Fixed Point Theory for 1-Set Weakly Contractive Operators in Banach Spaces

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Abstract. In this work, using an analogue of Sadovskii’s fixed point result and several important inequalities we investigate and give new existence theorems for the nonlinear operator equation $F(x) = \mu x$, $(\mu \geq 1)$ for some weakly sequentially continuous, weakly condensing and weakly 1-set weakly contractive operators with different boundary conditions. Correspondingly, we can obtain some applicable fixed point theorems of Leray-Schauder, Altman and Furi-Pera types in the weak topology setting which generalize and improve the corresponding results of [3, 15, 16].

Key Words: Weakly condensing, weakly sequentially continuous, fixed point theorem, operator equation.

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1 Introduction

The Study of fixed point problem for condensing operators and 1-set contractive operators especially for a closed convex subset into itself has been one of the main objects of research in nonlinear functional analysis and was started by [19], Petryshyn [17, 18], Nussbaum [14] and Browder [4]. These studies were mainly based on the potential tool of degree theory. Since then, whether a 1-set contractive mapping defined on the closure of bounded open subset of a Banach space has a fixed point, has become an interesting problem. For example, in [12] the author has defined the fixed point index of 1-set-contractive operators, introduced the concept of semi-closed 1-set-contractive operators and obtained some fixed point theorems of such a class of operators. Recently, Some existence theorems for fixed points of semi-closed 1-set-contractive type mappings

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and under several boundary conditions related to degree theory have been considered by some authors (see [13, 21, 23]). Because the weak topology is the convenient and natural setting to investigate the existence problems of fixed points and eigenvectors for operators and solutions of various kinds of nonlinear differential equations and nonlinear integral equations in Banach spaces, the above mentioned results cannot be easily applied. These equations can be transformed into fixed point problems and nonlinear operator equations involving a broader class of nonlinear operators, in which the operators have the property that the image of any set in a certain sense more weakly compact than the original set itself. The major problem to face is that an infinite dimensional Banach space equipped with its weak topology does not admit open bounded sets. That is, a weakly closed and bounded subset has a weak interior and thus coincides with its weak boundary which yields very difficult the verification of the boundary conditions. To this interest, we introduce the concept of weakly semi-closed operators at the origin (see Definition 2.5), and provide new existence theorems for the nonlinear operator equation $F(x) = \mu x$ ($\mu \geq 1$), for some class of weakly sequentially continuous, weakly condensing ($\beta$-condensing) and 1-set weakly contractive ($\beta$-nonexpansive) operators (see Definition 2.1) under several boundary conditions in particular of Furi-Pera type [9] and under weakest assumptions on operators and domains as it is known (here $\beta$ is the DeBlasi measure of weak noncompactness see [5]). Meanwhile analogues of fixed point theorems of Altman, Leray-Schauder and Furi-Pera types are given in the weak topology setting which generalize and extend relevant and recent ones (see [3, 15, 16]). In addition, our arguments and methods are elementary in the sense that without any recourse to degree theory or theory of homotopy-extensions.

2 Preliminaries

Let $\Omega_E$ be the family of bounded subsets of a Banach space $E$ and let $\mathcal{K}^w$ be the family of weakly compact subsets of $E$. Also let $B_E$ be the closed unit ball of $E$. The DeBlasi [5] measure of weak noncompactness is the map $\beta: \Omega_E \to [0, \infty)$ defined by

$$\beta(X) = \inf \{ t > 0 : \text{there exists } Y \in \mathcal{K}^w \text{ such that } X \subset Y + tB_E \}, \text{ here } X \in \Omega_E.$$

For convenience we recall some properties of $\beta$: Let $X_1, X_2 \in \Omega_E$. Then

(i) $X_1 \subset X_2$ implies $\beta(X_1) \leq \beta(X_2)$.

(ii) $\beta(X_1) = 0$ iff $\overline{X_1^w} \in \mathcal{K}^w$, here $\overline{X_1^w}$ is the weak closure of $X_1$ in $E$.

(iii) $\beta(X_1) = \beta(\overline{X_1^w})$.

(iv) $\beta(X_1 \cup X_2) = \max \{ \beta(X_1), \beta(X_2) \}$.

(v) $\beta(\lambda X_1) = \lambda \beta(X_1)$ for all $\lambda > 0$.

(vi) $\beta(\text{conv}(X_1)) = \beta(X_1)$.

(vii) $\beta(X_1 + X_2) \leq \beta(X_1) + \beta(X_2)$.
Definition 2.1. Let $D$ be a nonempty subset of the Banach space $E$. If $F$ maps $D$ into $E$, we say that

(a) $F$ is $\beta$-condensing if $F$ is bounded and $\beta(F(V)) < \beta(V)$ for all bounded subsets $V$ of $D$ with $\beta(V) > 0$,

(b) $F$ is $\beta$-nonexpansive if $F$ is bounded and $\beta(F(V)) \leq \beta(V)$ for all bounded subsets $V$ of $D$.

Definition 2.2. Let $E$ be a Banach space. An operator $F: E \rightarrow E$ is said to be weakly compact if $F(D)$ is relatively weakly compact for every bounded subset $D \subset E$.

Definition 2.3. Let $E$ be a Banach space. An operator $F: E \rightarrow E$ is said to be weakly sequentially continuous on $E$ iff for every sequence $(x_n)_n$ with $x_n \overset{\text{w}}{\rightarrow} x$, we have $F(x_n) \overset{\text{w}}{\rightarrow} F(x)$, here $\overset{\text{w}}{\rightarrow}$ denotes weak convergence.

Definition 2.4. A subset $D$ of a Banach space is called weakly sequentially closed if, whenever $x_n \in D$ for all $n \in \mathbb{N}$ and $x_n \overset{\text{w}}{\rightarrow} x$, then $x \in D$.

Definition 2.5. Let $D$ be a nonempty weakly closed set of a Banach space $E$ and $F: D \rightarrow E$ a weakly sequentially continuous operator. $F$ is said to be weakly semi-closed operator at $\theta$ iff the conditions $x_n \in D$, $x_n - F(x_n)$ converges weakly to $\theta$ imply that there exists $x \in D$ such that $F(x) = x$.

It should be noted that this class of operators, as special cases, includes the weakly sequentially continuous operators which are weakly compact, weakly contractive, $\beta$-condensing, $(I - F)(D)$ is weakly sequentially closed and others.

The following fixed point result stated in [3] is an analogue of Sadovskii’s fixed point result [1], will be used throughout this section. The proof follows from the O. Arino, S. Gautier and J. P. Penot Theorem [2].

Theorem 2.1. Let $\Omega$ be a non-empty, convex closed set in a Banach space $E$. Assume $F: \Omega \rightarrow \Omega$ is a weakly sequentially continuous and $\beta$-condensing mapping. In addition, suppose that $F(\Omega)$ is bounded. Then $F$ has a fixed point.
3 Main results

We start this section by stating some interesting facts of a weakly sequentially and β-condensing operators which are useful in the sequel.

**Lemma 3.1.** Let C be a nonempty weakly closed set of a Banach space E and F : C → E a weakly sequentially continuous and β-condensing operator with F(C) being bounded, then

(a) for all weakly compact subsets K of E, \((I - F)^{-1}(K)\) is weakly compact.

(b) \(I - F\) maps weakly closed subsets of C onto weakly sequentially closed sets in E.

**Proof.** (a) Let \(K \subseteq E\) be a nonempty weakly compact set and let \(D = (I - F)^{-1}(K)\). Since \(I - F\) is weakly sequentially continuous, \(D\) is weakly sequentially closed. Moreover, we have

\[
β(D) \leq β(K) + β(F(D)) = β(F(D)).
\]

Since \(F\) is β-condensing, it follows that \(β(D) = 0\). Let \(x \in C\) be weakly adherent to \(D\). Since \(\overline{D}^w\) is weakly compact, by Eberlein-Šmulian Theorem [6, Theorem 8.12.4, pp. 549], there exists a sequence \((x_n)_n \subseteq D\) such that \(x_n \xrightarrow{w} x\), so \(x \in D\). Hence \(\overline{D}^w = D\) and \(D\) is a weakly closed subset of \(C\). Therefore \(D\) is weakly compact.

(b) Let \(D \subseteq C\) be a weakly closed set and consider \(x_n \in (I - F)(D)\) such that \(x_n \xrightarrow{w} x\) in \(E\). We have \(x_n = (I - F)(u_n), \forall n \geq 1\) with \(u_n \in D\). The set \(K = \{x_n\}_n^w\) is weakly compact and so \((I - F)^{-1}(K)\) is weakly compact. Therefore, we may assume that \(u_n \xrightarrow{w} u\) in \(D\), for some \(u \in D\). Due to the weak sequential continuity of \(I - F\), we have \(x = (I - F)(u)\) and so \(x \in (I - F)(D)\). Accordingly \((I - F)(D)\) is weakly sequentially closed. \(\square\)

Now, we are ready to investigate a class of operator equations for a broader class of nonlinear weakly sequentially continuous operators, in which the operators have the property that the image of any set is in a certain sense more weakly compact than the original set itself.

**Theorem 3.1.** Let \(Ω\) be a closed convex subset of a Banach space \(E\), \(U\) a weakly open subset of \(Ω\) such that \(θ \in U\) and \(μ \geq 1\). Suppose that \(F : \overline{U}^w \to Ω\) is weakly sequentially continuous β-condensing operator and \(F(\overline{U}^w)\) is bounded. Then, either

(A1) the operator equation \(F(x) = μx\) has a solution in \(\overline{U}^w\), or

(A2) there is a point \(x \in \partial_Ω U\) (the weak boundary of \(U\) in \(Ω\)) and \(λ \in (0,1)\) with \(x = λF(x)/μ\).

**Proof.** Suppose \(A_2\) does not hold. We observe that supposition is satisfied also for \(λ = 0\) (since \(θ \in U\)). If \(A_2\) is satisfied for \(λ = 1\), then the theorem has obtained proof. In conclusion, we can consider that the supposition is not satisfied for any \(x \in \partial_Ω U\) and any \(λ \in [0,1]\). Let \(D\) the set defined by

\[
D = \{x \in \overline{U}^w : x = \frac{λ}{μ}F(x), \text{ for some } λ \in [0,1]\}.
\]
If a fixed point \( \beta \in D \) and \( F(\overline{U}^w) \) is bounded. We have \( D \subset \text{conv}\{(\theta) \cup F(D)\} \). So, \( \beta(D) \neq 0 \) implies

\[
\beta(D) \leq \beta(\text{conv}\{(\theta) \cup F(D)\}) \leq \beta(F(D)) < \beta(D),
\]

which is a contradiction. Hence, \( \beta(D) = 0 \) and \( D \) is relatively weakly compact. Now, we prove that \( D \) is weakly sequentially closed. The weak sequentially continuity of \( F \) implies that \( D \) is weakly sequentially closed. For that, let \( (x_n)_n \) a sequence of \( D \) such that \( x_n \overset{w}{\to} x, \) \( x \in \overline{U}^w \). For all \( n \in \mathbb{N} \), there exists a \( \lambda_n \in [0,1] \) such that \( x_n = \lambda_n F(x_n) / \mu. \lambda_n \in [0,1], \) we can extract a subsequence \( (\lambda_{n_j})_j \) such that \( \lambda_{n_j} \to \lambda \in [0,1]. \) So, \( \lambda_{n_j} F(x_{n_j}) / \mu \overset{w}{\to} \lambda F(x) / \mu. \) Hence \( x = \lambda F(x) / \mu \) and \( x \in D. \) Let \( x \in \overline{U}^w, \) be weakly adherent to \( D. \) Since \( \overline{D}^w \) is weakly compact, by the Eberlein-Šmulian Theorem [6, Theorem 8.12.4, pp. 549], there exists a sequence \( (x_n)_n \subset D \) such that \( x_n \overset{w}{\to} x, \) \( x \in D. \) Hence \( \overline{D}^w = D \) and \( D \) is a weakly closed subset of \( \overline{U}^w. \) Therefore \( D \) is weakly compact. Because \( E \) endowed with its weak topology is a Hausdorff locally convex space, we have that \( E \) is completely regular [20, pp. 16]. Since \( D \cap (\Omega \setminus U) = \emptyset, \) Then by [10, pp. 146], there is a weakly continuous function \( \varphi: \Omega \to [0,1], \) such that \( \varphi(x) = 1 \) for \( x \in D \) and \( \varphi(x) = 0 \) for \( x \in \Omega \setminus U. \) Let \( F^+: \Omega \to \Omega \) be the mapping defined by:

\[
F^*(x) = \frac{\varphi(x)}{\mu} F(x), \quad (\mu \geq 1).
\]

Because \( \Omega \) is convex, \( \theta \in \Omega \) and \( \mu \geq 1, \) \( F^* \) is well defined. Clearly, \( F^*(\Omega) \) is bounded. Because \( \partial \Omega U = \partial \overline{U}^w, \varphi \) is weakly continuous and \( F \) is weakly sequentially continuous, we have that \( F^* \) is weakly sequentially continuous. Let \( X \subset \Omega, \) bounded. Then, since

\[
F^*(X) \subset \text{conv}\{(\theta) \cup F(X \cap U)\},
\]

we have

\[
\beta(F^*(X)) \leq \beta(F(X \cap U)) \leq \beta(F(X)),
\]

and \( \beta(F^*(X)) < \beta(X) \) if \( \beta(X) \neq 0. \) So, \( F^* \) is \( \beta \)-condensing. Therefore by Theorem 2.1, \( F^* \) has a fixed point \( x_0 \in \Omega. \) If \( x_0 \not\in U, \varphi(x_0) = 0 \) and \( x_0 = 0, \) which contradicts the hypothesis \( \theta \in U. \) Then \( x_0 \in U \) and \( x_0 = \varphi(x_0) F(x_0) / \mu, \) which implies that \( x_0 \in D. \) Accordingly, \( \varphi(x_0) = 1 \) and so \( F(x_0) = \mu x_0 \) and the proof is complete.

**Remark 3.1.** When \( \mu = 1, \) we obtain Theorem 3.1 in [3]. When \( \mu = 1 \) Theorem 3.1 extends Theorem 2.3 in [16] and shows that the condition \( \overline{U}^w \) is weakly compact in the statement of this theorem is redundant.

**Corollary 3.1.** Let \( \Omega \) be a closed convex subset of a Banach space \( E, \) \( U \) a weakly open subset of \( \Omega \) such that \( \theta \in U \) and \( \mu \geq 1. \) Suppose that \( F: \overline{U}^w \to \Omega \) is weakly sequentially continuous operator and \( F(\overline{U}^w) \) is relatively weakly compact. Then, either

\( \quad (A_1) \quad \text{the operator equation } F(x) = \mu x \text{ has a solution in } \overline{U}^w, \) or
(A2) there is a point \(x \in \partial_\Omega U\) (the weak boundary of \(U\) in \(\Omega\)) and \(\lambda \in (0,1)\) with \(x = \frac{\lambda F(x)}{\mu}\).

**Corollary 3.2.** Let \(\Omega\) be a closed convex subset of a Banach space \(E, U\) a weakly open subset of \(\Omega\) such that \(\theta \in U\) and \(\mu \geq 1\). Suppose that \(F : U^w \to \Omega\) is weakly sequentially continuous \(\beta\)-condensing operator and \(F(U^w)\) is bounded. In addition we suppose that

\[
x \neq \frac{\lambda}{\mu} F(x), \quad \lambda \in (0,1), \quad x \in \partial_\Omega U.
\]

Then, the operator equation \(F(x) = \mu x\) has a solution in \(U^w\).

**Proof.** It suffices to apply Theorem 3.1. \(\square\)

**Theorem 3.2.** Let \(\Omega\) be a closed convex subset of a Banach space \(E, U\) a weakly open subset of \(\Omega\) such that \(\theta \in U\) and \(\mu \geq 1\). Suppose that \(F : U^w \to \Omega\) is weakly sequentially continuous \(\beta\)-nonexpansive operator and \(F(U^w)\) is bounded. In addition, assume that

(a) \(x \neq \frac{\lambda}{\mu} F(x), \lambda \in (0,1), x \in \partial_\Omega U\).

(b) \(F\) is weakly semi-closed at \(\theta\).

Then, the operator equation \(F(x) = \mu x\) has a solution in \(U^w\).

**Proof.** Let \(F_n = t_n F / \mu, n = 1, 2, \ldots\), where \((t_n)_n\) is a sequence of \((0,1)\) such that \(t_n \to 1\). Since \(\theta \in \Omega\) and \(\Omega\) is convex, it follows that \(F_n\) maps \(U^w\) into \(\Omega\). Suppose that \(\lambda_n F_n y_n = y_n\) for some \(y_n \in \partial_\Omega U\) and for some \(\lambda_n \in (0,1)\). Then we have \(\lambda_n t_n F_n y_n = y_n\) which contradicts the hypothesis (a) since \(\lambda_n t_n / \mu \in (0,1)\). Let \(X\) an arbitrary bounded subset of \(U^w\). Then we have

\[
\beta(F_n(X)) = \beta(t_n F(X)) \leq \frac{t_n}{\mu} \beta(F(X)) \leq t_n \beta(X).
\]

So, if \(\beta(X) \neq 0\) we have

\[
\beta(F_n(X)) < \beta(X).
\]

Therefore \(F_n\) is \(\beta\)-condensing on \(U^w\) (note that \(\mu \geq 1\)). From Corollary 3.2, \(F_n\) has a fixed point, say, \(x_n \in U^w\). Therefore,

\[
x_n - \frac{1}{\mu} F(x_n) = \frac{1}{\mu} (t_n - 1) F(x_n) \xrightarrow{n \to \infty} \theta\quad \text{as } n \to \infty,
\]

since \(t_n \to 1\) as \(n \to \infty\) and \(F(U^w)\) is bounded. Since \(F / \mu\) is either \(\beta\)-condensing (if \(\mu > 1\)) or \(\beta\)-nonexpansive (if \(\mu = 1\)), by Lemma 3.1 and condition (b), we obtain that there exists a point \(x_0\) in \(U^w\) such that

\[
\theta = \left( I - \frac{1}{\mu} F \right) (x_0).
\]

Thus \(F(x_0) = \mu x_0\). \(\square\)
As a consequence of the previous theorem we obtain an extension of Theorem 3.4 in [3]:

**Corollary 3.3.** Let \( \Omega \) be a closed convex subset of a Banach space \( E \) and \( U \) a weakly open subset of \( \Omega \) such that \( \theta \in U \). Suppose that \( F : \overline{U}^{w} \to \Omega \) is weakly sequentially continuous, \( \beta \)-nonexpansive and semi-weakly closed operator at \( \theta \) and \( F(\overline{U}^{w}) \) is bounded. In addition we suppose that \( F \) satisfies the Leray-Schauder boundary condition

\[
x \neq \lambda F(x), \quad \lambda \in (0,1), \quad x \in \partial \Omega U. \tag{3.1}
\]

Then, the operator \( F \) has a fixed point in \( \overline{U}^{w} \).

**Remark 3.2.** Corollary 3.3 is a nonlinear alternative of Leray-Schauder type for weakly sequentially continuous, \( \beta \)-nonexpansive and semi-weakly closed operator at \( \theta \).

The next lemma holds easily:

**Lemma 3.2.** When \( y > 1 \) and \( \alpha > 0 \), the following inequality holds:

\[
(y + 1)^{\alpha + 1} > y^{\alpha + 1} + 1. \tag{3.2}
\]

**Theorem 3.3.** Let \( \Omega \) be a closed convex subset of a Banach space \( E \), \( U \) a weakly open subset of \( \Omega \) such that \( \theta \in U \) and \( \mu \geq 1 \). Suppose that \( F : \overline{U}^{w} \to \Omega \) is weakly sequentially continuous \( \beta \)-nonexpansive operator, \( F(\overline{U}^{w}) \) is bounded and \( F \) is weakly semi-closed at \( \theta \). In addition, assume that there exist \( \gamma \geq 0 \) and \( \alpha > 0 \) such that

\[
\left[ \gamma \| \mu x \| + \| F(x) + \mu x \|^{\alpha} \right] \| F(x) + \mu x \|^{\alpha}
\leq \left[ \gamma \| \mu x \| + \| F(x) \|_{\alpha} \right] \| F(x) \| + \gamma \| \mu x \|^{2} + \| \mu x \|^{\alpha + 1} \tag{3.3}
\]

for every \( x \in \partial \Omega U \). Then the operator equation \( F(x) = \mu x \) has a solution in \( \overline{U}^{w} \).

**Proof.** We suppose that the operator equation \( F(x) = \mu x \) has no solution in \( \partial \Omega U \) (otherwise we are finished). In order to apply Theorem 3.2, we prove that

\[
x \neq \frac{\lambda}{\mu} F(x), \quad \lambda \in (0,1), \quad x \in \partial \Omega U. \tag{3.4}
\]

Suppose that (3.4) is not true. Then there exist \( \lambda_{0} \in (0,1) \) and \( x_{0} \in \partial \Omega U \), such that \( \lambda_{0} F(x_{0}) / \mu = x_{0} \). That is \( F(x_{0}) = \mu x_{0} / \lambda_{0} \). Inserting \( F(x_{0}) = \mu x_{0} / \lambda_{0} \) into (3.3), we obtain

\[
\left[ \gamma \| \mu x_{0} \| + \left( \frac{\mu}{\lambda_{0}} x_{0} + \mu x_{0} \right)^{\alpha} \right] \left( \frac{\mu}{\lambda_{0}} x_{0} + \mu x_{0} \right)
\leq \left[ \gamma \| \mu x_{0} \| + \left( \frac{\mu}{\lambda_{0}} x_{0} \right)^{\alpha} \right] \left( \frac{\mu}{\lambda_{0}} x_{0} \right) + \gamma \| \mu x_{0} \|^{2} + \| \mu x_{0} \|^{\alpha + 1}.
\]
This implies
\[\gamma \|\mu x_0\| \left\| \frac{\mu}{\lambda_0} x_0 + \mu x_0 \right\| + \left\| \frac{\mu}{\lambda_0} x_0 + \mu x_0 \right\|^{a+1} \leq \gamma \|\mu x_0\| \left\| \frac{\mu}{\lambda_0} x_0 \right\| + \left\| \frac{\mu}{\lambda_0} x_0 \right\|^{a+1} + \gamma \|\mu x_0\|^2 + \|\mu x_0\|^{a+1}.\]

That is
\[\gamma \left(\frac{1}{\lambda_0} + 1\right) \|\mu x_0\|^2 + \left(\frac{1}{\lambda_0} + 1\right)^{a+1} \|\mu x_0\|^{a+1} \leq \gamma \left(\frac{1}{\lambda_0} + 1\right) \|\mu x_0\|^2 + \frac{1}{\lambda_0^{a+1}} \|\mu x_0\|^{a+1} + \|\mu x_0\|^{a+1}.\]  \hspace{1cm} (3.5)

Since \(\mu \geq 1\), \(x_0 \in \partial \Omega U\), thus \(\mu x_0 \neq \theta\). Therefore, \(\|\mu x_0\|^{a+1} \neq 0\) and by (3.5), we obtain
\[\left(\frac{1}{\lambda_0} + 1\right)^{a+1} \leq \frac{1}{\lambda_0^{a+1}} + 1,\]
and this contradicts Lemma 3.2, since \(\lambda_0^{-1} \in (1, \infty)\). Hence
\[x \neq \frac{\lambda}{\mu} F(x), \quad \lambda \in (0, 1), \quad x \in \partial \Omega U.\]

Accordingly, by Theorem 3.2 the operator equation \(F(x) = \mu x\) has a solution in \(\overline{U^w}\). \(\square\)

As a consequence we have this fixed point result:

**Corollary 3.4.** Let \(\Omega\) be a closed convex subset of a Banach space \(E\) and \(U\) a weakly open subset of \(\Omega\) such that \(\theta \in U\). Suppose that \(F:\overline{U^w} \to \Omega\) is weakly sequentially continuous \(\beta\)-nonexpansive operator, \(F(\overline{U^w})\) is bounded and \(F\) is weakly semi-closed at \(\theta\). In addition, assume that assume that there exist \(\gamma \geq 0\) and \(\alpha > 0\) such that
\[\left[\gamma \|x\| + \|F(x) + x\|^\alpha\right] \|F(x) + x\| \leq \left[\gamma \|x\| + \|F(x)\|^\alpha\right] \|F(x)\| + \gamma \|x\|^2 + \|x\|^{\alpha+1}\]
for every \(x \in \partial \Omega U\). Then the operator \(F\) has a fixed point in \(\overline{U^w}\).

**Proof.** In fact, from Theorem 3.3 it suffices to set \(\mu = 1\). \(\square\)

**Lemma 3.3.** When \(y > 1\) and \(\alpha > 0\), the following inequality holds:
\[(y-1)^{\alpha+1} < y^{\alpha+1} - 1.\]
Theorem 3.4. Let $\Omega$ be a closed convex subset of a Banach space $E$, $U$ a weakly open subset of $\Omega$ such that $\theta \in U$ and $\mu \geq 1$. Suppose that $F : \overline{U^w} \to \Omega$ is weakly sequentially continuous $\beta$-nonexpansive operator, $F(\overline{U^w})$ is bounded and $F$ is weakly semi-closed at $\theta$. In addition, assume that there exist $\gamma \geq 0$ and $\alpha > 0$ such that
\[
\gamma \|\mu x\| + \|F(x) - \mu x\|^2 \geq \|F(x)\|^2 - \alpha \|x\|^2
\]
for every $x \in \partial \Omega U$. Then the operator equation $F(x) = \mu x$ has a solution in $\overline{U^w}$.

Remark 3.3. When $\gamma = 0$, $\alpha = 1$ and $\mu = 1$, Eq. (3.6) is that $\|F(x) - x\|^2 \geq \|F(x)\|^2 - \|x\|^2$. Thus, Theorem 3.4 is a generalization and an analogous of the famous Altman fixed point theorem in the case of weakly sequentially, $\beta$-nonexpansive and weakly semi-closed operator at $\theta$.

Using Lemma 3.2 and Lemma 3.3, we obtain:

Lemma 3.4. When $y > 1$ and $a > 0$, the following inequality holds:
\[
(y+1)^{a+1} - (y-1)^{a+1} > 2.
\]

Theorem 3.5. Let $\Omega$ be a closed convex subset of a Banach space $E$, $U$ a weakly open subset of $\Omega$ such that $\theta \in U$ and $\mu \geq 1$. Suppose that $F : \overline{U^w} \to \Omega$ is weakly sequentially continuous $\beta$-nonexpansive operator, $F(\overline{U^w})$ is bounded and $F$ is weakly semi-closed at $\theta$. In addition, assume that there exists $\alpha > 0$ such that
\[
\|F(x) + \mu x\|^{a+1} - \|F(x) - \mu x\|^{a+1} \leq 2 \|\mu x\|^{a+1},
\]
for every $x \in \partial \Omega U$. Then the operator equation $F(x) = \mu x$ has a solution in $\overline{U^w}$.

Proof. The Theorem can be proved using Lemma 3.4 and techniques used in the proof of Theorem 3.3.

We now investigate nonlinear operator equation and present fixed point results of Furi-Pera type [9] for weakly sequentially continuous operator $F : M \to \Omega$ in separable Banach space $E$ where $M$ and $\Omega$ are two closed convex subset of $E$ such that $M$ is a proper subset of $\Omega$ (note that the weak interior of $M$ may be empty). These results can be used to establish general existence principles for function, integral and differential equations.

Theorem 3.6. Let $E$ be a separable Banach space, $\Omega$ a weakly compact convex subset of $E$, and $M$ a proper closed convex subset of $\Omega$ with $\theta \in M$ and $\mu \geq 1$. Also, assume $F : M \to \Omega$ is a weakly sequentially continuous operator. In addition assume
\[
\text{there exists a weakly sequentially continuous retracion} \, : \, \Omega \to M, \quad (3.7a)
\]
if \( \{ (x_j, \lambda_j) \}_{j=1}^{\infty} \) is a sequence in $M \times \{0,1\}$ with $x_j \xrightarrow{w} x$, $\lambda_j \to \lambda$ and
\[
x = \frac{\lambda_j}{\mu} F(x_j) \text{ with } 0 \leq \lambda < 1, \text{ then } \frac{\lambda_j}{\mu} F(x_j) \in M \text{ for } j \text{ sufficiently large.} \quad (3.7b)
\]

Then the operator equation $F(x) = \mu x$ has a solution in $M$. 

Proof. Consider 

\[ N = \left\{ x \in \Omega, \ x = \frac{1}{\mu} Fr(x) \right\}. \]

We first show \( N \neq \emptyset \). To see this look at \( r \circ (F/\mu) \). We have \( r \circ (F/\mu) : M \to M \) is weakly sequentially continuous and \( \beta \)-condensing map (\( M \) is weakly compact) then by Theorem 2.1 there exists \( y \in M \) such that \( y = r(F(y)/\mu) \). Let 

\[ z = \frac{1}{\mu} F(y) \in \Omega, \ (\theta \in \Omega, \ \mu \geq 1 \text{ and } \Omega \text{ convex}). \]

Then, we obtain

\[ \frac{1}{\mu} Fr(z) = \frac{1}{\mu} Fr\left(\frac{1}{\mu} F(y)\right) = \frac{1}{\mu} F(y) = \frac{1}{\mu} \mu z = z. \]

Thus, \( N \neq \emptyset \). Now let \( (x_n) \) a sequence of \( N \) such that \( x_n \xrightarrow{w} x \in \Omega \). For all \( n \in \mathbb{N} \), we have 

\[ x_n = \frac{1}{\mu} Fr(x_n) \text{ and } r(x_n) \xrightarrow{w} r(x) \]

in \( M \). Because \( F/\mu \) is weakly sequentially continuous, \( x = Fr(x)/\mu \). Hence \( x \in N \) and \( N \) is weakly sequentially closed. Since \( \Omega \) is weakly compact, applying again the Eberlein-Šmulian theorem, we obtain that \( N \) is weakly compact. We now show \( M \cap N \neq \emptyset \). To do this, we argue by contradiction and we use some ideas in [9]. Suppose \( M \cap N = \emptyset \). Then since \( N \) is compact and \( M \) is closed we have by [8, pp. 65] that there exists \( \delta > 0 \) such that \( d(N,M) = \inf \{ \| x - y \| : x \in N, y \in M \} > \delta > 0 \). Because \( E \) is separable, the weak topology on \( \Omega \) is metrizable (see [22]), and let \( \rho \) denote this metric. Choose \( m \in \{1,2,\cdots\} \) such that \( 1 < \delta m \). For \( i \in \{m,m+1,\cdots\} \), let

\[ U_i = \left\{ x \in \Omega, \ \rho(x,M) < \frac{\epsilon}{i} \right\}. \]

We fix \( i \in \{m,m+1,\cdots\} \). Now \( U_i \) is open in \( \Omega \) with respect to the topology generated by \( \rho \), and so \( U_i \) is weakly open in \( \Omega \). Also we have

\[ \overline{U_i^c} = \overline{U_i^c} = \left\{ x \in \Omega, \ \rho(x,M) \leq \frac{\epsilon}{i} \right\} \]

and

\[ \partial \Omega U_i = \left\{ x \in \Omega, \ \rho(x,M) = \frac{\epsilon}{i} \right\}. \]

Since \( d(N,M) > \epsilon \), then \( N \subseteq \overline{U_i^c} = \emptyset \). Applying Theorem 3.3 in [3], we get that there exists \( \lambda_i \in (0,1) \) and \( y_i \in \partial \Omega U_i \) such that

\[ y_i = \frac{\lambda_i}{\mu} Fr(y_i) \quad (\mu \geq 1, \ \frac{1}{\mu} F \text{ is } \beta \text{-condensing}). \]
In particular, since \( y_i \in \partial \Omega U_i \), then
\[
\frac{\lambda_i}{\mu} Fr(y_i) \notin M \text{ for each } i \in \{m, m+1, \ldots\}. \tag{3.8}
\]

Now we investigate
\[
R = \left\{ x \in E \colon x = \frac{\lambda}{\mu} Fr(x), \text{ for some } \lambda \in [0, 1] \right\}.
\]

\( R \) is non-empty, because \( \theta \in R \). Because \( Fr \) is weakly sequentially continuous, \( \Omega \) is weakly compact, by the Eberlein-Šmulian theorem \( R \) is weakly sequentially closed. This together with

\[
\eta(y_j, M) = \epsilon_j, \quad \lambda_j \in [0, 1] \text{ for } j \in \{m, m+1, \ldots\}
\]

implies that we may assume without loss of generality that
\[
\lambda_j \to \lambda_0 \quad \text{and} \quad y_j \overset{w}{\to} y_0 \in \overline{M^w \cap \Omega \setminus M^w} = \partial \Omega M.
\]

Also since \( y_j = \lambda_j Fr(y_j) / \mu \) we have \( y_0 = \lambda_0 Fr(y_0) / \mu \). If \( \lambda_0 = 1 \), then \( y_0 = Fr(y_0) / \mu \), which contradicts \( M \cap N = \emptyset \). Thus, \( \lambda_0 \in [0, 1) \). But (3.7c) with \( x_j = r(y_j) \in M \), \( x = y_0 = r(y_0) \in \partial \Omega M \) implies \( \lambda_j Fr(y_j) / \mu \in M \) for \( j \) sufficiently large. This contradicts (3.8). Thus \( M \cap N \neq \emptyset \).

Consequently, there exists \( x \in M \) such that
\[
x = \frac{1}{\mu} Fr(x) = \frac{1}{\mu} F(x).
\]

Accordingly, there exists \( x \in M \) such that \( F(x) = \mu x \).

\( \square \)

**Remark 3.4.** Theorem 3.6 generalizes and improves Theorem 2.4 in [16] and Theorem 2.5 in [15].

When \( \mu = 1 \) we obtain this fixed point result of Furi-Pera type:

**Corollary 3.5.** Let \( E \) be a separable Banach space, \( \Omega \) a weakly compact convex subset of \( E \), and \( M \) a proper closed convex subset of \( \Omega \) with \( \theta \in M \) and \( \mu \geq 1 \). Also, assume \( F : M \to \Omega \) a weakly sequentially continuous operator. In addition assume

there exists a weakly sequentially continuous retraction \( r : \Omega \to M \),

if \( \{(x_j, \lambda_j)\}_{j=1}^\infty \) is a sequence in \( M \times [0, 1] \) with \( x_j \overset{w}{\to} x \), \( \lambda_j \to \lambda \) and \( x = \lambda F(x) \) with \( 0 \leq \lambda < 1 \), then \( \lambda_j F(x_j) \in M \) for \( j \) sufficiently large.

Then the operator \( F \) has a fixed point in \( M \).
Corollary 3.6. Let $E$ be a separable and reflexive Banach space $\Omega$ be a closed bounded convex of $E$ and $M$ a proper closed convex subset of $\Omega$ with $\theta \in M$ and $\mu \geq 1$. Also, assume $F : M \to \Omega$ a weakly sequentially continuous operator and Eq. (3.7c) holds. Then the operator equation $F(x) = \mu x$ has a solution in $M$.

Proof. Since a subset of a reflexive Banach space is weakly compact iff it is closed in the weak topology and bounded in the norm topology, then, by Theorem 3.6 it suffices to prove that Eq. (3.7a) is satisfied. By [15] there exists a weakly continuous retraction $r : E \to M$, so $r : \Omega \to M$ is a weakly sequentially retraction. □

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References