functions $x:\mathcal{J}_n\to\mathbf{R}$ that are Lebesgue integrable with the norm

$$\|x\|_{L^1}=\int_0^n|x(t)|dt.$$

Recall that $C(\mathcal{J}_n,\mathbf{R})$ is the Banach space of continuous functions from $\mathcal{J}_n$ into $\mathbf{R}$ endowed with the uniform norm,

$$\|x\|_n=\max\{|x(t)|:t\in\mathcal{J}_n\},$$

and $C^2=C\times C$ is the Banach space of continuous functions from J_n into $\mathbf{R}$ endowed with the uniform norm

$$\|(x,y)\|_n=\max\{\|x\|_n,\ \|y\|_n:(x,y)\in C^2,\ t\in\mathcal{J}_n\}.$$

The Arzela-Ascoli theorem and Schauder fixed point theorem are recalled in the following. They play important roles in this article and the reader is referred to [20,21].

Theorem 2.1. (Arzela-Ascoli Theorem). *Let U be a compact metric space and Ω any subset of $C(U)$. Then the following statements are equivalent:*

(i) Ω *is a compact subset of the metric space* $C(U)$ *equipped with the uniform metric.*

(ii) Ω *is closed, bounded, and equicontinuous.*

Corollary 2.1. If $\{x_n\}$ is a sequence in $C[a,b]$ and the functions in $\{x_n\}$ form a bounded equicontinuous subset of $C[a,b]$, then $\{x_n\}$ has a subsequence which converges uniformly to some function in $C[a,b]$.

Theorem 2.2. (Schauder Fixed Point Theorem). *Let* K *be a closed, bounded and convex subset of a Banach space. If* $F:K\to K$ *is a compact mapping, then* F *has a fixed point.*

Definitions of Caputo and Riemann-Liouville fractional derivative/integral and their relation are given bellow.

Definition 2.1. For a function x defined on an interval $[a,b]$, the Riemann-Liouville fractional integral of f of order $\alpha>0$ is defined by

$$I^{\alpha}_{a^+}x(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)ds, \quad t>a,$$

and the Remann-Liouville fractional derivative of x of order $\alpha>0$ is defined by

$$D^{\alpha}_{a^+}x(t)=\frac{d^n}{dt^n}\{I^{n-\alpha}_{a^+}x(t)\},$$

where $n-1<\alpha\le n$, while the Caputo fractional derivative of x of order $\alpha>0$ is defined by

$${}^cD^{\alpha}_{a^+}x(t)=I^{n-\alpha}_{a^+}\{x^{(n)}(t)\}.$$

An important relation among Caputo fractional derivative and Riemann-Lioville fractional derivative is the following expression

$$D^{\alpha}_{a^+}x(t)={}^cD^{\alpha}_{a^+}x(t)+\sum_{j=1}^{n-1}\frac{x^{(j)}(a)}{\Gamma(j-\alpha+1)}(t-a)^{j-\alpha}. \tag{2.1}$$

We denote ${}^cD^{\alpha}_{a^+}x(t)$ by ${}^cD^{\alpha}_{a}x(t)$ and $I^{\alpha}_{a^+}x(t)$ by $I^{\alpha}_{a}x(t)$ simply. Further ${}^cD^{\alpha}_{0^+}x(t)$ and $I^{\alpha}_{0^+}x(t)$ are referred to ${}^cD^{\alpha}x(t)$ and $I^{\alpha}x(t)$, respectively.

Theorem 2.3. *Let* $y\in C^m([0,b],\mathbf{R})$ *and* $\alpha,\beta\in(m-1,m)$, $m\in\mathbf{N}$ *and* $x\in C^1([0,b],\mathbf{R})$*. Then*

(1) ${}^cD^{\alpha}I^{\alpha}x(t)=x(t)$;

(2) $I^{\alpha}I^{\beta}x(t)=I^{\alpha+\beta}x(t)$;

(3) $\lim_{t\to0^+}\{{}^cD^{\alpha}y(t)\}=\lim_{t\to0^+}\{I^{\alpha}y(t)\}$;

(4) ${}^cD^{\alpha}\lambda=0$, *where* λ *is a constant;*

(5) $I^{\alpha}\{{}^cD^{\alpha}y(t)\}=y(t)-\sum_{k=0}^{m-1}\frac{y^{(k)}(0)}{k!}t^k$.

Part (1) and (2) can be shown by using the semigroup properties of the Caputo derivative and Theorem 3.1 in [7]. For the proof of the last part, the reader is also referred to Theorem 2.22 in [10].

Proposition 2.1. Let $y\in\mathcal{C}([0,\infty),\mathbf{R})$, $n\in\mathbf{N}$ and $\alpha>0$, $\beta>0$, then

$$\begin{aligned}&(\mathbf{i})\;\; I^{\alpha}(ty(t))=tI^{\alpha}y(t)-\alpha I^{\alpha+1}y(t),\\&(\mathbf{ii})\;\; I^{\alpha}\{tI^{\beta}y(t)\}=tI^{\alpha+\beta}y(t)-\alpha I^{\alpha+\beta+1}y(t).\end{aligned}$$

Proof. (i) can be found in [pp. 53, [4]] and (ii) is an immediate consequence of (i) and Theorem 2.5 (2). $\square$

Lemma 2.1. (Lemma 2.22 [7]). *Let $\alpha>0$, then $I^{\alpha}({}^cD^{\alpha}x(t))=x(t)+c_0+c_1t+c_2t^2+\cdots+c_{r-1}t^{r-1}$ for some $c_i\in\mathbf{R}$, $i=0,1,\cdots,r-1,r=[\alpha]+1$.*

3 Main result

Let $n\in\mathbf{N}$, $\gamma>0$, $\eta>0$ and $1<\alpha,\beta\leq 2$. Consider the system of boundary value problem

$$ {}^cD^{\alpha}x(t)=tI^{\gamma}f(t,y(t))+f(t,y(t)),\quad t\in\mathfrak{I}_n, \tag{3.1a}$$
$$ {}^cD^{\beta}y(t)=tI^{\eta}g(t,x(t))+g(t,x(t)),\quad t\in\mathfrak{I}_n, \tag{3.1b}$$
$$ x(0)=x_0,\quad x'(n)=0,\quad y(0)=y_0,\quad y'(n)=0, \tag{3.1c}$$

where $f,g:[0,\infty)\times\mathbf{R}\to\mathbf{R}$ are given continuous functions.

In this section, we first discuss the system of nonlinear fractional differential equations (3.1) which has at least one solution.

Proposition 3.1. Assume that $x,y\in C([0,n],\mathbf{R})$, then the system of boundary valued problem (3.1) is equivalent to the following system of Volterra fractional integral equation

$$\begin{aligned}x(t)&=-c_0-c_1t+tI^{\alpha+\gamma}f(t,y(t))-\alpha I^{\alpha+\gamma+1}f(t,y(t))+I^{\alpha}f(t,y(t)),\\y(t)&=-c_0-c_1t+tI^{\beta+\eta}g(t,x(t))-\beta I^{\beta++\eta+1}g(t,y(t))+I^{\beta}g(t,x(t)).\end{aligned}$$

Proof. By integrating both sides of Eqs. (3.1a)-(3.1b) of order α, β, respectively and using Proposition 2.1 together with Lemma 2.1, the lemma is proved. $\square$

The next lemma shows that the solvability of the system of boundary value problem (3.1) is equivalent to the solvability of a system of the fractional integral equation.

Lemma 3.1. *Assume that $f,g\in C(\mathfrak{I}_n\times\mathbf{R},\mathbf{R})$ and consider the linear system of fractional order differential equation*

$$ {}^cD^{\alpha}x(t)=tI^{\gamma}f(t,y(t))+f(t,y(t)),\quad t\in\mathfrak{I}_n,\quad 1<\alpha\leq 2, \tag{3.2a}$$
$$ {}^cD^{\beta}y(t)=tI^{\eta}g(t,x(t))+g(t,x(t)),\quad t\in\mathfrak{I}_n,\quad 1<\beta\leq 2, \tag{3.2b}$$
$$ x(0)=x_0,\quad x'(n)=0,\quad y(0)=y_0,\quad y'(n)=0. \tag{3.2c}$$

Then $x,y\in C(\mathcal{I}_n,\mathbf{R})$ *is a solution* (3.2a)-(3.2c) *if and only if* x,y *is a solution of the system of the fractional integral equation:*

$$x(t)=x(0)+\int_0^n G_n(t,s)f(s,y(s))ds, \tag{3.3a}$$

$$y(t)=y(0)+\int_0^n H_n(t,s)g(s,x(s))ds, \tag{3.3b}$$

where $G_n(t,s)$, $H_n(t,s)$ *are the Green's functions defined by*

$$G_n(t,s)=\begin{cases} \dfrac{t(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}-\dfrac{\alpha(t-s)^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}+\dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\mathcal{G}(t,s), & 0\le s\le t\le n,\\ \mathcal{G}(t,s), & 0\le t\le s\le n, \end{cases} \tag{3.4}$$

with

$$\mathcal{G}(t,s)=\frac{-t(n-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}-\frac{n(n-s)^{\alpha+\gamma-2}}{\Gamma(\alpha+\gamma-1)}+\frac{\alpha t(n-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}-\frac{t(n-s)^{\alpha-2}}{\Gamma(\alpha-1)}, \tag{3.5a}$$

$$H_n(t,s)=\begin{cases} \dfrac{t(t-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta)}-\dfrac{\beta(t-s)^{\beta+\eta}}{\Gamma(\beta+\eta+1)}+\dfrac{(t-s)^{\beta-1}}{\Gamma(\beta)}+\mathcal{H}(t,s), & 0\le s\le t\le n,\\ \mathcal{H}(t,s), & 0\le t\le s\le n, \end{cases} \tag{3.5b}$$

where

$$\mathcal{H}(t,s)=\frac{-t(n-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta)}-\frac{n(n-s)^{\beta+\eta-2}}{\Gamma(\beta+\eta-1)}+\frac{\beta(n-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta)}-\frac{t(n-s)^{\beta-2}}{\Gamma(\beta-1)}. \tag{3.6}$$

Proof. Let $x,y\in C(\mathcal{I}_n,\mathbf{R})$ be a solution of Eqs. (3.2a) and (3.2b) respectively. In view of Proposition 3.1, we have

$$x(t)=tI^{\alpha+\gamma}f(t,y(t))-\alpha I^{\alpha+\gamma+1}f(t,y(t))+I^{\alpha}f(t,y(t))-c_0-c_1t, \tag{3.7a}$$

$$y(t)=tI^{\beta+\eta}g(t,x(t))-\beta I^{\beta++\eta+1}g(t,x(t))+I^{\beta}g(t,x(t))-d_0-d_1t, \tag{3.7b}$$

for arbitrary constants c_0 and c_1. By differentiating (3.2a) and (3.2b), we get

$$x'(t)=\int_0^t\Big\{\frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}+\frac{t(t-s)^{\alpha+\gamma-2}}{\Gamma(\alpha+\gamma-1)}\Big\}-\frac{\alpha(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}+\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big\}f(s,y(s))ds-c_1, \tag{3.8a}$$

$$y'(t)=\int_0^t\Big\{\frac{(t-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta)}+\frac{t(t-s)^{\beta+\eta-2}}{\Gamma(\beta+\eta-1)}-\frac{\beta(t-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta)}+\frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)}\Big\}g(s,x(s))ds-c_1. \tag{3.8b}$$

Hence using the boundary conditions (3.2c) into (3.8a) and (3.8b), we obtain $c_0=-x_0$, $d_0=-y_0$ and

$$c_1=\int_0^n\Big\{\frac{(n-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}+\frac{n(n-s)^{\alpha+\gamma-2}}{\Gamma(\alpha+\gamma-1)}-\frac{\alpha(n-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}+\frac{(n-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big\}f(s,y(s))ds,$$

$$d_1=\int_0^n\Big\{\frac{(n-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta)}+\frac{n(n-s)^{\beta+\eta-2}}{\Gamma(\beta+\eta-1)}-\frac{\beta(n-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta)}+\frac{(n-s)^{\beta-2}}{\Gamma(\beta-1)}\Big\}g(s,x(s))ds.$$

Substituting the values $c_0=-x_0$, $d_0=-y_0$ and the above values of c_1 and c_2 into (3.7a) and (3.7b), we get

$$\begin{aligned} x(t)=x_0-&\int_0^n\Big\{\frac{nt(n-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma-1)}-\frac{\alpha t(n-s)^{\alpha+\gamma}}{\Gamma(\alpha+\gamma)}-\frac{t(n-s)^{\alpha-2}}{\Gamma(\alpha-1)}\Big\}f(s,y(s))ds\\ &+\int_0^t\Big\{\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}-\frac{(t-s)^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}\Big\}f(s,y(s))ds, \qquad (3.9a)\\ y(t)=y_0-&\int_0^n\Big\{\frac{nt(n-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta-1)}-\frac{\beta t(n-s)^{\beta+\eta}}{\Gamma(\beta+\eta)}-\frac{t(n-s)^{\beta-2}}{\Gamma(\beta-1)}\Big\}g(s,x(s))ds\\ &+\int_0^t\Big\{\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}+\frac{t(t-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta)}-\frac{(t-s)^{\beta+\eta}}{\Gamma(\beta+\eta+1)}\Big\}g(s,x(s))(s)ds, \qquad (3.9b) \end{aligned}$$

and then

$$\begin{aligned} x(t)&=x_0+\int_t^n\mathcal{G}(t,s)f(s,y(s))ds+\int_0^t\Big\{\mathcal{G}(t,s)+\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{t(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\\ &\qquad-\frac{\alpha(t-s)^{\alpha+\gamma}}{\Gamma(\alpha+\gamma+1)}\Big\}f(s,y(s))ds\\ &=x(0)+\int_0^n G_n(t,s)f(s,y(s))ds,\\ y(t)&=y_0+\int_t^n\mathcal{H}(t,s)g(s,x(s))ds+\int_0^t\Big\{\mathcal{H}(t,s)+\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}+\frac{t(t-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta)}\\ &\qquad-\frac{\beta(t-s)^{\beta+\eta}}{\Gamma(\beta+\eta+1)}\Big\}g(s,x(s))ds\\ &=y(0)+\int_0^n H_n(t,s)g(s,x(s))ds, \end{aligned}$$

where $\mathcal{G}(t,s)$ and $\mathcal{H}(t,s)$ are as before.

Conversely, suppose that $x,y\in C(\mathcal{I}_n,\mathbf{R})$ satisfying in (3.3a)-(3.3b), then x, y satisfying in Eq. (3.9a) and Eq. (3.9b), thus $x(0)=x_0$, $y(0)=y_0$. By differentiating of Eq. (3.9a) and Eq. (3.9b), we have

$$\begin{aligned} x'(t)&=\int_t^n\frac{\partial\mathcal{G}(t,s)}{\partial t}f(s,y(s))ds-\mathcal{G}(t,t-0)+\mathcal{G}(t,t-0)\\ &\qquad+\int_0^t\Big\{\frac{\partial\mathcal{G}(t,s)}{\partial t}+\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}-\frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\\ &\qquad+\frac{t(t-s)^{\alpha+\beta-2}}{\Gamma(\alpha+\gamma-1)}-\frac{\alpha(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\Big\}f(s,y(s))ds\\ &=\int_0^t\Big\{\frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}+\frac{t(t-s)^{\alpha+\gamma-2}}{\Gamma(\alpha+\gamma-1)}-\frac{\alpha(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\Big\}f(s,y(s))ds\\ &\qquad-\int_0^n\Big\{\frac{(n-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}+\frac{(n-s)^{\alpha+\gamma-2}}{\Gamma(\alpha+\gamma-1)}-\frac{(n-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\Big\}f(s,y(s))ds\\ &\qquad+\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds-\int_0^n\frac{(n-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,y(s))ds. \end{aligned}$$

And similarly,

$$\begin{aligned}y'(t)=&\int_t^n \frac{\partial\mathcal{H}(t,s)}{\partial t}g(s,x(s))ds-\mathcal{H}(t,t-0)+\mathcal{H}(t,t-0)\\&+\int_0^t\Big\{\frac{\partial\mathcal{H}(t,s)}{\partial t}+\frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)}-\frac{(t-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta)}\\&+\frac{t(t-s)^{\beta+\eta-2}}{\Gamma(\beta+\eta-1)}-\frac{\beta(t-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta)}\Big\}g(s,x(s))ds\\=&\int_0^t\Big\{\frac{(t-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta)}+\frac{t(t-s)^{\beta+\eta-2}}{\Gamma(\beta+\eta-1)}-\frac{\beta(t-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta)}\Big\}g(s,x(s))ds\\&-\int_0^n\Big\{\frac{(n-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta)}+\frac{(n-s)^{\beta+\eta-2}}{\Gamma(\beta+\eta-1)}-\frac{(n-s)^{\beta+\eta-1}}{\Gamma(\beta+\eta)}\Big\}g(s,x(s))ds\\&+\int_0^t\frac{(t-s)^{\beta-2}}{\Gamma(\beta-1)}g(s,x(s))ds-\int_0^n\frac{(n-s)^{\beta-2}}{\Gamma(\beta-1)}g(s,x(s))ds.\end{aligned}$$

Thus $x'(n)=0$, $y'(n)=0$ and

$$\begin{aligned}{}^cD^\alpha x(t)=&{}^cD^{\alpha-1}x'(t)={}^cD^{\alpha-1}\int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}f(s,x(s))ds\\&+{}^cD^{\alpha-1}\int_0^t\Big\{\frac{(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}+\frac{t(t-s)^{\alpha+\gamma-2}}{\Gamma(\alpha+\gamma-1)}-\frac{\alpha(t-s)^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)}\Big\}f(s,x(s))ds\\=&{}^cD^{\alpha-1}\big\{I^{(\alpha-1)}f(t,y(t)\big\}+{}^cD^{\alpha-1}\Big\{\frac{d}{dt}\big[{}^cI^\alpha(tI^\alpha f(t,y(t))\big]\Big\}\\=&f(t,y(t))+tI^\alpha f(t,y(t)).\end{aligned}$$

Hence ${}^cD^\alpha x(t)-tI^\gamma f(t,y(t))=f(t,y(t))$ and similarly we have ${}^cD^\alpha y(t)-tI^\gamma g(t,x(t))=g(t,x(t))$. The proof is therefore complete. □

Remark 3.1. For each $t\in\mathcal{J}_n$, denote the functions

$$g_n(t)=\int_0^n|G_n(t,s)|ds,\quad h_n(t)=\int_0^n|H_n(t,s)|ds.$$

Then g_n,h_n are continuous on $\mathcal{J}_n$ and hence bounded. Let

$$\overset{*}{G}_n=\max\big\{g_n(t):t\in\mathcal{J}_n\big\},\quad \overset{*}{H}_n=\max\big\{h_n(t):t\in\mathcal{J}_n\big\}.$$

Theorem 3.1. *Assume that $f(t,\cdot)$ and $g(t,\cdot)$ are continuous on $[0,\infty)\times\mathbf{R}\to\mathbf{R}$ and there exist four continuous and nondecreasing functions $\omega,\sigma,v,\mu:[0,\infty)\to\mathbf{R}^+$ such that*

(H1) $|f(t,u)|\le\omega(t)\sigma(|u|)$, $|g(t,u)|\le v(t)\mu(|u|)$ for each $t\in[0,\infty)$ and $u\in\mathbf{R}$,

(H2) There exist two positive constants r, ρ such that

$$r\ge|x_0|+\overset{*}{\omega}_n\sigma(r)\overset{*}{G}_n,\quad \rho\ge|y_0|+\overset{*}{v}_n\mu(\rho)\overset{*}{H}_n,\tag{3.10}$$

where $\overset{*}{\omega}_n=\max\{\omega(t):t\in\mathcal{J}_n\}$ and $\overset{*}{v}_n=\max\{v(t):t\in\mathcal{J}_n\}$.

Then the system (1.1a)-(1.1c) *has at least one solution* $(x(t),y(t))$ *on* $[0,\infty)$ *such that* $|x(t)|\le r$ *and* $|y(t)|\le\rho$.

Before starting the proof of Theorem 3.1 we need to prove the following lemma.

Lemma 3.2. *Assume that* $f(t,\cdot)$ *and* $g(t,\cdot)$ *are continuous on* $[0,\infty)\times\mathbf{R}\to\mathbf{R}$ *and there exist four continuous and nondecreasing functions* $\omega,\sigma,\eta,\mu:[0,\infty)\to\mathbf{R}^+$ *such that* (H1)-(H2) *hold.*

Let

$$\mathcal{C}=C(\mathcal{I}_n,\mathbf{R})\times C(\mathcal{I}_n,\mathbf{R}) \quad \textit{and} \quad \Omega=\big\{(x,y)\in\mathcal{C}:\|(x,y)\|_n<\mathcal{R}\big\},$$

where

$$\|(x,y)\|_n=\max\big\{\|x(t)\|_n,\ \|y(t)\|_n,\ t\in\mathcal{I}_n\big\} \quad \textit{and} \quad \mathcal{R}=\max\{r,\rho\},$$

so that r, ρ *are the constants from* (H2). *Consider the operator* $F:\mathcal{C}\to\mathcal{C}$ *defied by*

$$(F(x,y))(t)=\big((Tx)(t),(Uy)(t)\big),$$

where

$$(Tx)(t)=x(0)+\int_0^n G_n(t,s)f(s,y(s))ds, \tag{3.11a}$$

$$(Uy)(t)=y(0)+\int_0^n H_n(t,s)g(s,x(s))ds. \tag{3.11b}$$

Then the following statements hold:

(I) Ω *is a closed, convex sub set of* $\mathcal{C}$,

(II) F *is continuous,*

(III) F *maps* Ω *into a bounded set of* $\mathcal{C}$,

(IV) F *maps* Ω *into an equicontinuous set of* $\mathcal{C}$,

(V) F *is completely continuous,*

(VI) $F(\Omega)\subset\Omega$.

Proof. (I) is clear so we try to prove (II). Let $\{(x_l,y_l)\}\in\mathcal{C}$ be a sequence such that $\{(x_l,y_l)\}\to(x,y)\in\mathcal{C}$ and let $L=\max\{\|x_l\|<L_1,\|y_l\|<L_2,\|x\|<L_3$ and $\|y\|<L_4\}$, then for each $t\in\mathcal{I}_n$, it is sufficient to show that $\|Tx_l-Tx\|_n\to0$ and $\|Uy_l-Uy\|_n\to0$ as $l\to\infty$. For each $t\in\mathcal{I}_n$ by (H1) we have

$$\begin{aligned}
|(Tx_l)(t)-(Tx)(t)|&\le\int_0^n|G_n(t,s)||f(s,x_l(s))-f(s,x(s))|ds\\
&\le\int_0^n\omega(s)|G_n(t,s)|[\sigma(|x_l(s)|)+\sigma(|x(s)|)]ds\\
&\le2\overset{*}{\omega}_n\sigma(R)\int_0^n|G_n(t,s)|ds\le2\overset{*}{G}_n\sigma(R)\overset{*}{\omega}_n,
\end{aligned}$$

where $\|m\|=\max\{|\omega(t)|: t\in \mathfrak{I}_n\}$. Thus the Lebesgue dominated convergence theorem implies that $\|Tx_l-Tx\|_n\to 0$ as $l\to\infty$. The proof of continuity of U is similar to that of T which was done in above.

(III) Let $(x,y)\in\Omega$ then $\|F(x,y)\|_n=\max\{\|Tx\|_n,\ \|Uy\|_n\}$ and for each $t\in\mathfrak{I}_n$, using (H1) we have

$$\begin{aligned}|(Tx)(t)|&\le|x_0|+\int_0^n|G_n(t,s)||f(s,x(s))|ds\\&\le|x_0|+\int_0^n|G_n(t,s)|\omega(s)\sigma(|x(s)|)ds\\&\le|x_0|+\overset{*}{\omega}_n\sigma(\|x\|_n)\int_0^n|G_n(t,s)|ds=|x_0|+\overset{*}{\omega}_n\sigma(\|x\|_n)\overset{*}{G}_n:=M_1,\end{aligned}$$

and

$$|(Uy)(t)|\le|y_0|+\int_0^n|H_n(t,s)||g(s,y(s))|ds\le|y_0|+\overset{*}{v}_n\mu(\|x\|_n)\overset{*}{H}_n:=M_2.$$

Let $M=\max\{M_1,M_2\}$, then $\|F(x,y)\|_n\le M$. That is to say, $F(\Omega)$ is uniformly bounded.

(IV) Since $G_n(t,s)$ and $H_n(t,s)$ are continuous on $\mathfrak{I}_n\times\mathfrak{I}_n$, they are uniformly continuous on $\mathfrak{I}_n\times\mathfrak{I}_n$. Thus, for fixed $s\in\mathfrak{I}_n$ and any $\epsilon>0$ there exists a constant $\delta>0$ such that for any $t_1,t_2\in\mathfrak{I}_n$ and $|t_1-t_2|<\delta$,

$$|G_n(t_1,s)-G_n(t_2,s)|<\frac{\epsilon}{2\sqrt{2}n\sigma(R)\overset{*}{\omega}_n},$$

and

$$|H_n(t_1,s)-H_n(t_2,s)|<\frac{\epsilon}{2\sqrt{2}n\eta(R)\overset{*}{\mu}_n}.$$

Then

$$|(Tx)(t_2)-(Tx)(t_1)|\le\int_0^n|G_n(t_2,s)-G_n(t_1,s)||f(s,x(s))|ds<\frac{\epsilon}{2\sqrt{2}}. \tag{3.12}$$

Similarly

$$|(Uy)(t_2)-(Uy)(t_1)|\le\int_0^n|H_n(t_2,s)-H_n(t_1,s)||g(s,y(s))|ds<\frac{\epsilon}{2\sqrt{2}}. \tag{3.13}$$

Using (H1), Eqs. (3.12), (3.13) and for the Euclidean distance d on $\mathbf{R}^2$, we have that if $t_1,t_2\in\mathfrak{I}_n$ are such that $|t_1-t_2|<\Omega$, then

$$\begin{aligned}d\big(F(x,y)(t_2)-F(x,y)(t_1)\big)&=\sqrt{\big[(Tx)(t_2)-(Tx)(t_1)\big]^2+\big[(Uy)(t_2)-(Uy)(t_1)\big]^2}\\&<\sqrt{2}\{|(Tx)(t_2)-(Tx)(t_1)|+|(Uy)(t_2)-(Uy)(t_1)|\}\\&<\epsilon.\end{aligned}$$

That is to say, $F(\Omega)$ is equicontinuous.

(V) It is a consequence of (I) - (III) together with Theorem 2.1 and combining Corollary 2.1.

(IV) Let $(x,y)\in\Omega$, that is $\|(x,y)\|_n<\mathcal{R}$ with $\mathcal{R}=\min\{r,\rho\}$. We prove that $F(x,y)\in\Omega$. For each $t\in\mathcal{J}_n$ and using (H1)-(H2) we have

$$\begin{aligned}\|F(x,y)\|_n&=\max\{\|Tx\|_n,\|Uy\|_n\}\\&\le\max\Big\{|x_0|+\int_0^n|G_n(t,s)||f(s,x(s))|ds,|y_0|+\int_0^n|H_n(t,s)||g(s,y(s))|ds\Big\}\\&\le\max\Big\{\overset{*}{\omega}_n\sigma(\|x\|_n)\overset{*}{G}_n,\overset{*}{v}_n\mu(\|x\|_n)\overset{*}{H}_n\Big\}\le\max\{r,\rho\}=\mathcal{R}.\end{aligned}$$

We complete the proof of Lemma 3.2. □

Proof of Theorem 3.1: Necessary conditions of Schauder's fixed point theorem for the operator $F:\mathcal{C}\to\mathcal{C}$ was obtained in Lemma 3.2, therefore F has fixed points (x_n,y_n) in Ω, hence by Lemma 3.1, the fixed points of F are solutions of the system of the boundary valued problem:

$$^cD^{\alpha}x(t)=I^{\gamma}f(t,y(t))+f(t,y(t)),\quad t\in\mathcal{J}_n,\quad 1<\alpha\le2, \tag{3.14a}$$

$$^cD^{\beta}y(t)=I^{\eta}g(t,x(t))+g(t,x(t)),\quad t\in\mathcal{J}_n,\quad 1<\beta\le2, \tag{3.14b}$$

$$x(0)=x_0,\quad x'(n)=0,\quad y(0)=y_0,\quad y'(n)=0. \tag{3.14c}$$

Using diagonalization process, we prove the system (1.1a)-(1.1c) has a bounded solution on $[0,\infty)$.

For $k\in\mathbf{N}$, assume that (x_k,y_k) is a solution of the boundary valued problem (3.14a)-(3.14c) on $[0,n_k]$ and $\{n_k\}_k\in\overset{*}{\mathbf{N}}$ is a sequence satisfying $0<n_1<n_2<\cdots<n_k<\cdots\uparrow\infty$. Let

$$(X_k(t),Y_k(t))=\begin{cases}(x_k(t),y_k(t)), & t\in[0,n_k],\\(x(n_k),y(n_k)), & t\in[n_k,\infty).\end{cases} \tag{3.15}$$

If we consider

$$\mathcal{S}=\{(X_1,Y_1),(X_2,Y_2),\cdots\},$$

then for each $t\in[0,n_1]$ and $k\in\mathbf{N}$ we have

$$\begin{aligned}\|(X_k,Y_k)\|&=\max\{\|X_k\|,\|Y_k\|\}\\&=\max\big\{\max\{|x_k(t)|:t\in[0,n_1]\},\ \max\{|y_k(t|:t\in[0,n_1]\}\big\}\\&=\max\big\{\|x_k\|,\|y_k\|\big\}\le\max\{r,\rho\}=\mathcal{R},\end{aligned}$$

and

$$X_{n_k}(t)=x_0+\int_0^{n_1}G_{n_1}(t,s)f(s,Y_{n_k}(s))ds, \tag{3.16a}$$

$$Y_{n_k}(t)=y_0+\int_0^{n_1}H_{n_1}(t,s)g(s,X_{n_k}(s))ds. \tag{3.16b}$$

Thus, for each $t,\tau\in[0,n_1]$ and $k\in\mathbf{N}$, from the system (3.16a)-(3.16b) and by (H1)-(H2) we get

$$|X_{n_k}(t)-X_{n_k}(\tau)|\le\lambda_1\int_0^{n_1}[G_{n_1}(t,s)-G_{n_1}(\tau,s)]ds,$$
$$|Y_{n_k}(t)-Y_{n_k}(\tau)|\le\lambda_1\int_0^{n_1}[H_{n_1}(t,s)-H_{n_1}(\tau,s)]ds,$$

where

$$\lambda_1=\max\{\overset{*}{\omega}_1\sigma(r),\ \overset{*}{v}_1\mu(\rho)\}.$$

Hence the Arzela-Ascoli Theorem guarantees that there is a subsequence $\mathcal{N}_1$ of $\mathbf{N}$ and two functions $u_1,v_1\in C([0,n_1],\mathbf{R})$ such that $(X_{n_k},Y_{n_k})\to(u_1,v_1)\in C([0,n_1],\mathbf{R})$ as $k\to\infty$ through $\mathcal{N}_1$.

Let $\overset{*}{\mathcal{N}}_1=\mathcal{N}_1-\{1\}$. Notice that $\|(X_{n_k},Y_{n_k})\|\le\mathcal{R}$ for each $t\in[0,n_2]$ and $k\in\mathbf{N}$. With repetition of the above process on the interval $[0,n_2]$, that is for each $t\in[0,n_2]$ and $k\in\mathbf{N}$ from the system (3.16a)-(3.16b) and by (H1)-(H2), we have

$$|X_{n_k}(t)-X_{n_k}(\tau)|\le\lambda_2\int_0^{n_1}[G_{n_1}(t,s)-G_{n_1}(\tau,s)]ds,$$
$$|Y_{n_k}(t)-Y_{n_k}(\tau)|\le\lambda_2\int_0^{n_1}[H_{n_1}(t,s)-H_{n_1}(\tau,s)]ds,$$

where $\lambda_2=\max\{\overset{*}{\omega}_2\sigma(r),\overset{*}{v}_2\mu(\rho)\}$. Hence the Arzela-Ascoli Theorem guarantees that there is a subsequence $\mathcal{N}_2$ of $\overset{*}{\mathcal{N}}_1$ and two functions $u_2,v_2\in C([0,n_2],\mathbf{R})$ such that $(X_{n_k},Y_{n_k})\to(u_2,v_2)\in C([0,n_2],\mathbf{R})$ as $k\to\infty$ through $\mathcal{N}_2$. It is clear that $(u_1(t),v_1(t))=(u_2(t),v_2(t))$ for each $t\in[0,n_1]$, as $\mathcal{N}_2\subseteq\overset{*}{\mathcal{N}}_1$.

Let $\overset{*}{\mathcal{N}}_2=\mathcal{N}_2-\{2\}$. Proceed inductively to obtain for $m\in\{3,4,\cdots\}$ a subsequence $\mathcal{N}_m$ of $\overset{*}{\mathcal{N}}_{m-1}$ and two functions $u_m,v_m\in C([0,n_m],\mathbf{R})$ such that $(X_{n_k},Y_{n_k})\to(u_m,v_m)\in C([0,n_m],\mathbf{R})$ as $k\to\infty$ through $\mathcal{N}_m$.

Let $\overset{*}{\mathcal{N}}_m=\mathcal{N}_m-\{m\}$. We define two functions x,y on $(0,\infty)$ as follows.

Fix $t\in(0,\infty)$ and let $m\in\mathbf{N}$ with $s\le n_m$. Then define $x(t)=X_m(t)$ and $y(t)=Y_m(t)$. Then $x,y\in C([0,\infty),\mathbf{R})$, $x(0)=x_0$, $y(0)=y_0$ and $|x(t)|\le\mathcal{R}$, $|y(t)|\le\mathcal{R}$ for $t\in[0,\infty)$. Again fix $t\in[0,\infty)$ and let $m\in\mathbf{N}$ with $s\le n_m$. Then for $n\in\overset{*}{\mathcal{N}}_m$ we have

$$X_{n_k}(t)=x_0+\int_0^{n_m}G_{n_m}(t,s)f(s,Y_{n_k}(s))ds,$$
$$Y_{n_k}(t)=y_0+\int_0^{n_m}H_{n_m}(t,s)g(s,X_{n_k}(s))ds.$$

Let $n_k \to \infty$ through $\overset{*}{\mathcal{N}}_m$ to obtain

$$X_m(t) = x_0 + \int_0^{n_m} G_{n_m}(\tau,s) f(s,Y_m(s)) ds,$$
$$Y_m(t) = y_0 + \int_0^{n_m} H_{n_m}(\tau,s) g(s,X_m(s)) ds,$$

that is

$$x(t) = x_0 + \int_0^{n_m} G_m(\tau,s) f(s,y(s)) ds,$$
$$y(t) = y_0 + \int_0^{n_m} H_m(\tau,s) g(s,x(s)) ds.$$

We can use this method for each $\tau \in [0,n_m]$, and for each $m \in \mathbf{N}$. Thus

$$\begin{aligned} &{}^cD^\alpha x(t) = I^\gamma f(t,y(t)) + f(t,y(t)), \qquad && t \in [0,n_m], \\ &{}^cD^\beta y(t) = I^\eta g(t,x(t)) g(t,x(t)), && t \in [0,n_m], \end{aligned}$$

for each $m \in \mathbf{N}$ and $\alpha,\beta \in (1,2]$ and the constructed functions x,y are a solution of the system (1.1). This completes the proof of the theorem.

Example 3.1. Consider the boundary value problem

$$\begin{aligned} &{}^cD^{\frac{3}{2}} x(t) - t I^{\frac{1}{2}} \Big(\frac{\sqrt[3]{y(t)}}{1+t^2} \Big) = \frac{\sqrt[3]{y(t)}}{1+t^2}, \qquad && t > 0, \\ &{}^cD^{\frac{4}{3}} y(t) - t I^{\frac{1}{3}} \Big(\frac{\sqrt{x(t)}}{1+e^t} \Big) = \frac{\sqrt{x(t)}}{1+e^t}, && t > 0, \\ &x(0) = 1, \ y(0) = 1, \ x \text{ and } y \text{ are bounded on } [0,\infty). \end{aligned}$$

Here,

$$\begin{aligned} &f(t,u) = \frac{\sqrt[3]{u}}{1+t^2}, \qquad && \omega(t) = \frac{1}{1+t^2}, \qquad && \sigma(u) = \sqrt[3]{u}, \\ &g(t,u) = \frac{\sqrt{u}}{1+e^t}, && v(u) = \frac{1}{1+e^t}, && \mu(u) = \sqrt{u}, \end{aligned}$$

f and g are continuous for each $(t,u) \in [0,\infty) \times \mathbf{R}$. The four functions ω,σ,η and μ are continuous on $[0,\infty)$ and satisfying (H1), that is $|f(t,u)| \le \omega(t)\sigma(|u|)$ and $|g(t,u)| \le v(t)\mu(|u|)$ for each $t \in [0,\infty)$ and $u \in \mathbf{R}$. We have

$$\overset{*}{\omega}_n = \sup\{\omega(t) : t \in \mathcal{J}_n\} = 1 \quad \text{and} \quad \overset{*}{v}_n = \sup\{v(t) : t \in \mathcal{J}_n\} = \frac{1}{2}.$$

The Green functions for this example are by Eq. (3.4)

$$G_n(t,s)ds=\begin{cases} t(t-s)-\dfrac{\alpha(t-s)^2}{2}+\dfrac{\sqrt{t-s}}{\Gamma(3/2)}+\mathcal{G}(t,s), & 0\le s\le t\le n,\\ \mathcal{G}(t,s), & 0\le t\le s\le n,\end{cases}$$

where

$$\mathcal{G}(t,s)=(\alpha-1)t(n-s)-n-\frac{t}{\Gamma(1/2)\sqrt{n-s}}.$$

Hence

$$G_n=\sup\Big\{\int_0^n |G_n(t,s)|ds,\ \ t\in\mathcal{I}_n\Big\}$$

exists. Since

$$\lim_{M\to\infty}\frac{M}{1+\overset{*}{\omega}_n\,\sigma(M)\,\overset{*}{G}_n}=\lim_{M\to\infty}\frac{M}{\sigma(M)}=\lim_{M\to\infty}\frac{M}{\sqrt[3]{M}}=\infty,$$

then there exists $r>0$ such that

$$\frac{r}{1+\overset{*}{\omega}_n\,\sigma(r)\,\overset{*}{G}_n}\ge 1.$$

On the other hand, Eq. (3.5b) yields

$$H_n(t,s)ds=\begin{cases} \dfrac{t(t-s)^{\frac{2}{3}}}{\Gamma(\frac{5}{3})}-\dfrac{\beta(t-s)^{\frac{5}{3}}}{\Gamma(\frac{8}{3})}+\dfrac{(t-s)^{\frac{1}{3}}}{\Gamma(\frac{4}{3})}+\mathcal{H}(t,s), & 0\le s\le t\le n,\\ \mathcal{H}(t,s), & 0\le t\le s\le n,\end{cases}$$

where

$$\mathcal{H}(t,s)=\frac{-t(n-s)^{\frac{2}{3}}}{\Gamma(\frac{5}{3})}-\frac{n(n-s)^{\frac{-1}{3}}}{\Gamma(\frac{2}{3})}+\frac{\beta t(n-s)^{\frac{2}{3}}}{\Gamma(\frac{5}{3})}-\frac{t(n-s)^{-\frac{2}{3}}}{\Gamma(\frac{-1}{4})}.$$

Hence

$$\overset{*}{H}_n=\sup\Big\{\int_0^n |H_n(t,s)|ds\Big\}$$

exists. Since

$$\lim_{N\to\infty}\frac{N}{1+\overset{*}{v}_n\,\mu(N)\,\overset{*}{H}_n}=\lim_{M\to\infty}\frac{N}{\mu(N)}=\lim_{M\to\infty}\frac{N}{\sqrt{N}}=\infty,$$

then there exists $\rho>0$ such that

$$\frac{\rho}{1+\overset{*}{v}_n\,\mu(\rho)\,\overset{*}{H}_n}\ge 1.$$

Hence this example satisfies in (H2). Therefore by Theorem 3.1 the system of this example has a bounded solution $(x,y)\in\Omega\subseteq\mathcal{C}$.

Remark 3.2. Proposition 2.1 (i) can be generalized, that is, if p is nonnegative integrable, then (see [5], pp. 53)

$$I^{\alpha}(t^p y(t)) = \sum_{k=0}^{p} \binom{-\alpha}{k} \left[D^{(k)} t^n\right] \left[I^{\alpha+k} y(t)\right] = \sum_{k=0}^{p} \binom{-\alpha}{k} \frac{\Gamma(p+1) t^{p-k}}{\Gamma(p-k+1)} I^{\alpha+k} y(t).$$

Hence, using Theorem 2.3 (2) the above equation yields

$$I^{\alpha}\left\{t^p I^{\beta} y(t)\right\} = \sum_{k=0}^{p} \binom{-\alpha}{k} \frac{\Gamma(p+1) t^{p-k}}{\Gamma(p-k+1)} I^{\alpha+\beta+k} y(t),$$

where

$$\binom{-\alpha}{k} = (-1)^k \times \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} = (-1)^k \times \frac{\Gamma(\alpha+k)}{k!\Gamma(\alpha)}.$$

Therefore we can prove that the system of nonlinear fractional differential equation:

$$\begin{aligned}
&{}^c D^{\alpha} x(t) = t^p I^{\gamma} f(t, y(t)) + f(t, y(t)), \quad t \in (0,\infty), \\
&{}^c D^{\beta} y(t) = t^p I^{\eta} g(t, x(t)) + g(t, x(t)), \quad t \in (0,\infty), \\
&x(0) = x_0,\ \ y(0) = y_0,\ \ x(t) \text{ and } y(t) \text{ are bounded on } [0,\infty),
\end{aligned}$$

under which the conditions (H1) and (H2) have at least one bounded solution on $[0,\infty)$, where p is a nonnegative integer.

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