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Boundedness for Multilinear Commutators of Marcinkiewicz Integral on Morrey-Herz Spaces with Non Doubling Measures

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Abstract. In this paper, the authors establish the boundedness of multilinear commutators generated by a Marcinkiewicz integral operator and a RBMO(μ) function on homogeneous Morrey-Herz spaces with non doubling measures.

Key Words: Marcinkiewicz integral, commutator, Morrey-Herz space, non doubling measure, RBMO function.

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1 Introduction and preliminaries

As an analogy of the classical Littlewood-Paley g function, Marcinkiewicz [1] introduced the operator

$$\mathcal{M}(f)(x)=\Big(\int_0^{\pi}\frac{|F(x+t)+F(x-t)-2F(x)|^2}{t^3}\mathrm{d}t\Big)^{\frac{1}{2}},\quad x\in[0,2\pi],$$

where $F(x)=\int_0^x f(t)\mathrm{d}t$. This operator is now called the Marcinkiewicz integral. Zygmund [2] proved that the operator $\mathcal{M}$ is bounded on the Lebesgue space $L^p([0,2\pi])$ for $p\in(1,\infty)$. Stein [3] generalized the above Marcinkiewicz integral to the following higher-dimensional case. Let Ω be homogeneous of degree zero in $\mathbf{R}^d$ for $d\geq 2$, integrable and have mean value zero on the unit sphere S^{d-1}. The higher-dimensional Marcinkiewicz integral is defined by

$$\mathcal{M}_\Omega(f)(x)=\Big(\int_0^{\infty}\Big|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{d-1}}f(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}},\quad x\in\mathbf{R}^d.$$

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Stein in [3] proved that if $\Omega\in\mathrm{Lip}_\delta(S^{d-1})$ for some $\delta\in(0,1]$, then $\mathcal{M}_\Omega$ is bounded on $L^p(\mathbf{R}^d)$ for any $p\in(1,2]$, and is also bounded from $L^1(\mathbf{R}^d)$ to $L^{1,\infty}(\mathbf{R}^d)$. Since then, a lot of papers focus on this operator. For some recent development, we mention that Al-Salman et al. in [4] obtained the $L^p(\mathbf{R}^d)$-boundedness for $p\in(1,\infty)$ of $\mathcal{M}_\Omega$ if $\Omega\in L(\log L)^{1/2}(S^{d-1})$; Fan and Sato in [5] proved that $\mathcal{M}_\Omega$ is bounded from the Lebesgue space $L^1(\mathbf{R}^d)$ to the weak Lebesgue space $L^{1,\infty}(\mathbf{R}^d)$ if $\Omega\in L\log L(S^{d-1})$. There are many other interesting works for this operator, among them we refer to [6,7] and their references. On the other hand, Torchinsky and Wang in [8] first introduced the commutator generated by the Marcinkiewicz integral $\mathcal{M}_\Omega$ and the classical $\mathrm{BMO}(\mathbf{R}^d)$ function, and established its $L^p(\mathbf{R}^d)$-boundedness for $p\in(1,\infty)$ when $\Omega\in\mathrm{Lip}_\delta(S^{d-1})$ for some $\delta\in(0,1]$. Such boundedness of this commutator is further discussed in [9,10] when Ω only satisfies certain size conditions. Moreover, its weak type endpoint estimate is obtained in [11,12] when $\Omega\in\mathrm{Lip}_\delta(S^{d-1})$ for some $\delta\in(0,1]$, and its weight weak type endpoint estimate is obtained in [13,14] when Ω satisfies a kind of Dini conditions. Also see [15–17] et al. for more informations.

Motivated by the work above, the main purpose of this paper is to establish a similar theory for the multilinear commutator generated by a Marcinkiewicz integral operator and a $\mathrm{RBMO}(\mu)$ function or $\mathrm{Osc}_{\exp L^r}(\mu)$ function on $\mathbf{R}^d$ with a positive Radon measure which may be non doubling.

To be precise, let μ be a positive Radon measure on $\mathbf{R}^d$ which only satisfies the following growth condition that for all $x\in\mathbf{R}^d$ and all $r>0$,

$$\mu(B(x,r))\le C_0r^n, \tag{1.1}$$

where $C_0>0$ and n are some positive constants, $0<n\le d$, and $B(x,r)$ is the open ball centered at x and having radius r. We recall that μ is said to be a doubling measure, if there is a positive constant C such that for any $x\in\mathrm{supp}\mu$ and $r>0$,

$$\mu(B(x,2r))\le C\mu(B(x,r)),$$

and that the doubling condition is a key assumption in the classical theory of harmonic analysis. In recent years, many classical results concerning the theory of Calderón-Zygmund operators and function spaces have been proved to be still valid if the Lebesgue measure is substituted by a measure μ as in (1.1); see [18–25]. We mention that the analysis on non-homogeneous spaces play an essential role in solving the long-standing open Painlevé's problem by Tolsa in [21].

To outline the structure of this paper, we first recall some notation and definitions. For a cube $Q\subset\mathbf{R}^d$, we mean a closed cube whose sides parallel to the coordinate axes, and we denote its side length by $l(Q)$ and its center by x_Q. Let $\gamma>1$ and $\beta>\gamma^n$. We say that a cube Q is an (γ,β)-doubling cube if $\mu(\gamma Q)\le\beta\mu(Q)$, where γQ denotes the cube with the same center as Q and $l(\gamma Q)=\gamma l(Q)$. For definiteness, if γ and β are not specified, by a doubling cube we mean a $(2,2^{d+1})$-doubling cube. Especially, for any given cube Q, we denote by $\tilde{Q}$ the smallest doubling cube which contains Q and has the same center as Q.

Given two cubes $Q_1 \subset Q_2$ in $\mathbf{R}^d$, set

$$K_{Q_1,Q_2} = 1 + \sum_{k=1}^{N_{Q_1,Q_2}} \frac{\mu(2^k Q_1)}{[l(2^k Q_1)]^n},$$

where N_{Q_1,Q_2} is the smallest positive integer k such that

$$l(2^k Q_1) \geq l(Q_2).$$

The concept of K_{Q_1,Q_2} is first appeared in [20], where some useful properties of K_{Q_1,Q_2} can be found. The following space RBMO(μ) is introduced by Tolsa in [20].

Definition 1.1. (see [20]) Let $\rho > 1$ be a fixed constant. A function $b \in L^1_{\rm loc}(\mu)$ is said to be in the space RBMO(μ) if there exists some constant $B > 0$ such that

(i) for any cube Q centered at some point of supp(μ),

$$\sup_Q \frac{1}{\mu(\rho Q)} \int_Q \big|b(x) - m_{\tilde{Q}}(b)\big| d\mu(x) \leq B < \infty.$$

(ii) for any two doubling cubes $Q_1 \subset Q_2$,

$$|m_{Q_1}(b) - m_{Q_2}(b)| \leq B K_{Q_1,Q_2}.$$

Where the supremum is taken over all cubes centered at some point of supp(μ), and $m_Q(b)$ denotes the mean value of b over the cube Q. The minimal constant B as above is defined to be the norm of b in the space RBMO(μ) and denoted by

$$\|b\|_{{\rm RBMO}(\mu)} = \|b\|_*.$$

Tolsa in [20] proved that the definition of the space RBMO(μ) is independent of the choice of ρ. The definition of the following function space of Orlicz type is a variant with a non doubling measure of the space ${\rm Osc}_{\exp L^r}$ in [22].

Definition 1.2. (see [22]) For $r \geq 1$, a function $b \in L^1_{\rm loc}(\mu)$ is said to be in the space ${\rm Osc}_{\exp L^r}(\mu)$ if there is a constant $B_1 > 0$ such that

(i) for any Q,

$$\big\|b - m_{\tilde{Q}}(b)\big\|_{\exp L^r, Q, \mu/\mu(2Q)} = \inf\Big\{\gamma > 0 : \frac{1}{\mu(2Q)} \int_Q \exp\Big(\frac{|b - m_{\tilde{Q}}(b)|}{\gamma}\Big)^r d\mu \leq 2\Big\} \leq B_1.$$

(ii) for any doubling cubes $Q_1 \subset Q_2$,

$$|m_{Q_1}(b) - m_{Q_2}(b)| \leq B_1 K_{Q_1,Q_2}.$$

The minimal constant B_1 satisfying (i) and (ii) is the norm of b in the space ${\rm Osc}_{\exp L^r}(\mu)$ and denoted by $\|b\|_{{\rm Osc}_{\exp L^r}(\mu)}$.

Obviously, for any $r\geq 1$, $\mathrm{Osc}_{\exp L^r}(\mu)\subset \mathrm{RBMO}(\mu)$. Moreover, from John-Nirenberg's inequality in [20], it follows that $\mathrm{Osc}_{\exp L^1}(\mu)=\mathrm{RBMO}(\mu)$. In [26], Pérez and Trujillo-González point that if μ is a Lebesgue measure in $\mathbf{R}^d$, the counterpart of the space $\mathrm{Osc}_{\exp L^r}(\mu)$ when $r>1$ is a proper subspace of the classical space $\mathrm{BMO}(\mathbf{R}^d)$. However, it is still unknown whether the space $\mathrm{Osc}_{\exp L^r}(\mu)$ is a proper subspace of the space $\mathrm{RBMO}(\mu)$ when μ is a non doubling measure.

We now introduce the Marcinkiewicz integral related to the measure μ as in (1.1). Let K be a locally integrable function on $\mathbf{R}^d\times\mathbf{R}^d\setminus\{(x,y):x=y\}$. Assume that there exists a constant $C>0$ such that for all $x,y,y'\in\mathbf{R}^d$ with $x\neq y$,

$$|K(x,y)|\leq C|x-y|^{-(n-1)} \tag{1.2}$$

and

$$\int_{|x-y|\geq 2|y-y'|}\frac{|K(x,y)-K(x,y')|+|K(y,x)-K(y',x)|}{|x-y|}\mathrm{d}\mu(x)\leq C. \tag{1.3}$$

The Marcinkiewicz integral $\mathcal{M}(f)$ associated to the above kernel K and the measure μ as in (1.1) is defined by

$$\mathcal{M}(f)(x)=\Big(\int_0^\infty\Big|\int_{|x-y|\leq t}K(x,y)f(y)\mathrm{d}\mu(y)\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}},\quad x\in\mathbf{R}^d. \tag{1.4}$$

Obviously, if μ is the d-dimensional Lebesgue measure in $\mathbf{R}^d$, and

$$K(x,y)=\frac{\Omega(x-y)}{|x-y|^{d-1}}$$

with Ω homogeneous of degree zero and $\Omega\in\mathrm{Lip}_\delta(S^{d-1})$ for some $(\delta\in(0,1]$, then it is easy to verify that K satisfies (1.2) and (1.3), and $\mathcal{M}$ in (1.4) is just the Marcinkiewicz integral $\mathcal{M}_\Omega$ introduced by Stein in [3]. Thus, $\mathcal{M}$ in (1.4) is a natural generalization of the classical Marcinkiewicz integral in the current setting.

To state the main result, we also need to introduce the following notation. As in [26], given any positive integer m, for all $i\in[1,m]$, we denote by $\mathcal{C}_i^m$ the family of all finite subsets $\sigma=\{\sigma(1),\sigma(2),\cdots,\sigma(i)\}$ of $\{1,2,\cdots,m\}$ with i different elements. For any $\sigma\in\mathcal{C}_i^m$, we define the complementary sequence $\sigma'=\{1,2,\cdots,m\}\setminus\sigma$.

Let $\vec{b}=(b_1,b_2,\cdots,b_m)$ be a finite family of locally integrable functions. For all $1\leq i\leq m$ and $\sigma=\{\sigma(1),\sigma(2),\cdots,\sigma(i)\}\in\mathcal{C}_i^m$, we will denote $\vec{b}_\sigma=(b_{\sigma(1)},b_{\sigma(2)},\cdots,b_{\sigma(i)})$ and the product $b_\sigma=b_{\sigma(1)}b_{\sigma(2)}\cdots b_{\sigma(i)}$. With this notation, we write

$$\big(b(x)-b(y)\big)_\sigma=\big(b_{\sigma(1)}(x)-b_{\sigma(1)}(y)\big)\cdots\big(b_{\sigma(i)}(x)-b_{\sigma(i)}(y)\big),$$

and

$$\big(b_Q-b(y)\big)_\sigma=\big((b_{\sigma(1)})_Q-b_{\sigma(1)}(y)\big)\cdots\big((b_{\sigma(i)})_Q-b_{\sigma(i)}(y)\big),$$

where Q is any cube in $\mathbf{R}^d, x,y\in\mathbf{R}^d$, and

$$f_Q=\frac{1}{|Q|}\int_Q f(y)\mathrm{d}y.$$

In particular, for $b_i\in\mathrm{RBMO}(\mu)$ $(1\le i\le m)$, we write

$$\|\vec{b}_\sigma\|_*=\|b_{\sigma(1)}\|_*\|b_{\sigma(2)}\|_*\cdots\|b_{\sigma(i)}\|_*.$$

If $\sigma=\{1,2,\cdots,m\}$, then σ' is an empty set, we denote $\|\vec{b}_\sigma\|_*$ simply by $\|\vec{b}\|_*$.

Let m be a positive integer, $b,b_i\in\mathrm{RBMO}(\mu)$ $(1\le i\le m)$ and $\vec{b}=(b_1,b_2,\cdots,b_m)$, we define the multilinear commutators $\mathcal{M}_{\vec{b}}$ by

$$\mathcal{M}_{\vec{b}}(f)(x)=\Big(\int_0^\infty\Big|\int_{|x-y|\le t}K(x,y)f(y)\times\prod_{i=1}^m\Big(b_i(x)-b_i(y)\Big)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac{1}{2}} \tag{1.5}$$

for $x\in\mathbf{R}^d$ with kernel K satisfying (1.2) and the following Hörmander-type condition that

$$\sup_{\substack{|y-y'|\le r\\ r>0,y,y'\in\mathbf{R}^d}}\sum_{l=1}^\infty l^m\int_{2^l r<|x-y|\le 2^{l+1}r}(|K(x,y)-K(x,y')|+|K(y,x)-K(y',x)|)\frac{1}{|x-y|}\mathrm{d}\mu(x)$$
$$\le C, \tag{1.6}$$

which is slightly stronger than (1.3). In what follows, if $m=1$ and $\vec{b}=b$, we denote $\mathcal{M}_{\vec{b}}(f)$ simply by $\mathcal{M}_b(f)$; and when $b_1=b_2=\cdots=b_m=b$, we denote $\mathcal{M}_{\vec{b}}(f)$ simply by $\mathcal{M}_{b,m}(f)$ which is called the mth order commutator.

Let $B_k=\{x\in\mathbf{R}^d:|x|\le 2^k\}$ and $A_k=B_k\backslash B_{k-1}$ for $k\in\mathbf{Z}$. And let $\chi_k=\chi_{A_k}$ for $k\in\mathbf{Z}$ be the characteristic function of the set A_k.

Definition 1.3. (see [23]) Let $\alpha\in\mathbf{R}$, $0<p\le\infty$, $0<q<\infty$ and $\lambda\ge 0$. The homogeneous Morrey-Herz spaces $M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)$ are defined by

$$M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)=\{f\in L^q_{loc}(\mathbf{R}^d\backslash\{0\},\mu):\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}<\infty\},$$

where

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}=\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mu)}\Big)^{\frac{1}{p}}$$

with the usual modifications made when $p=\infty$.

Compare the homogeneous Morrey-Herz spaces $M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)$ with the homogeneous Herz spaces $\dot{K}^{\alpha,p}_q(\mu)$ (see [25]), where

$$\dot{K}^{\alpha,p}_q(\mu)=\Big\{f\in L^q_{loc}(\mathbf{R}^d\backslash\{0\},\mu):\sum_{k=-\infty}^\infty 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mu)}<\infty\Big\}.$$

Obviously, $M\dot{K}^{\alpha,0}_{p,q}(\mu)=\dot{K}^{\alpha,p}_{q}(\mu)$. Moreover, it is easy to observe that $\dot{K}^{0,q}_{q}(\mu)=L^q(\mu)$.

Throughout this paper, C denotes a constant that is independent of the main parameters involved but whose value may differ from line to line. For any index $p\in[1,\infty]$, we denote by p' its conjugate index, namely, $1/p+1/p'=1$. For $A\sim B$, we mean that there is a constant $C>0$ such that $C^{-1}B\le A\le CB$.

2 Main result and its proof

The following theorem is the main result of this paper, which is new even when $b_1=b_2=\cdots=b_m=b$, namely, Theorem 2.1 is also new even for the commutator of the m-th order.

Theorem 2.1. *Let $\lambda\ge 0$, $0<p<\infty$, $1<q<\infty$. If $\mathcal{M}$ in* (1.4) *is bounded on $L^2(\mu)$ when $K(x,y)$ satisfies* (1.2) *and* (1.6), *then for any positive integer m and $b_i\in \mathrm{RBMO}(\mu)$ $(1\le i\le m)$, the multilinear commutator $\mathcal{M}_{\vec{b}}$ in* (1.5) *is bounded on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)$ with*

$$-\frac{n}{q}+\lambda<\alpha<n\Big(1-\frac{1}{q}\Big)+\lambda.$$

Proof. Let $f\in M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)$. Write

$$f(x)=\sum_{j=-\infty}^{\infty} f(x)\chi_j(x)\equiv\sum_{j=-\infty}^{\infty} f_j(x).$$

Then, we have

$$\begin{aligned}
\|\mathcal{M}_{\vec{b}}(f)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}
&=\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\|\mathcal{M}_{\vec{b}}(f)\chi_k\|^p_{L^q(\mu)}\Big)^{\frac{1}{p}}\\
&\le C\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\Big(\sum_{j=-\infty}^{k+1}\|\chi_k\mathcal{M}_{\vec{b}}(f_j)\|_{L^q(\mu)}\Big)^p\Big)^{\frac{1}{p}}\\
&\quad+C\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\Big(\sum_{j=k+2}^{\infty}\|\chi_k\mathcal{M}_{\vec{b}}(f_j)\|_{L^q(\mu)}\Big)^p\Big)^{\frac{1}{p}}\\
&\equiv E_1+E_2.
\end{aligned}$$

To estimate E_1, we first consider

$$\begin{aligned}
\|\chi_k\mathcal{M}_{\vec{b}}(f_j)\|_{L^q(\mu)}
&\le\Big(\int_{A_k}\Big(\int_0^{\infty}\Big|\int_{|x-y|\le t}K(x,y)f_j(y)\cdot\prod_{i=1}^m(b_i(x)-b_i(y))\mathrm{d}\mu(y)\Big|^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac{q}{2}}\mathrm{d}\mu(x)\Big)^{\frac{1}{q}}\\
&\le\Big(\int_{A_k}\Big(\int_0^{|x|}\Big(\int_{|x-y|\le t}|K(x,y)f_j(y)|\cdot\prod_{i=1}^m|b_i(x)-b_i(y)|\mathrm{d}\mu(y)\Big)^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac{q}{2}}\mathrm{d}\mu(x)\Big)^{\frac{1}{q}}\\
&\quad+\Big(\int_{A_k}\Big(\int_{|x|}^{\infty}\Big(\int_{|x-y|\le t}|K(x,y)f_j(y)|\cdot\prod_{i=1}^m|b_i(x)-b_i(y)|\mathrm{d}\mu(y)\Big)^2\frac{\mathrm{d}t}{t^3}\Big)^{\frac{q}{2}}\mathrm{d}\mu(x)\Big)^{\frac{1}{q}}\\
&=E_{11}+E_{12}.
\end{aligned}$$

Note that when $x\in A_k, y\in A_j$ and $j\le k+1$, we have $|x|\sim|x-y|$. Therefore, for $x\in A_k$, by the mean-value theorem of differentials, we have

$$\Big|\frac{1}{|x|^2}-\frac{1}{|x-y|^2}\Big|\le C\frac{|y|}{|x-y|^3}. \tag{2.1}$$

Let Q_j be the smallest cube which contains A_j with center at the origin. For $j\le k+1$, by (1.2), (2.1), $|x|\sim|x-y|$, Minkowski's inequality and with the aid of the fact

$$\prod_{i=1}^m\big(b_i(x)-b_i(y)\big)=\sum_{i=0}^m\sum_{\sigma\in\mathfrak{C}_i^m}\Big(b(x)-m_{\tilde{Q}_j}(b)\Big)_\sigma\Big(m_{\tilde{Q}_j}(b)-b(y)\Big)_{\sigma'},$$

we have

$$\begin{aligned}E_{11}\le& C\Big(\int_{A_k}\Big(\int_{A_j}\frac{|f(y)|\prod_{i=1}^m|b_i(x)-b_i(y)|}{|x-y|^{n-1}}\frac{|y|^{\frac12}}{|x-y|^{\frac32}}\mathrm{d}\mu(y)\Big)^q\mathrm{d}\mu(x)\Big)^{\frac1q}\\
\le& C2^{\frac j2-k(n+\frac12)}\Big(\int_{A_k}\Big(\int_{A_j}|f(y)|\prod_{i=1}^m|b_i(x)-m_{\tilde{Q}_j}(b_i)|\mathrm{d}\mu(y)\Big)^q\mathrm{d}\mu(x)\Big)^{\frac1q}\\
&+C2^{\frac j2-k(n+\frac12)}\Big(\int_{A_k}\sum_{i=1}^{m-1}\sum_{\sigma\in\mathfrak{C}_i^m}\Big(\int_{A_j}\Big|\Big(b(x)-m_{\tilde{Q}_j}(b)\Big)_\sigma\\
&\times\Big(m_{\tilde{Q}_j}(b)-b(y)\Big)_{\sigma'}\Big||f(y)|\mathrm{d}\mu(y)\Big)^q\mathrm{d}\mu(x)\Big)^{\frac1q}\\
&+C2^{\frac j2-k(n+\frac12)}\Big(\int_{A_k}\Big(\int_{A_j}|f(y)|\prod_{i=1}^m|m_{\tilde{Q}_j}(b_i)-b_i(y)|\mathrm{d}\mu(y)\Big)^q\mathrm{d}\mu(x)\Big)^{\frac1q}\\
=&E_{111}+E_{112}+E_{113}.\end{aligned}$$

We first estimate the term E_{111}. With the aid of the fact $K_{\tilde{Q}_j,\tilde{Q}_k}\le C(k-j)$ (see Lemma 2.1 in [20]), by (1.1), Minkowski's inequality, Hölder's inequality and the property of RBMO function, we have

$$\begin{aligned}E_{111}\le& C2^{\frac j2-k(n+\frac12)}\Big(\int_{A_j}|f(y)|\Big(\int_{A_k}\prod_{i=1}^m|b_i(x)-m_{\tilde{Q}_j}(b_i)|^q\mathrm{d}\mu(x)\Big)^{\frac1q}\mathrm{d}\mu(y)\Big)\\
\le& C2^{\frac j2-k(n+\frac12)}\Big(\int_{A_j}|f(y)|\prod_{i=1}^m\Big(\int_{A_k}|b_i(x)-m_{\tilde{Q}_j}(b_i)|^{r_iq}\mathrm{d}\mu(x)\Big)^{\frac{1}{r_iq}}\mathrm{d}\mu(y)\Big)\\
\le& C2^{\frac j2-k(n+\frac12)}\Big(\int_{A_j}|f(y)|\prod_{i=1}^m\Big(2^{\frac{kn}{r_iq}}\|b_i\|_*+2^{\frac{kn}{r_iq}}K_{\tilde{Q}_j,\tilde{Q}_k}\|b_i\|_*\Big)\mathrm{d}\mu(y)\Big)\\
\le& C\prod_{i=1}^m\|b_i\|_*(k-j)^m2^{(j-k)(\frac12+\frac{n}{q'})}\|f_j\|_{L^q(\mu)},\end{aligned}$$

where $1/r_1+\cdots+1/r_m=1$ ($r_i>1$, $i\in[1,m]$).

Now, let us consider E_{112}. Similar to the estimate for E_{111}, we have

$$\begin{aligned}E_{112}\le& C2^{\frac{j}{2}-k(n+\frac{1}{2}-\frac{n}{q})}\Big(\sum_{i=1}^{m-1}\sum_{\sigma\in\mathcal{C}_i^m}(k-j)^i\|\vec{b}_\sigma\|_*\int_{A_j}\big|(m_{\tilde{Q}_j}(b)-b(y))_{\sigma'}\big|\big|f(y)\big|\mathrm{d}\mu(y)\Big)\\ \le& C(k-j)^m2^{(j-k)(\frac{1}{2}+\frac{n}{q'})}\Big(\sum_{i=1}^{m-1}\sum_{\sigma\in\mathcal{C}_i^m}\|\vec{b}_\sigma\|_*\|\vec{b}_{\sigma'}\|_*\|f_j\|_{L^q(\mu)}\Big)\\ \le& C\prod_{i=1}^m\|b_i\|_*(k-j)^m2^{(j-k)(\frac{1}{2}+\frac{n}{q'})}\|f_i\|_{L^q(\mu)}.\end{aligned}$$

For E_{113}, similar to the estimate for E_{111} and E_{112}, we also have

$$E_{113}\le C\prod_{i=1}^m\|b_i\|_*(k-j)^m2^{(j-k)(\frac{1}{2}+\frac{n}{q'})}\|f_i\|_{L^q(\mu)}.$$

Combining the estimates above then gives

$$E_{11}\le C\prod_{i=1}^m\|b_i\|_*(k-j)^m2^{(j-k)(\frac{1}{2}+\frac{n}{q'})}\|f_i\|_{L^q(\mu)}.$$

For E_{12}, similar to the estimate for E_{11}, we can get

$$E_{12}\le C\prod_{i=1}^m\|b_i\|_*(k-j)^m2^{(j-k)(\frac{1}{2}+\frac{n}{q'})}\|f_i\|_{L^q(\mu)}.$$

Then, when $j\le k+1$, we obtain

$$\|\chi_k\mathcal{M}_{\vec{b}}(f_j)\|_{L^q(\mu)}\le C\prod_{i=1}^m\|b_i\|_*(k-j)^m2^{(j-k)(\frac{1}{2}+\frac{n}{q'})}\|f_i\|_{L^q(\mu)}.$$

Therefore, using the fact for

$$\|f_j\|_{L^q(\mu)}^p\le 2^{-j\alpha p}\sum_{i=-\infty}^{j}2^{i\alpha p}\|f_i\|_{L^q(\mu)}^p,$$

we get

$$\begin{aligned}E_1\le& C\prod_{i=1}^m\|b_i\|_*\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\Big(\sum_{j=-\infty}^{k+1}(k-j)^m2^{(j-k)\frac{n}{q'}}\|f_j\|_{L^q(\mu)}\Big)^p\Big)^{\frac{1}{p}}\\ \le& C\prod_{i=1}^m\|b_i\|_*\sup_{k_0\in\mathbf{Z}}2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0}2^{k\lambda p}\Big)^{\frac{1}{p}}\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mu)}\\ \le& C\prod_{i=1}^m\|b_i\|_*\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mu)}.\end{aligned}$$

An argument similar to the estimate for E_1, and note that when $x\in A_k, y\in A_j$ and $j\geq k+2$, $|y|\sim|x-y|$. For $x\in A_k$, via the mean-value theorem of differentials gives

$$\Big|\frac{1}{|y|^2}-\frac{1}{|x-y|^2}\Big|\leq C\frac{|x|}{|x-y|^3}. \tag{2.2}$$

We thus obtain

$$E_2\leq C\prod_{i=1}^m\|b_i\|_*\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mu)}.$$

Combining the estimate above for E_1 and E_2, we complete the proof of Theorem 2.1. □

The result of Theorem 2.1 for $\lambda=0$ is also new on homogeneous Herz spaces $\dot{K}^{\alpha,p}_q(\mu)$. Furthermore, when $\alpha=\lambda=0$ and $p=q$ in Theorem 2.1 we can obtain the following corollary.

Corollary 2.1. Let $1<q<\infty$. If $\mathcal{M}$ in (1.4) is bounded on $L^2(\mu)$ when $K(x,y)$ satisfies (1.2) and (1.6), then for any positive integer m and $b_i\in\mathrm{RBMO}(\mu)$ $(1\leq i\leq m)$, the multilinear commutator $\mathcal{M}_{\vec{b}}$ in (1.5) is bounded on $L^q(\mu)$.

Remark 2.1. The result above is also new for any $b_i\in\mathrm{Osc}_{\exp L^{r_i}}(\mu)\subset\mathrm{RBMO}(\mu)$, where $1\leq r_i<\infty$ and $i=1,2,\cdots,m$.

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