Endpoint Estimates for Hardy Operator’s Conjugate Operator with Power Weight on $n$-Dimensional Space

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Abstract. In this paper, we establish two integral inequalities for Hardy operator’s conjugate operator at the endpoint on $n$-dimensional space. The operator $H_n^*$ is bounded from $L^{1\alpha}(G^n)$ to $L^{q\beta}(G^n)$ with the bound explicitly worked out and the similar result holds for $H_n^\ast$.

Key Words: Conjugate operator, power weight, endpoint estimate.

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1 Introduction

Let $f$ be a non-negative integrable function on $G := (0, \infty)$. The classical Hardy operator is defined by

$$Hf(x) := \frac{1}{x} \int_0^x f(t) dt,$$

and its conjugate operator

$$H^*f(x) := \int_x^\infty \frac{f(t)}{t} dt$$

for all $x > 0$.

For $n$-dimensional case with $n \geq 2$, Hardy operators can be defined on product space as

$$H_n f(x) := \frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(t_1, \ldots, t_n) dt_1 \cdots dt_n,$$

(1.1)

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and its conjugate operator defined as

$$H_n^* f(x) := \int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} \frac{f(t_1, \ldots, t_n)}{t_1 \cdots t_n} \, dt_1 \cdots dt_n,$$

(1.2) for \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n = (0, \infty)^n\), where \(f\) is any measurable function on \(\mathbb{R}^n\).

Another definition is given by Christ and Grafakos in [2] as follows

$$\mathcal{H}_n f(x) = \frac{1}{\omega_n |x|^n} \int_{|y|<|x|} f(y) \, dy,$$

(1.3) and

$$\mathcal{H}_n^* f(x) = \frac{1}{\omega_n} \int_{|y|>|x|} \frac{f(y)}{|y|^n} \, dy,$$

(1.4) for \(x \in \mathbb{R}^n \setminus \{0\}\), where \(f\) is any measurable function on \(\mathbb{R}^n\) and \(\omega_n = \pi^{n/2} / \Gamma(1+n/2)\) is the volume of the unit ball in \(\mathbb{R}^n\).

For the case \(1 < p \leq q < \infty\), the boundedness of the operators \(H_n^*\) and \(H_n\) from \(L^p_w(\mathbb{R}^n)\) to \(L^q_w(\mathbb{R}^n)\) were discussed in many papers (cf. [1, 13, 15, 17]). The estimates on the endpoint for the operator \(H_n\) and \(\mathcal{H}_n\) were systematically studied in [18]. It should be pointed out that the operator \(H_{\alpha,n}^*\), \(n \geq 2\), defined by (1.1) fails to be of weak type of \((1,1)\), however, \(H_n^*\) is bounded from \(L^1_w(\mathbb{R}^n)\) to \(L^{1/2}_w(\mathbb{R}^n)\) for arbitrary \(n \in \mathbb{N}\). This shows that some power weight can change the boundedness of the operator \(H_n\) on the endpoint. Motivated by the idea of the reference [18], it is natural for us to discuss the boundedness of the operator \(H_n^*\) on the endpoint.

The purpose of this paper is to establish the boundedness of the operators \(H_n^*\) and \(\mathcal{H}_n^*\) on the endpoint.

Throughout the paper, we have the following notations. For two \(n\)-dimensional vectors \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\) and \(\beta = (\beta_1, \beta_2, \ldots, \beta_n)\), \(\alpha \cdot \beta = \alpha_1 \beta_1 + \alpha_2 \beta_2 + \cdots + \alpha_n \beta_n\), \(\alpha < \beta\) means each \(\alpha_i < \beta_i\), \(i = 1, \ldots, n\), and \(x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}\), \(x \in \mathbb{R}^n\). For some bold type figures and letters, we have \(I = (1, \ldots, 1)\), \(p = (p, \ldots, p)\). It is clear that \(x^1 = x_1 \cdots x_n\).

Now we first formulate our main results as follows.

**Theorem 1.1.** Suppose that \(f\) is any non-negative measurable function on \(\mathbb{R}^n\) and \(1 \leq q < \infty\). If \(\alpha\) and \(\beta\) are two \(n\)-tuples in \(\mathbb{R}^n\) such that \(\alpha + 1 > 0\) and \(\beta + 1 = q(\alpha + 1)\), then the following inequality

$$\left( \int_{\mathbb{R}^n} (H_n^* f(x))^q x^\beta \, dx \right)^{1/q} \leq \left( \prod_{i=1}^n \frac{1}{(\beta_i + 1)} \right)^{1/q} \int_{\mathbb{R}^n} f(x) x^\alpha \, dx$$

(1.5) holds for the operator \(H_n^*\) defined by (1.2), that is, \(H_n^*\) is bounded from \(L^1_w(\mathbb{R}^n)\) to \(L^q_w(\mathbb{R}^n)\) with the norm of \(H_n^*\) satisfying

$$\|H_n^*\| \leq \left( \prod_{i=1}^n \frac{1}{(\beta_i + 1)} \right)^{1/q}.$$
Theorem 1.2. Suppose that $f$ is any non-negative measurable function on $\mathbb{R}^n$ and $1 \leq q < \infty$. If $\alpha$ and $\beta$ are two real numbers such that $\alpha/n + 1 > 0$ and $\beta/n + 1 = q(\alpha/n + 1)$, then the following inequality
\[
\left( \int_{\mathbb{R}^n} (\mathcal{H}_n^* f(x))^q |x|^\beta dx \right)^{1/q} \leq \frac{1}{\omega_n} \left( \frac{v_n}{\beta + n} \right)^{\frac{1}{q}} \int_{\mathbb{R}^n} f(x) |x|^\alpha dx
\] (1.6)
holds for the Hardy operators $\mathcal{H}_n^*$ defined by (1.4).

2 Auxiliary and some lemmas

To prove our theorems, we first provide some definitions and lemmas which will be used in the sequel. Some lemmas can be found in classic literatures and here we omit their proofs.

Lemma 2.1. (I) The multiplicative group $G = (0, \infty)$ is a locally compact group; so is $G^n$.

(II) For the multiplicative group $G^n$ with the group operation
\[ xy = (x_1y_1, \cdots, x_ny_n), \]
the Haar measure $\mu$ on $G^n$ is $dx/x_1 \cdots x_n$, and $\mu$ is a $\sigma$-finite measure.

The proof of Lemma 2.1 can be found in [10].

Definition 2.1. Define
\[ L^p(G, \mu) = \left\{ f : \| f \|_{L^p(G, \mu)} = \left( \int_G |f(x)|^p d\mu(x) \right)^{1/p} < +\infty \right\}, \]
for $1 \leq p < +\infty$; when $p = \infty$,
\[ L^\infty(G, \mu) = \{ f : \| f \|_{L^\infty} < +\infty \}. \]

It is well-known that Hölder’s inequality holds in $L^p(G, \mu)$. Moreover, Fubini’s theorem also holds, provided that $\mu$ is a $\sigma$-finite measure on $G$. In the paper, $L^p(G^n, \mu)$ denotes the function space on the locally compact group $G^n$ with respect to the Haar measure $d\mu = dx/x_1 \cdots x_n$. Others such as $L^p(G)$, $L^p(\mathbb{R}^n)$ and $L^p(G^n)$ are ones with respect to the Lebesgue measure.

Definition 2.2. Let $f \in L^1(G, \mu)$ and $g \in L^p(G, \mu)$, $1 \leq p \leq +\infty$. Define the convolution of $f$ and $g$ by
\[ f \ast g(x) = \int_G f(y)g(y^{-1}x)d\mu(y). \]
By the reference [6], we know that $f \ast g$ is well defined and in $L^p(G, \mu)$.

Note that $f \ast g = g \ast f$, if $G$ is an abelian group. The fundamental inequality involving convolutions is the following.
Lemma 2.2 (Young’s inequality). Let \( 1 \leq p, q, r \leq \infty \) satisfy \( 1/q + 1 = 1/p + 1/r \), and \( \mu \) be a \( \sigma \)-finite measure on \( G \). Then for \( f \in L^p(G, \mu) \) and \( g \in L^q(G, \mu) \) satisfying

\[
\|f\|_{L^p(G, \mu)} \leq \|g\|_{L^q(G, \mu)},
\]

we have

\[
\|f * g\|_{L^r(G, \mu)} \leq \|g\|_{L^q(G, \mu)} \|f\|_{L^p(G, \mu)},
\]

where \( \tilde{g}(x) = g(x^{-1}) \).

We clearly conclude that

\[
\|g\|_{L^r(G^n, \mu)} = \|\tilde{g}\|_{L^r(G^n, \mu)},
\]

holds for any \( g \in L^r(G^n, \mu) \). In fact, for any Borel measurable \( A \) in \( G^n \), \( \mu(A) = \mu(A^{-1}) \), where \( A^{-1} = \{x^{-1} : x \in A\} \).

The proof of Lemma 2.2 can be found in [6].

Lemma 2.3. Suppose that \( f \) is a non-negative measurable function on \( G \) and \( 1 \leq p \leq q < \infty \). The operator \( Y^n_s \) is defined by

\[
Y^n_s f(x) := x^{q-1} \int_0^\infty f(t) \frac{1}{t^s} \, dt, \quad x > 0.
\]  

(2.1)

If \( a \) and \( \beta \) are real numbers such that \( a + 1 + p(s-1) > 0 \) and \( \beta + 1 = q(a+1)/p \), then the following inequality

\[
\left( \int_G (Y^n_s f(x))^{q/p} x^p \, dx \right)^{1/p} \leq \left( \frac{p}{r(a+1+p(s-1))} \right)^{1/p} \left( \int_G f(x)^{q/p} x^{q/p} \, dx \right)^{1/p},
\]

(2.2)

holds, where \( 1/r = 1/q - 1/p + 1 \).

Proof. Set

\[
f_1(x) = f(x) x^{a+1/p}, \quad f_2(x) = x^{\frac{\beta+1+q(s-1)}{q}} \chi_{(0,1]}(x).
\]

We calculate the convolution of two functions \( f_1 \) and \( f_2 \) with respect to Haar measure on \( G \). Since

\[
\beta + 1 = \frac{q}{p}(a+1) \quad \text{and} \quad a + 1 + p(s-1) > 0,
\]

it follows from simple computation that

\[
f_1 * f_2(x) = \int_G f(y) y^{a+1/p} (y^{-1} x)^{\frac{\beta+1+q(s-1)}{q}} \chi_{(0,1]}(y^{-1} x) \, dy \]
\[
= \int_0^\infty f(y) y^{a+1/p} x^{\frac{\beta+1+q(s-1)}{q} - 1} \chi_{(0,1]}(y^{-1} x) \, dy \]
\[
= x^{\frac{\beta+1+q(s-1)}{q}} \int_0^\infty f(y) y^{-s} \, dy = x^{\frac{\beta+1+q(s-1)}{q}} \int_0^\infty f(y) y^{q/p} \, dy \]
\[
= x^{\frac{a+1}{p}} Y^n_s f(x).
\]
Hence, by the simple computation, we conclude that

$$
\|f_1 * f_2\|_{L^q(G, \mu)} = \left( \int_G \left( \frac{x^{\frac{\beta+1}{q}}}{x} Y_s^* f(x) \right)^q \frac{dx}{x} \right)^{\frac{1}{q}} = \left( \int_G (Y_s^* f(x))^q x^\beta \frac{dx}{x} \right)^{\frac{1}{q}},
$$

(2.3a)

$$
\|f_1\|_{L^p(G, \mu)} = \left( \int_G \left( f(x) x^{\frac{\alpha+1}{p}} \right)^p \frac{dx}{x} \right)^{\frac{1}{p}} = \left( \int_G f^p(x) x^\alpha dx \right)^{\frac{1}{p}},
$$

(2.3b)

Note that $\beta+1 = q(\alpha+1)/p$ and $\alpha+1+p(s-1)>0$. It follows that

$$
\|f_2\|_{L^r(G, \mu)} = \left( \int_0^1 x^{\frac{\beta+1+q(s-1)}{q}} \chi_{(0,1)}(x) \right)^{\frac{1}{q}} = \left( \int_0^1 x^{\frac{\beta+1+q(s-1)}{q}} \frac{dx}{x} \right)^{\frac{1}{q}} < \infty.
$$

Assuming that

$$\frac{1}{r} = \frac{1}{q} - \frac{1}{p} + 1,$$

it implies from Lemma 2.2 that

$$
\|f_1 * f_2\|_{L^q(G, \mu)} \leq \|f_2\|_{L^r(G, \mu)} \|f_1\|_{L^p(G, \mu)},
$$

(2.4)

that is, we have

$$
\left( \int_G (Y_s^* f(x))^q x^\beta dx \right)^{\frac{1}{q}} \leq \left( \frac{p}{r(\alpha+1+p(s-1))} \right)^{\frac{1}{p}} \left( \int_G f^p(x) x^\alpha dx \right)^{\frac{1}{p}}.
$$

So, the lemma is proved.

In order to study the boundedness of the operator $\mathcal{H}_n^*$ at the endpoint, we first introduce the following definition.

**Definition 2.3.** Let $f$ be a measurable function defined on $\mathbb{R}^n$. Define

$$
R_f(x) := \frac{1}{v_n} \int_{S^{n-1}} f(|x|\xi) d\sigma(\xi),
$$

where $v_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit ball in $\mathbb{R}^n$ and $d\sigma$ is the induced Lebesgue measure on $S^{n-1}$. Observe that if $f$ is a radial function, then we easily have $R_f = f$.

As to $R_f$, we have the following lemmas.

**Lemma 2.4.** If $f$ is a non-negative measurable function on $\mathbb{R}^n$, then

$$
\mathcal{H}_n^*(R_f)(x) = \mathcal{H}_n^* f(x), \quad x \in \mathbb{R}^n \setminus \{0\}.
$$

(2.5)
Proof. Obviously, $R_f$ is a radial function. For $x \in \mathbb{R}^n \setminus \{0\}$, it follows that

$$
\mathcal{H}_n^*(R_f)(x) = \frac{1}{\omega_n} \frac{1}{|y|} \int_{S^{n-1}} f(|y|) d\sigma(\xi) dy
$$

$$
= \frac{1}{\omega_n} \frac{1}{|y|} \int_{S^{n-1}} f(|y|) d\sigma d\sigma(\xi)
$$

$$
= \frac{1}{\omega_n} \int_{S^{n-1}} \int_{S^{n-1}} \int_{|x|}^{\infty} f(r\xi) r^{-1} dr d\sigma(\theta) d\sigma(\xi)
$$

$$
= \frac{1}{\omega_n} \int_{S^{n-1}} \int_{|x|}^{\infty} f(r\xi) r^{-1} dr d\sigma(\xi)
$$

$$
= \mathcal{H}_n^* f(x).
$$

(2.6)

Note that in (2.6) we set $y = |y| \theta$.

Lemma 2.5. Let $f$ be under the same conditions as in Lemma 2.4. Then the following inequality

$$
\left( \int_{\mathbb{R}^n} (R_f(x))^p |x|^a dx \right)^{\frac{1}{p}} \leq \left( \int_{\mathbb{R}^n} |f(x)|^p |x|^a dx \right)^{\frac{1}{p}}
$$

(2.7)

holds for $1 \leq p < \infty$ and any real number $a$.

The proof of Lemma 2.5 can be found in [18].

Remark 2.1. By Lemma 2.4 and Lemma 2.5, to prove Theorem 1.2, we merely need to consider the radial function.

3 Proofs of main theorems

Proof of Theorem 1.1. Set

$$
f_1(x) = f(x)x^{\alpha + 1}, \quad f_2(x) = x^{\beta} \chi_{(0,1]}(x).
$$

Using Lemma 2.3, we calculate the convolution of two functions $f_1$ and $f_2$ with respect to Haar measure on $G^n$. Since $\beta + 1 = q(\alpha + 1)$ and $\alpha + 1 > 0$, it follows that

$$
f_1 * f_2(x) = \int_{G^n} f(y) y^{\beta + 1} x (y^{-1} x) \frac{dy}{y_1 \cdots y_n} = x^{\beta + 1} \mathcal{H}_n^* f(x).
$$

Hence we conclude that

$$
\|f_1 * f_2\|_{L^q(G^n, \mu)} = \left( \int_{G^n} (x^{\beta + 1} \mathcal{H}_n^* f(x))^q \frac{dx}{x_1 \cdots x_n} \right)^{\frac{1}{q}} = \|\mathcal{H}_n^* f\|_{L^q(G^n)},
$$

(3.1a)

$$
\|f_1\|_{L^1(G^n, \mu)} = \int_{G^n} f(x)x^{\alpha + 1} \frac{dx}{x_1 \cdots x_n} = \|f\|_{L^1_{\alpha}(G^n)},
$$

(3.1b)
Since \( r \) satisfies \( 1/r = 1 + 1/q - 1/p \), we clearly have \( r = q \), when \( p = 1 \).

Consequently, using Lemma 2.3 again, we have
\[
\|f_2\|_{L^r(G^n, \mu)} = \left( \prod_{i=1}^{n} \frac{1}{(\beta_i + 1)} \right)^{1/r},
\]
(3.2)
where \( \| \cdot \|_{L^1(G^n, \mu)} \) denotes \( L^1 \)-norm with respect to Haar measure on \( G^n \) and \( \| \cdot \|_{L^1_{x^\alpha}(G^n)} \) denotes \( L^1 \)-norm with the power weight \( x^\alpha \) with respect to Lebesgue measure on \( G^n \).

Therefore, combining Lemma 2.2 with the inequalities (3.1a), (3.1b) and (3.2) yields that
\[
\|H_n f\|_{L^q x^\beta(G^n)} \leq \left( \prod_{i=1}^{n} \frac{1}{(\beta_i + 1)} \right)^{1/q} \|f\|_{L^1 x^\alpha(G^n)}.
\]
This finishes the proof of Theorem 1.1. \( \square \)

**Proof of Theorem 1.2.** Without loss of generality, let \( f \) be a non-negative radial function on \( \mathbb{R}^n \), thus \( f \) can be denoted as the composite function of \( f_0 \) and \( h \), where \( f_0 \) is a measurable function defined on \( [0, \infty) \) and \( h(x) = |x|, x \in \mathbb{R}^n \), say \( f(x) = f_0(|x|) \),
\[
\left( \int_{\mathbb{R}^n} (H_n f(x))^q |x|^\beta dx \right)^{1/q} = \left( \int_{\mathbb{R}^n} \left( \frac{1}{\omega_n} \int_{|y|>|x|} f(y) dy \right)^q |x|^\beta dx \right)^{1/q} = \frac{1}{\omega_n} \left( \int_{S^{n-1}} \int_0^\infty r^{n-1} \left( \int_{S^{n-1}} \int_r^\infty f_0(t) s^{-1} dt ds \right)^q r^\beta dr ds(\xi) \right)^{1/q} = \frac{1}{\omega_n} \nu_n^{1/2} \nu_n \left( \int_0^\infty r^{n+\beta-1} \left( \int_r^\infty \frac{f_0(t)}{t} dt \right)^q dr \right)^{1/q} \leq \frac{1}{\omega_n} \nu_n \left( \frac{\nu_n}{\beta+n} \right)^{1/2} \int_0^\infty f_0(r) r^{\nu} r^{n-1} dr = \frac{1}{\omega_n} \nu_n \left( \frac{\nu_n}{\beta+n} \right)^{1/2} \int_{S^{n-1}} \int_0^\infty f_0(r) r^{\nu} r^{n-1} dr = \frac{1}{\omega_n} \nu_n \left( \frac{\nu_n}{\beta+n} \right)^{1/2} \int_{\mathbb{R}^n} f(x) |x|^\nu dx.
\]
(3.3)

Observe that the inequality (3.3) follows from the inequality (2.2) in Lemma 2.3 with \( s = 1 \) and \( p = 1 \).

This completes the proof of Theorem 1.2. \( \square \)

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References

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