A Note on a Theorem of J. Szabados

Laiyi Zhu and Yang Tan*

School of Information, Renmin University of China, Beijing 100872, China

Received 29 April 2013; Accepted (in revised version) 30 May 2013
Available online 30 September 2013

Abstract. In this note, we establish a companion result to the theorem of J. Szabados on the maximum of fundamental functions of Lagrange interpolation based on Chebyshev nodes.

Key Words: Lagrange interpolation, Chebyshev polynomial, fundamental function of interpolation.

AMS Subject Classifications: 41A05, 41A10

1 Introduction

Let $T_n(x) = \cos(n \arccos x)$ be the Chebyshev polynomial of degree $n$ with the roots

$$x_{k,n} = \cos t_{k,n}, \quad t_{k,n} = \frac{2k-1}{2n} \pi, \quad k = 1,2,\ldots,n,$$

and let

$$l_{k,n}(x) = \frac{(-1)^{k-1} \cos nt_{k,n}}{n \cos t_{k,n} - \cos t_{k,n}}, \quad x = \cos t, \quad k = 1,2,\ldots,n, \quad (1.1)$$

be the fundamental polynomials of Lagrange interpolation based on the Chebyshev nodes. Setting

$$l_{k,n} = \max_{|x| \leq 1} l_{k,n}(x), \quad k = 1,2,\ldots,n,$$

$$M_n(x) = \max_{1 \leq k \leq n} l_{k,n}(x), \quad |x| \leq 1,$$

$$\overline{M}_n = \max_{|x| \leq 1} M_n(x),$$

$$\underline{M}_n = \min_{1 \leq k \leq n} l_{k,n},$$

$$\overline{M}_n^* = \min_{1 \leq k \leq n} l_{k,n},$$

*Corresponding author. Email addresses: zhulaiyi@ruc.edu.cn (L. Y. Zhu), shutongtan@sina.com (Y. Tan)

it is easy to see that
\[ M_n \leq \overline{M}_n \leq M^*_n, \]
and
\[ M^*_n \leq \overline{M}_n. \]

In [1], Erdős and Grünwald proved the following theorem.

**Theorem 1.1.** We have
\[ |l_{k,n}(x)| < \frac{4}{\pi}, \quad |x| \leq 1, \quad 1 \leq k \leq n, \quad n = 1, 2, \cdots. \quad (1.2) \]

Moreover,
\[ \lim_{n \to \infty} l_{1,n}(1) = \lim_{n \to \infty} l_{n,n}(-1) = \frac{4}{\pi}. \quad (1.3) \]

It follows from Theorem 1.1 that
\[ \lim_{n \to \infty} \overline{M}_n = \lim_{n \to \infty} \overline{M}_n = \frac{4}{\pi}. \]

In [2], J. Szabados proved the following theorem.

**Theorem 1.2.** We have
\[ \lim_{n \to \infty} M_n = \frac{2}{\pi} \cos \frac{2 - \sqrt{3}}{2} \pi = 0.580 \cdots. \quad (1.4) \]

It is natural to ask that which of \( M^*_n \) and \( M_n \) is bigger and what is the behavior of \( \overline{M}_n \)? In this note we prove the following theorem.

**Theorem 1.3.** We have
\[ \lim_{n \to \infty} M^*_n = 1. \quad (1.5) \]

### 2 Proof of Theorem 1.3

For convenience, we denote \( t_{k,n}, x_{k,n}, l_{k,n}(x) \) and \( l_k, x_k, l_k(x) \), and \( l_k \) respectively and denote \( l_k(\cos t) \) by \( f_k(t) \), \( k = 1, 2, \cdots, n \). In order to prove Theorem 1.3, we need the following lemmas.

**Lemma 2.1.** For \( k = 2, 3, \cdots, [(n + 1)/2] \), \( n > 2, t \in [0, t_{k-1}] \cap [t_{k+1}, \pi] \), we have
\[ |f_k(t)| \leq \frac{2}{\pi}. \quad (2.1) \]
Proof. For \( t \in [0, t_{k-1}] \), by the monotonicity of the function \( 1/(\cos t - \cos k) \), we have

\[
|f_k(t)| = \frac{\sin t_k}{n} \frac{|\cos nt|}{\cos t - \cos k_n} \leq \frac{\sin t_k}{n} \frac{\sin \pi}{2n} \frac{\cos \frac{t_{k-1}+t_k}{2} + \cos \frac{t_{k-1}+t_k}{2}}{2n \sin \frac{\pi}{2n} \sin \frac{t_{k-1}+t_k}{2}}.
\]

Since

\[
\frac{\pi}{2n} \leq \frac{t_{k-1}+t_k}{2} \leq \frac{\pi}{2}, \quad \tan \frac{\pi}{2n} \leq \tan \frac{t_{k-1}+t_k}{2},
\]

we get

\[
\sin \frac{\pi}{2n} \cos \frac{t_{k-1}+t_k}{2} \leq \cos \frac{\pi}{2n} \sin \frac{t_{k-1}+t_k}{2},
\]

which implies

\[
|f_k(t)| \leq \left( n \tan \frac{\pi}{2n} \right)^{-1} \leq \frac{2}{\pi}.
\]

For \( t \in [t_{k+1}, \pi] \), it follows from the monotonicity of the function \( (\cos t - \cos k)^{-1} \) and

\[
\cos k - \cos k_{k+1} \geq \cos k_{k-1} - \cos k \quad \text{for} \ t \leq \frac{\pi}{2},
\]

that

\[
|f_k(t)| \leq \frac{\sin t_k}{n} \frac{1}{\cos k - \cos t} \leq \frac{\sin t_k}{n (\cos k - \cos k_{k+1})} \leq \frac{\sin t_k}{n (\cos k_{k-1} - \cos k)} \leq \frac{2}{\pi}.
\]

This completes the proof of Lemma 2.1. \( \square \)

**Lemma 2.2.** For sufficient large \( n \), we have

\[
l_m = \max_{t \in [\theta_m, l_m]} f_m(t), \quad (2.2)
\]

where \( m = \lceil (n+1)/2 \rceil, \theta_m = t_m - \pi/2n^2 \).

Proof. It is clear that there exists only one \( \tau_m \in (t_{m-1}, t_{m+1}) \) for which \( f'_m(\tau_m) = 0 \), and \( l_m = f_m(\tau_m) \). An easy calculation yields

\[
f'_m(t_m) = -\frac{\cos t_m}{2\sin t_m}.
\]

Hence \( l_m = f_m(t_m) = 1 \), for \( n = 2m - 1 \), and \( f'_m(t_m) < 0 \), for \( n = 2m \). On the other hand, we have

\[
f'_m(\theta_m) = (-1)^{m-1} \frac{\sin t_m}{n} \frac{l_m(\theta_m)}{n (\cos \theta_m - \cos t_m)^2}.
\]
where
\[ h_m(\theta_m) = -n \sin n\theta_m (\cos \theta_m - \cos t_m) + \sin \theta_m \cos n\theta_m \]
\[ = (-1)^{m-1} \left[ -n \cos \frac{\pi}{2n} (\cos \theta_m - \cos t_m) + \sin \theta_m \sin \frac{\pi}{2n} \right]. \]

Writing
\[ g_m(\theta_m) = -n \cos \frac{\pi}{2n} (\cos \theta_m - \cos t_m) + \sin \theta_m \sin \frac{\pi}{2n}, \]
we have
\[ g_m(\theta_m) = \cos \frac{\pi}{2n} \left[ -2n \sin \frac{\pi}{4n^2} \sin \left( t_m - \frac{\pi}{2n} \right) + \sin \left( t_m - \frac{\pi}{2n} \right) \tan \frac{\pi}{2n} \right]. \]

Thus, when \( n = 2m - 1, t_m = \frac{\pi}{2}, \)
\[ g_m(\theta_m) = \cos \frac{\pi}{2n} \left[ -2n \sin \frac{\pi}{4n^2} \sin \left( t_m - \frac{\pi}{2n} \right) + \sin \left( t_m - \frac{\pi}{2n} \right) \tan \frac{\pi}{2n} \right] > 0, \]
and when \( n = 2m, t_m = \frac{\pi}{2} - \frac{\pi}{2n}, \)
\[ g_m(\theta_m) = \sin \frac{\pi}{2n} \cos \left( \frac{\pi}{2n} + \frac{\pi}{2n^2} \right) - 2n \sin \frac{\pi}{4n^2} \cos \frac{\pi}{2n} \cos \left( \frac{\pi}{2n} + \frac{\pi}{2n^2} \right) \]
\[ = -\frac{\pi^3}{24n^3} + o \left( \frac{1}{n^3} \right), \]
for sufficient large \( n. \)

Therefore, for sufficient large \( n, \tau_m \in [\theta_m, t_m], \) and this completes the proof of Lemma 2.2. \( \square \)

Proof of Theorem 1.3. Firstly, by the well-known relations
\[ l_k(x_i) = \begin{cases} 1, & i = k, \\ 0, & i \neq k, \end{cases} \quad i, k = 1, 2, \ldots, n, \] (2.3)
we have
\[ M_n^* \geq 1, \quad n = 1, 2, \ldots. \] (2.4)

Secondly,
\[ M_n^* \leq l_m, \quad n = 1, 2, \ldots, \] (2.5)
where \( m = \left[ (n+1)/2 \right]. \) It follows from Lemma 2.2 and the mean-value theorem that
\[ l_m = l_m(\tau_m) \]
\[ = l_m(t_m) + l_m'(\zeta_m) (\tau_m - t_m) \]
\[ = 1 + l_m'(\zeta_m) (\tau_m - t_m), \]
where \( \zeta_m \in (\tau_m, t_m). \)

Finally, the Bernstein inequality, Theorem 1.1 and Lemma 2.2 imply that
\[ l_m \leq 1 + \frac{2}{n}. \] (2.6)
Thus, the assertion follows immediately from (2.4)-(2.6). \( \square \)
References