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Simultaneous Approximation for Szász-Mirakian-Stancu-Durrmeyer Operators

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Abstract. The aim of this work is to generalize Szász-Mirakian operator in the sense of Stancu-Durrmeyer operators. We obtain approximation properties of these operators. Here we study asymptotic as well as rate of convergence results in simultaneous approximation for these modified operators.

Key Words: Szász-Mirakian-Stancu-Durrmeyer operator, simultaneous approximation, asymptotic, rate of convergence.

AMS Subject Classifications: 41A20, 41A25, 41A35

1 Introduction

Let α and β be two non-negative parameters satisfying the condition $0\leq\alpha\leq\beta$. For any nonnegative integer n,

$$f\in C[0,\infty)\to S_n^{(\alpha,\beta)}f,$$

the Stancu type Szász-Mirakian-Durrmeyer operators are defined by

$$S_{n,r}^{(\alpha,\beta)}(f,x)=n\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^\infty s_{n,k+r}(t)f\Big(\frac{nt+\alpha}{n+\beta}\Big)dt, \tag{1.1}$$

where

$$s_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!}.$$

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For $\alpha=\beta=0$ these operators become the well known Szász-Mirakian-Durrmeyer operators

$$S_n^{(0,0)}(f,x)=S_n(f,x)$$

introduced by Mazhar and Totik [3]. In [1] the author established some direct results in simultaneous approximation for this special case. Gupta et al. [2] estimated the rate of convergence for functions having derivatives of bounded variation for this special case $\alpha=\beta=r=0$. Also for this special case [4] estimated the rate of convergence for the Bézier variant of Szász-Mirakian-Durrmeyer operators.

The purpose of this paper is to study approximation properties of the Stancu type Szász-Mirakian-Durrmeyer operators. We give the rate of convergence and Voronovskaya type asymptotic result for the same operators.

2 Basic results

In this section we establish a recurrence formula for the moments.

For simultaneous approximation, we need the following form of the operators (1.1)

$$S_{n,r}^{(\alpha,\beta)}(f,x)=n\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^{\infty}s_{n,k+r}(t)f\Big(\frac{nt+\alpha}{n+\beta}\Big)dt.$$

Lemma 2.1. *For $n,m\in\mathbf{N}\cup\{0\}$, $0\le\alpha\le\beta$, let us consider*

$$\mu_{n,m,r}^{(\alpha,\beta)}(x)=S_{n,r}^{(\alpha,\beta)}\big((t-x)^m,x\big)=n\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^{\infty}s_{n,k+r}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt,$$

we get

$$\begin{aligned}&\mu_{n,0,r}^{(\alpha,\beta)}(x)=1,\qquad \mu_{n,1,r}^{(\alpha,\beta)}(x)=\frac{\alpha+r+1-\beta x}{n+\beta},\\ &\mu_{n,2,r}^{(\alpha,\beta)}(x)=\frac{\beta^2x^2+2(n-\alpha\beta-\beta-\beta r)x+(\alpha+r+1)(\alpha+r+2)-\alpha}{(n+\beta)^2},\end{aligned}$$

and

$$\begin{aligned}(n+\beta)\mu_{n,m+1,r}^{(\alpha,\beta)}(x)=&x[\mu_{n,m,r}^{(\alpha,\beta)}(x)]'+(m+\alpha+r+1-\beta x)\mu_{n,m,r}^{(\alpha,\beta)}(x)\\ &+m\Big(\frac{2(n+\beta)x-\alpha}{n+\beta}\Big)\mu_{n,m-1,r}^{(\alpha,\beta)}(x).\end{aligned}\tag{2.1}$$

Proof. By simple calculation we can easily obtain

$$xs'_{n,k}(x)=(k-nx)s_{n,k}(x).$$

We have from the definition of $\mu_{n,m}^{(\alpha,\beta)}(x)$

$$\begin{aligned}x[\mu_{n,m,r}^{(\alpha,\beta)}(x)]'=&n\sum_{k=0}^{\infty}xs'_{n,k}(x)\int_0^\infty s_{n,k+r}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt\\&-nmx\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^\infty s_{n,k+r}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^{m-1}dt\\=&n\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^\infty (k-nx)s_{n,k+r}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt-mx\mu_{n,m-1,r}^{(\alpha,\beta)}(x).\end{aligned}$$

Now

$$\begin{aligned}&x\Big[[\mu_{n,m,r}^{(\alpha,\beta)}(x)]'+m\mu_{n,m-1,r}^{(\alpha,\beta)}(x)\Big]\\=&n\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^\infty ts'_{n,k+r}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt+n^2\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^\infty ts_{n,k+r}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt\\&-n(nx+r)\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^\infty s_{n,k+r}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt.\end{aligned}$$

Putting

$$t=\Big(\frac{n+\beta}{n}\Big)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)-\frac{\alpha}{n}+\Big(\frac{n+\beta}{n}\Big)x,$$

we have

$$\begin{aligned}&x\Big[[\mu_{n,m,r}^{(\alpha,\beta)}(x)]'+m\mu_{n,m-1}^{(\alpha,\beta)}(x)\Big]\\=&n\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^\infty s'_{n,k+r}(t)\Big(\frac{n+\beta}{n}\Big)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^{m+1}dt\\&-n\Big(\frac{\alpha-(n+\beta)x}{n}\Big)\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^\infty s'_{n,k+r}(t)\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt\\&+n^2\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^\infty s_{n,k+r}(t)\Big(\frac{n+\beta}{n}\Big)\Big\{\frac{nt+\alpha}{n+\beta}-\frac{\alpha-(n+\beta)x}{n}-x\Big\}\Big(\frac{nt+\alpha}{n+\beta}-x\Big)^m dt\\&-(nx+r)\mu^{(\alpha,\beta)}{}_{n,m,r}(x)\\=&-\Big(\frac{n+\beta}{n}\Big)(m+1)\mu_{n,m,r}^{(\alpha,\beta)}(x)\Big(\frac{n}{n+\beta}\Big)-\Big\{\frac{\alpha-(n+\beta)x}{n}\Big\}(-m)\mu_{n,m-1,r}^{(\alpha,\beta)}(x)\Big(\frac{n}{n+\beta}\Big)\\&+n\Big(\frac{n+\beta}{n}\Big)\mu_{n,m+1,r}^{(\alpha,\beta)}(x)-n\Big\{\frac{\alpha-(n+\beta)x}{n}\Big\}\mu_{n,m,r}^{(\alpha,\beta)}(x)-nx\mu_{n,m,r}^{(\alpha,\beta)}(x)\\=&-(m+1)\mu_{n,m,r}^{(\alpha,\beta)}(x)+m\Big\{\frac{\alpha-(n+\beta)x}{n+\beta}\Big\}\mu_{n,m-1,r}^{(\alpha,\beta)}(x)+(n+\beta)\mu_{n,m+1,r}^{(\alpha,\beta)}(x)\\&-\{\alpha-(n+\beta)x\}\mu_{n,m,r}^{(\alpha,\beta)}(x)-(nx+r)\mu_{n,m,r}^{(\alpha,\beta)}(x).\end{aligned}$$

Hence

$$(n+\beta)\mu^{(\alpha,\beta)}_{n,m+1,r}(x)=x[\mu^{(\alpha,\beta)}_{n,m,r}(x)]'+(m+\alpha+r+1-\beta x)\mu^{(\alpha,\beta)}_{n,m,r}(x)$$
$$+m\left\{\frac{2(n+\beta)x-\alpha}{n+\beta}\right\}\mu^{(\alpha,\beta)}_{n,m-1,r}(x).$$

This completes the proof. □

Remark 2.1. From Lemma 2.1, for $n\geq\beta^2+(\alpha+r)^2+3r+2\alpha+2$ and any $x\in(0,\infty)$, we have

$$\mu^{(\alpha,\beta)}_{n,2,r}(x)\leq\frac{(x+1)^2}{n+\beta}.$$

Remark 2.2. Applying Cauchy-Schwarz inequality and Remark 2.1, for $n\geq\beta^2+(\alpha+r)^2+3r+2\alpha+2$, we have

$$S^{(\alpha,\beta)}_{n,r}(|t-x|,x)\leq\left[\mu^{(\alpha,\beta)}_{n,2,r}(x)\right]^{\frac{1}{2}}\leq\frac{x+1}{\sqrt{n+\beta}}.$$

Lemma 2.2. *Suppose that $x\in(0,\infty)$, then for $n\geq r^2+3r+2$, we have*

$$\lambda_{n,r}(x,y)=n\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^y s_{n,k+r}(t)dt\leq\frac{(x+1)^2}{n(x-y)^2},\qquad 0\leq y<x,$$
$$1-\lambda_{n,r}(x,z)=n\sum_{k=0}^{\infty}s_{n,k}(x)\int_z^{\infty}s_{n,k+r}(t)dt\leq\frac{(x+1)^2}{n(z-x)^2},\qquad x<z<\infty.$$

Proof. The result follows directly from Remark 2.1 in the case $\alpha=\beta=0$, as for the first inequality, we have

$$\lambda_{n,r}(x,y)=n\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^y s_{n,k+r}(t)dt=\frac{S^{(0,0)}_{n,r}\left((t-x)^2,x\right)}{(y-x)^2}\leq\frac{(x+1)^2}{n(x-y)^2}.$$

Similarly, we can prove the second inequality. □

Lemma 2.3. *Let f be s times differentiable on $[0,\infty)$ such that $f^{(s-1)}(t)=\mathcal{O}(t^q)$, as $t\to\infty$ where q is a positive integer. Then for any $r,s\in N^0$ and $n>\max\{q,r+s+1\}$, we have*

$$D^sS^{(\alpha,\beta)}_{n,r}(f,x)=\left(\frac{n}{n+\beta}\right)^sS^{(\alpha,\beta)}_{n,r+s}(D^sf,x),\quad D\equiv\frac{d}{dx}.$$

Proof. First, by simple computation, we have

$$D[s_{n,k}(x)]=n[s_{n,k-1}(x)-s_{n,k}(x)]. \tag{2.2}$$

The identity (2.2) is true even for the case $k=0$, as we observe that for $r<0$, $s_{n,r}(x)=0$. We shall prove the result by using the principle of mathematical induction. Using (2.2), we have

$$\begin{aligned}D\big[S_{n,r}^{(\alpha,\beta)}(f,x)\big] &= n\sum_{k=0}^{\infty} Ds_{n,k}(x)\int_0^\infty s_{n,k+r}(t)f\Big(\frac{nt+\alpha}{n+\beta}\Big)dt\\ &= n\sum_{k=0}^{\infty} n\big[s_{n,k-1}(x)-s_{n,k}(x)\big]\int_0^\infty s_{n,k+r}(t)f\Big(\frac{nt+\alpha}{n+\beta}\Big)dt\\ &= n^2\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty \big[s_{n,k+r+1}(t)-s_{n,k+r}(t)\big]f\Big(\frac{nt+\alpha}{n+\beta}\Big)dt.\end{aligned}$$

Using (2.2), and integrating by parts we have

$$\begin{aligned}DS_{n,r}^{(\alpha,\beta)}(f,x) &= n^2\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty -\frac{D[s_{n,k+r+1}(t)]}{n}f\Big(\frac{nt+\alpha}{n+\beta}\Big)dt\\ &= \frac{n^2}{n+\beta}\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty s_{n,k+r+1}(t)f^{(1)}\Big(\frac{nt+\alpha}{n+\beta}\Big)dt\\ &= \frac{n}{n+\beta}S_{n,r+1}^{(\alpha,\beta)}(Df,x),\end{aligned}$$

which means that the identity is satisfied for $s=1$. Let us suppose that the result holds for $s=l$ i.e.,

$$\begin{aligned}D^l S_{n,r}^{(\alpha,\beta)}(f,x) &= \Big(\frac{n}{n+\beta}\Big)^l S_{n,r+l}^{(\alpha,\beta)}(D^l f,x)\\ &= n\Big(\frac{n}{n+\beta}\Big)^l\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty s_{n,k+r+l}(t)D^l f\Big(\frac{nt+\alpha}{n+\beta}\Big)dt.\end{aligned}$$

Now,

$$\begin{aligned}D^{l+1}S_{n,r}^{(\alpha,\beta)}(f,x) &= n\Big(\frac{n}{n+\beta}\Big)^l\sum_{k=0}^{\infty} Ds_{n,k}(x)\int_0^\infty s_{n,k+r+l}(t)D^l f\Big(\frac{nt+\alpha}{n+\beta}\Big)dt\\ &= n\Big(\frac{n}{n+\beta}\Big)^l\sum_{k=0}^{\infty} n\big[s_{n,k+r+l-1}(x)-s_{n,k+r+l}(x)\big]\int_0^\infty s_{n,k+r+l}(t)D^l f\Big(\frac{nt+\alpha}{n+\beta}\Big)dt\\ &= n^2\Big(\frac{n}{n+\beta}\Big)^l\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty \big[s_{n,k+r+l+1}(t)-s_{n,k+r+l}(t)\big]D^l f\Big(\frac{nt+\alpha}{n+\beta}\Big)dt\\ &= n^2\Big(\frac{n}{n+\beta}\Big)^l\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty -\frac{D[s_{n,k+r+l+1}(t)]}{n}D^l f\Big(\frac{nt+\alpha}{n+\beta}\Big)dt.\end{aligned}$$

Integrating by parts for the last integral, we get

$$D^{l+1}S_{n,r}^{(\alpha,\beta)}(f,x) = n\Big(\frac{n}{n+\beta}\Big)^{l+1}\sum_{k=0}^{\infty} s_{n,k}(x)\int_0^\infty s_{n,k+r+l+1}(t)D^{l+1}f\Big(\frac{nt+\alpha}{n+\beta}\Big)dt.$$

Therefore,

$$D^{l+1}S_{n,r}^{(\alpha,\beta)}(f,x)=\Big(\frac{n}{n+\beta}\Big)^{l+1}S_{n,r+l+1}^{(\alpha,\beta)}\big(D^{l+1}f(x)\big).$$

Thus the result is true for

$$s=l+1,$$

hence by mathematical induction, the lemma is valid. □

3 Rate of converence

The class of absolutely continuous functions f defined on $(0,\infty)$ is defined by $B_q(0,\infty)$, $q>0$ and satisfying:

(i) $|f(t)|\le C_1t^q$, $C_1>0$,

(ii) having a derivative f' on the interval $(0,\infty)$ which coincides a.e. with a function of bounded variation on every finite sub-interval of $(0,\infty)$. It can be observed that for all functions $f\in B_q(0,\infty)$ possess for each $c>0$ the representation

$$f(x)=f(c)+\int_c^x\psi(t)dt,\quad x\ge c.$$

Theorem 3.1. *Let $f\in B_q(0,\infty)$, $q>0$ and $x\in(0,\infty)$. Then for n sufficiently large, we have*

$$\begin{aligned}&\Big|S_{n,r}^{(\alpha,\beta)}(f,x)-f(x)\Big|\\ \le&\frac{(x+1)^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x+x/k}((f')_x)+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((f')_x)+\frac{(1+1/x)^2}{n}(|f(2x)-f(x)-xf'(x^+)|\\ &+|f(x)|)+\mathcal{O}(n^{-q})+|f'(x^+)|\frac{(x+1)^2}{n}+\frac{1}{2}\frac{x+1}{\sqrt{n+\beta}}|f'(x^+)-f'(x^-)|\\ &+\frac{\alpha+r+1-\beta x}{2(n+\beta)}|f'(x^+)+f'(x^-)|,\end{aligned}$$

where $\bigvee_a^b f(x)$ denotes the total variation of f_x on $[a,b]$, and the auxiliary function f_x is defined by

$$f_x(t)=\begin{cases}f(t)-f(x^-), & 0\le t<x,\\ 0, & t=x,\\ f(t)-f(x^+), & x<t<\infty.\end{cases}$$

Proof. Using the identity

$$\begin{aligned}f'(u)=&(f')_x(u)+\frac{f'(x^+)+f'(x^-)}{2}+\frac{f'(x^+)-f'(x^-)}{2}\mathrm{sgn}(u-x)\\ &+\Big[f'(x)-\frac{f'(x^+)+f'(x^-)}{2}\Big]\chi_x(u),\end{aligned}\tag{3.1}$$

where

$$\chi_x(u)=\begin{cases} 1, & u=x, \\ 0, & u\neq x. \end{cases}$$

Applying the mean value theorem, we get

$$S_{n,r}^{(\alpha,\beta)}(f,x)-f(x)=S_{n,r}^{(\alpha,\beta)}\Big(\int_x^t f'(u)du,x\Big). \tag{3.2}$$

Now, by using the above identity (3.1) in (3.2) and the fact that

$$S_{n,r}^{\alpha,\beta}\Big(\int_x^t \chi_x(u)du,x\Big)=0,$$

after simple computation, we have

$$\begin{aligned}
\Big|S_{n,r}^{\alpha,\beta}(f,x)-f(x)\Big|\le&\Big|\int_x^\infty\Big(\int_x^t (f')_x(u)du\Big)n\sum_{k=0}^\infty s_{n,k}(x)s_{n,k+r}(t)dt\\
&+\int_0^x\Big(\int_x^t (f')_x(u)du\Big)n\sum_{k=0}^\infty s_{n,k}(x)s_{n,k+r}(t)\mathrm{d}t\Big|\\
&+\frac{\Big|f'(x^+)+f'(x^-)\Big|}{2}\mu_{n,1,r}^{(\alpha,\beta)}(x)+\frac{\Big|f'(x^+)-f'(x^-)\Big|}{2}[\mu_{n,2,r}^{(\alpha,\beta)}(x)]^{1/2}\\
=&|A_{n,r}(f,x)+B_{n,r}(f,x)|+\frac{\Big|f'(x^+)+f'(x^-)\Big|}{2}\mu_{n,1,r}^{(\alpha,\beta)}(x)\\
&+\frac{\Big|f'(x^+)-f'(x^-)\Big|}{2}[\mu_{n,2,r}^{(\alpha,\beta)}(x)]^{1/2}.
\end{aligned} \tag{3.3}$$

Applying Remark 2.1 and Remark 2.2 to (3.3), we have

$$\begin{aligned}
\Big|S_{n,r}^{\alpha,\beta}(f,x)-f(x)\Big|\le&|A_{n,r}(f,x)|+|B_{n,r}(f,x)|+\frac{|f'(x^+)-f'(x^-)|}{2}\frac{x+1}{\sqrt{n+\beta}}\\
&+\frac{|f'(x^+)+f'(x^-)|}{2}\frac{\alpha+r+1-\beta x}{(n+\beta)}.
\end{aligned} \tag{3.4}$$

The estimation of the terms $A_{n,r}(f,x)$ and $B_{n,r}(f,x)$ will lead to proof of the theorem.

First,

$$\begin{aligned}|A_{n,r}(f,x)|=&\Big|\int_x^{\infty}\Big(\int_x^t (f')_x(u)du\Big)n\sum_{k=0}^{\infty}s_{n,k}(x)s_{n,k+r}(t)dt\Big|\\=&\Big|\int_{2x}^{\infty}\Big(\int_x^t (f')_x(u)du\Big)n\sum_{k=0}^{\infty}s_{n,k}(x)s_{n,k+r}(t)dt\\&+\int_x^{2x}\Big(\int_x^t (f')_x(u)du\Big)d_t(1-\lambda_{n,r}(x,t))\Big|\\\le&\Big|n\sum_{k=0}^{\infty}s_{n,k}(x)\int_{2x}^{\infty}(f(t)-f(x))s_{n,k+r}(t)dt\Big|\\&+|f'(x^+)|\Big|n\sum_{k=0}^{\infty}s_{n,k}(x)\int_{2x}^{\infty}s_{n,k+r}(t)(t-x)dt\Big|\\&+\Big|\int_x^{2x}(f')_x(u)du\Big||1-\lambda_{n,r}(x,2x)|+\int_x^{2x}|(f')_x(t)||1-\lambda_{n,r}(x,t)|dt.\end{aligned}$$

Applying Remark 2.1 with $\alpha=\beta=0$, we have

$$\begin{aligned}&|A_{n,r}(f,x)|\\\le& n\sum_{k=0}^{\infty}s_{n,k}(x)\int_{2x}^{\infty}s_{n,k+r}(t)C_1t^{2q}dt+\frac{|f(x)|}{x^2}n\sum_{k=0}^{\infty}s_{n,k}(x)\int_{2x}^{\infty}s_{n,k+r}(t)(t-x)^2dt\\&+|f'(x^+)|\int_{2x}^{\infty}n\sum_{k=0}^{\infty}s_{n,k}(x)s_{n,k+r}(t)|t-x|dt+\frac{(1+1/x)^2}{n}|f(2x)-f(x)-xf'(x^+)|\\&+\frac{(x+1)^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_x^{x+\frac{x}{k}}((f')_x)+\frac{x}{\sqrt{n}}\bigvee_x^{x+\frac{x}{\sqrt{n}}}((f')_x).\end{aligned}\tag{3.5}$$

To estimate the integral $n\sum_{k=0}^{\infty}s_{n,k}(x)\int_{2x}^{\infty}s_{n,k+r}(t)C_1t^{2q}dt$ in (3.5) above, we proceed as follows:

Obviously $t\ge 2x$ implies that $t\le 2(t-x)$ and it follows from Lemma 2.1, that

$$\begin{aligned}n\sum_{k=0}^{\infty}s_{n,k}(x)\int_{2x}^{\infty}s_{n,k+r}(t)t^{2q}dt\le& 2^{2q}\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^{\infty}s_{n,k+r}(t)(t-x)^{2q}dt\\=&2^{2q}\mu_{n,2q,r}^{(\alpha,\beta)}(x)=\mathcal{O}(n^{-q}),\quad n\to\infty.\end{aligned}$$

Applying Schwarz inequality and Remark 2.1 ($\alpha=\beta=0$), the third term in right hand side of (3.5) is estimated as follows:

$$\begin{aligned}&|f'(x^+)|n\sum_{k=0}^{\infty}s_{n,k}(x)\int_{2x}^{\infty}s_{n,k+r}(t)|t-x|dt\\\le&\frac{|f'(x^+)|}{x}n\sum_{k=0}^{\infty}s_{n,k}(x)\int_0^{\infty}s_{n,k+r}(t)(t-x)^2dt=|f'(x^+)|\frac{(x+1)^2}{nx}.\end{aligned}$$

Thus by Lemma 2.1 and Remark 2.1 ($\alpha=\beta=0$), we have

$$\begin{aligned}|A_{n,r}(f,x)|\le&\mathcal{O}(n^{-q})+|f'(x^+)|\cdot\frac{(x+1)^2}{nx}\\&+\frac{(1+1/x)^2}{n}(|f(2x)-f(x)-xf'(x^+)|+|f(x)|)\\&+\frac{(x+1)^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+\frac{x}{k}}((f')_x)+\frac{x}{\sqrt{n}}\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}((f')_x).\end{aligned}\tag{3.6}$$

Applying, Lemma 2.2 with $y=x-x/\sqrt{n}$, and integrating by parts, we have

$$\begin{aligned}|B_{n,r}(f,x)|&=\Big|\int_0^x\int_x^t(f')_x(u)du\mathrm{d}_t(\lambda_{n,r}(x,t))\Big|\\&=\int_0^x\lambda_{n,r}(x,t)(f')_x(t)dt\le\Big(\int_0^y+\int_y^x\Big)|(f')_x(t)||\lambda_{n,r}(x,t)|dt\\&\le\frac{(x+1)^2}{n}\int_0^y\bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt+\int_y^x\bigvee_t^x((f')_x)dt\\&\le\frac{(x+1)^2}{n}\int_0^y\bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}((f')_x).\end{aligned}$$

Let $u=x/(x-t)$, then we have

$$\begin{aligned}\frac{(x+1)^2}{n}\int_0^y\bigvee_t^x((f')_x)\frac{1}{(x-t)^2}dt&=\frac{(x+1)^2}{n}\int_1^{\sqrt{n}}\bigvee_{x-\frac{x}{u}}^{x}((f')_x)du\\&\le\frac{(x+1)^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}((f')_x).\end{aligned}$$

Thus

$$|B_{n,r}(f,x)|\le\frac{(x+1)^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-\frac{x}{k}}^{x}((f')_x)+\frac{x}{\sqrt{n}}\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}((f')_x).\tag{3.7}$$

The required result is obtained by combining (3.4), (3.6) with (3.7). □

As a consequence of Lemma 2.3, we have the following corollary:

Corollary 3.1. Let $f^{(s)}\in DB_q(0,\infty)$, $q>0$ and $x\in(0,\infty)$. Then for n sufficiently large, we have

$$\begin{aligned}
&\left|D^s S_{n,r}^{(\alpha,\beta)}(f,x)-f^{(s)}(x)\right|\\
\le&\frac{(x+1)^2}{nx}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-x/k}^{x+x/k}((D^{s+1}f)_x)+\frac{x}{\sqrt{n}}\bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}((D^{s+1}f)_x)\\
&+\frac{(1+1/x)^2}{n}(|D^sf(2x)-D^sf(x)-xD^{s+1}f(x^+)|+|D^sf(x)|)+\mathcal{O}(n^{-q})\\
&+\frac{(x+1)^2}{nx}|D^{s+1}f(x^+)|+\frac{1}{2}\frac{x+1}{\sqrt{n+\beta}}|D^{s+1}f(x^+)-D^{s+1}f(x^-)|\\
&+\frac{1}{2}|D^{s+1}f(x^+)+D^{s+1}f(x^-)|\frac{\alpha+r+1-\beta x}{n+\beta},
\end{aligned}$$

where $\bigvee_a^b f(x)$ denotes the total variation of f_x on $[a,b]$, and f_x is defined by

$$D^{s+1}f_x(t)=\begin{cases} D^{s+1}f(t)-D^{s+1}f(x^-), & 0\le t<x,\\ 0, & t=x,\\ D^{s+1}f(t)-D^{s+1}f(x^+), & x<t<\infty.\end{cases}$$

4 Asymptotic formula

We consider the class $L[0,\infty)$ of all measurable functions defined on $[0,\infty)$ such that

$$L[0,\infty):=\Big\{f:\int_0^\infty e^{-nt}f(t)dt<\infty \text{ for some positive integer } n\Big\}.$$

It can be observed that this class is bigger than that of all integrable functions on $[0,\infty)$.

Further we consider

$$L_\alpha[0,\infty):=\big\{f\in L[0,\infty):f(t)=\mathcal{O}(e^{\alpha t}),\ t\to\infty,\ \alpha>0\big\}.$$

We have the following asymptotic formula by using Lemma 2.1.

Theorem 4.1. *Let $f\in L_\alpha[0,\infty)$ and suppose it is bounded on every finite subinterval of $[0,\infty)$ having a derivative of order $r+2$ at a point $x\in(0,\infty)$, then we have*

$$\lim_{n\to\infty}n[(S_{n,r}^{(\alpha,\beta)})^{(r)}(f,x)-f^{(r)}(x)]=(\alpha+r+1-\beta x)f^{(r+1)}(x)+xf^{(r+2)}(x).$$

The proof follows along the line of [1].

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