Fixed Point Theorem of \{a,b,c\} Contraction and Nonexpansive Type Mappings in Weakly Cauchy Normed Spaces

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Abstract. Let \(C\) be a closed convex weakly Cauchy subset of a normed space \(X\). Then we define a new \(\{a,b,c\}\) type nonexpansive and \(\{a,b,c\}\) type contraction mapping \(T\) from \(C\) into \(C\). These types of mappings will be denoted respectively by \(\{a,b,c\}\)-ntype and \(\{a,b,c\}\)-ctype. We proved the following:

1. If \(T\) is \(\{a,b,c\}\)-ntype mapping, then \(\inf\{\|T(x)−x\|:x\in C\}=0\), accordingly \(T\) has a unique fixed point. Moreover, any sequence \(\{x_n\}_{n\in\mathbb{N}}\) in \(C\) with \(\lim_{n\to\infty}\|T(x_n)−x_0\|=0\) has a subsequence strongly convergent to the unique fixed point of \(T\).

2. If \(T\) is \(\{a,b,c\}\)-ctype mapping, then \(T\) has a unique fixed point. Moreover, for any \(x\in C\) the sequence of iterates \(\{T^n(x)\}_{n\in\mathbb{N}}\) has subsequence strongly convergent to the unique fixed point of \(T\).

This paper extends and generalizes some of the results given in [2, 4, 7] and [13].

Key Words: Fixed point, generalized type of contraction and nonexpansive mappings, normed space.

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1 Introduction

Let \(C\) be a closed convex subset of a normed space \(X\) and \(T\) be a mapping from \(C\) into \(C\) which satisfies \(\|T(x)−T(y)\|≤a\|x−y\|+b\|y−T(y)\|+c\|x−T(x)\|\) for all \(x,y\in C\) and for some real numbers \(a,b,c\in [0,1]\).

When \(0<a<1,\ b=c=0\), \(T\) becomes a contraction mapping. If \(X\) is complete, S. Banach gave his famous Banach contraction mappings principle, namely, \(T\) has a unique fixed point.

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When $a = 1$, $b = c = 0$, $T$ becomes a nonexpansive mapping, if $C$ is a bounded closed convex subset of a Banach space $X$, W. A. Kirk proved fixed point theorems concerning this type mappings, [7].

Recently, the existence of fixed points of $T$ when the domain of $T$ is unbounded discussed in [5].

When $a = 0$, we have the Kannan maps, see [6].

When $a + b + c < 1$ a unique fixed point of mappings $T$ defined on a closed convex subset of a weakly Cauchy normed space is proved [2].

**Theorem 1.1** (see [2]). Let $X$ be a normed space, $C$ be a closed convex and weakly Cauchy subset of $X$ and $T$ be a mapping from $C$ into $C$ which satisfies $\| T(x) - T(y) \| \leq a \| x - y \| + b \| y - T(y) \| + c \| x - T(x) \|$ for all $x, y \in C$ and for some real numbers $a, b, c \in [0, 1]$ with $a + b + c < 1$. Then $T$ has a unique fixed point $y \in C$.

When $0 < a < 1$, $b, c \geq 0$, and $a + b + c = 1$, $T$ becomes Gregus type mapping, M. Gregus proved the existence of a unique fixed point of such mappings provided that $C$ is closed convex subset of a Banach space $X$ [4].

**Theorem 1.2** (see [4]). Let $C$ be a closed convex subset of a Banach space $X$ and $T$ be a mapping from $C$ into $C$ which satisfies $\| T(x) - T(y) \| \leq a \| x - y \| + b \| y - T(y) \| + c \| x - T(x) \|$ for all $x, y \in C$ and for some real numbers $a, b, c \in [0, 1]$ with $0 < a < 1$ and $a + b + c = 1$. Then $T$ has a unique fixed point $y \in C$.

More general contraction type mapping was given in [3, 8], and [9]. It is proved that

**Theorem 1.3** (see [3]). Let $(X, d)$ be a complete metric space and $T$ be a mapping from $X$ into $X$ which satisfies $d(T(x), T(y)) \leq ad(x, y) + bd(y, T(y)) + cd(x, T(x)) + ed(T(x), y) + fd(T(y), x)$ for all $x, y \in X$ and for some real numbers $a, b, c, e, f \in [0, 1]$ with $a + b + c + e + f < 1$. Then $T$ has a unique fixed point.

When $a + b + c = 1$, $T$ becomes $\{a, b, c\}$-Generalized nonexpansive type mapping, Sahar Mohamed Ali proved the existence of a unique fixed point of such mappings when $C$ is closed convex, containing a contraction point, and weakly Cauchy subset of a normed space $X$ [12].

Some other generalizations have been given in [11] and [10].

This paper extends and generalizes some of the results given in [2, 4, 13], and [7] to the $\{a, b, c\}$-type mappings defined on a closed convex weakly Cauchy subset of a normed space not necessarily Banach in general.

### 2 Notations and basic definitions

We have the following:
Lemma 3.1. Let X be a normed space, C be a closed convex subset of X, \( \{x_n\}_{n \in \mathbb{N}} \) be a Cauchy sequence in C that has subsequence converging weakly to some point y in X. Then \( \{x_n\}_{n \in \mathbb{N}} \) has a subsequence converging strongly to y and y \( \in \) C.

We also have the following:

Lemma 3.2. Let X be a normed space and T be a mapping from X into X, if there is a real number \( t, t < 1 \) which satisfies that for every \( x \in X \) there exists \( y \in X \) such that \( \|T(y) - y\| \leq t\|T(x) - x\| \), then

\[
\inf\left\{\|T(x) - x\| : x \in X\right\} = 0.
\]

Proof. Let \( x_1 \) be an arbitrary element in X. Then there is \( x_2 \in X \) such that \( \|T(x_2) - x_2\| \leq t\|T(x_1) - x_1\| \) and for such \( x_2 \) pick \( x_3 \in X \) such that \( \|T(x_3) - x_3\| \leq t\|T(x_2) - x_2\| \), hence \( \|T(x_3) - x_3\| \leq t^2\|T(x_1) - x_1\| \), constructing successively a sequence \( \{x_n\}_{n \in \mathbb{N}} \) in X such that \( \|T(x_{n+1}) - x_{n+1}\| \leq t\|T(x_n) - x_n\| \) proves the existence of a sequence \( \{x_n\}_{n \in \mathbb{N}} \) such that

\[
\|T(x_{n+1}) - x_{n+1}\| \leq t\|T(x_1) - x_1\|. \tag{3.1}
\]

Taking the limit of both sides of the inequalities (3.1) as \( n \to \infty \) proves that

\[
\lim_{n \to \infty} \|T(x_n) - x_n\| = 0. \tag{3.2}
\]

The limit (3.2) is sufficient to prove that \( \inf\{\|T(x) - x\| : x \in X\} = 0 \).

Lemma 3.3. Let X be a normed space, T be a mapping from X into X, T satisfy \( \|T(x) - T(y)\| \leq a\|x - y\| + b\|y - T(y)\| + c\max\{\|x - T(x)\|, \|x - T(y)\|\} \) for all \( x, y \in X \) and for some positive real numbers \( a, b, c, c \neq 1 \). Then for any \( x \in X \), the sequence of iterates \( \{T_n(x)\}_{n \in \mathbb{N}} \) satisfies

\[
\|T^{n+1}(x) - T^n\| \leq \left(\frac{a+b}{1-c}\right)^n\|T(x) - x\|. \tag{3.3}
\]
Proof. Let $x \in X$, we have

$$
\|T(T(x)) - T(x)\| \leq a \|T(x) - x\| + b \|x - T(x)\| + c \max \left\{ \|T(x) - T(T(x))\|, \|T(x) - T(x)\| \right\}
$$

$$
= (a + b) \|T(x) - x\| + c \|T(x) - T(T(x))\|,
$$

hence

$$
\|T^2(x) - T(x)\| \leq \left( \frac{a + b}{1 - c} \right) \|T(x) - x\|. \tag{3.4}
$$

Since $\|T^3(x) - T^2(x)\| = \|T(T^2(x)) - T(T(x))\|$, we use (3.4) to get

$$
\|T^3(x) - T^2(x)\| \leq \left( \frac{a + b}{1 - c} \right) \|T^2(x) - T(x)\|.
$$

Using the inequalities (3.4) again, we see that

$$
\|T^3(x) - T^2(x)\| \leq \left( \frac{a + b}{1 - c} \right)^2 \|T(x) - (x)\|.
$$

Continuing this inductive process proves (3.3). \qed

We are interested in the following:

**Theorem 3.1.** Let $C$ be a closed convex and weakly Cauchy subset of a normed space $X$, $T$ be a $\{a,b,c\}$-n-type mapping, then $\inf \{ \|T(x) - x\| : x \in C \} = 0$, accordingly $T$ has a unique fixed point. Moreover, any sequence $\{x_n\}_{n \in \mathcal{N}}$ in $C$ with $\lim_{n \to \infty} \|T(x_n) - x_n\| = 0$ has a subsequence strongly convergent to the unique fixed point of $T$.

**Proof.** Using Lemma 3.3 with the fact that $(a + b)/(1 - c) = 1$, the inequalities (3.3) insure that for every $x \in C$ and $n \in \mathcal{N}$, we have

$$
\|T^{n+1}(x) - T^n(x)\| \leq \|T^n(x) - T^{n-1}(x)\| \leq \|T(x) - x\|.
$$

On the other hand,

$$
\|T^3(x) - T^2(x)\| \leq a \|T^2(x) - x\| + b \|x - T(x)\|
$$

$$
+ c \max \{ \|T^2(x) - T^3(x)\|, \|T^2(x) - T^2(x)\| \}
$$

$$
= a \|T^2(x) - x\| + b \|x - T(x)\| + c \|T^2(x) - T(x)\|
$$

$$
\leq (a + c) \|T^2(x) - T(x)\| + (a + b) \|x - T(x)\|
$$

$$
\leq (2a + b + c) \|x - T(x)\| = (a + 1) \|x - T(x)\|.
$$

Since $C$ is convex, the element $y = \frac{1}{2} (T^2(x) + T^3(x))$ is in $C$, one has

$$
\|y - T(x)\| \leq \frac{1}{2} \|T(x) - T^2(x)\| + \|T(x) - T^3(x)\| \leq \frac{1}{2} (a + 2) \|x - T(x)\|,
$$

$$
\|y - T^2(x)\| = \frac{1}{2} \|T^3(x) - T^2(x)\| \leq \frac{1}{2} \|x - T(x)\|,
$$

$$
\|y - T^3(x)\| = \frac{1}{2} \|T^3(x) - T^2(x)\| \leq \frac{1}{2} \|x - T(x)\|,
$$

therefore $y$ is a fixed point of $T$.
then
\[
2\|T(y) - y\| \leq \|T(y) - T^2(x)\| + \|T(y) - T^3(x)\| \\
\leq a\|y - T(x)\| + b\|T(x) - T^2(x)\| + c\max\{|y - T(y)|, \|y - T^2(x)\|\} \\
\quad + a\|y - T^2(x)\| + b\|T^3(x) - T^2(x)\| + c\max\{|y - T(y)|, \|y - T^3(x)\|\} \\
\leq \left(\frac{1}{2}a + (2b + \frac{1}{2}a)\right)\|T(x) - x\| + 2c\max\{|y - T(y)|, \frac{1}{2}\|T(x) - x\|\} \\
= \left(\frac{1}{2}\right)(a^2 + 3a + 4b)\|T(x) - x\| + 2c\max\{|y - T(y)|, \frac{1}{2}\|T(x) - x\|\} \\
= \left(\frac{1}{2}\right)a(a-1) + 4(a + b)\|T(x) - x\| + 2c\max\{|y - T(y)|, \frac{1}{2}\|T(x) - x\|\}.
\]

We have two cases, the first case is when
\[
\max\{|y - T(y)|, \frac{1}{2}\|T(x) - x\|\} = \frac{1}{2}\|T(x) - x\|.
\]
in this case, we have
\[
2\|T(y) - y\| \leq \left(\frac{1}{2}\right)a(a-1) + 4(a + b)\|T(x) - x\| + 2c\left(\frac{1}{2}\right)\|T(x) - x\| \\
= \frac{1}{2}(a(a-1) + 4(a + b) + 2c)\|T(x) - x\| \\
= \frac{1}{2}(a(a-1) + 2(a + b) + 2(a + b + c))\|T(x) - x\| \\
\leq \frac{1}{2}(a(a-1) + 2(a + b) + 2)\|T(x) - x\| \\
\leq [a + b + 1]\|T(x) - x\| \\
= (1 - c + 1)\|T(x) - x\| \\
= (2 - c)\|T(x) - x\|.
\]

Therefore,
\[
\|y - T(y)\| \leq \left[\frac{2 - c}{2}\right]\|T(x) - x\| = \left(1 - \frac{c}{2}\right)\|T(x) - x\|. \tag{3.5}
\]
The second case is when
\[
\max\{|y - T(y)|, \frac{1}{2}\|T(x) - x\|\} = \|y - T(y)\|,
\]
in this case we have
\[
\|T(y) - y\| \leq \left(\frac{1}{4}\right)(a^2 + 3a + 4b)\|T(x) - x\| + c\|y - T(y)\|.
\]
Consequently, we see that
\[
\|T(y) - y\| \leq \frac{(a^2 + 3a + 4b)}{4(1-c)} \|T(x) - x\|
\]
\[
= \left[ \frac{a^2 - a + 4(a + b)}{4(1-c)} \right] \|T(x) - x\|
\]
\[
= \left[ \frac{(a^2 - a) + 4(1-c)}{4(1-c)} \right] \|T(x) - x\|.
\]
Therefore,
\[
\|T(y) - y\| \leq \left[ 1 - \frac{a(1-a)}{4(1-c)} \right] \|T(x) - x\|. \tag{3.6}
\]
In both cases, one can write,
\[
\|T(y) - y\| \leq t \|T(x) - x\|, \tag{3.7}
\]
where \(t\) is a positive real number with \(t < 1\). Now, using Lemma 3.2, we see that \(\inf\{\|T(x) - x\| : x \in C\} = 0\). Pick any sequence \(\{x_n\}_{n \in \mathbb{N}}\) with \(\lim_{n \to \infty} \|T(x_n) - x_n\| = 0\), we claim that such a sequence is Cauchy sequence in \(C\). In fact, we have
\[
\|x_m - x_n\| \leq \|T(x_m) - x_n\| + \|x_n - T(x_m)\| + \|T(x_m) - T(x_n)\|
\]
\[
\leq \|T(x_m) - x_n\| + \|x_n - T(x_m)\|
\]
\[
+ \frac{1}{1-c} [a\|x_m - x_n\| + b\|x_n - T(x_m)\| + c\|x_n - T(x_m)\|]
\]
\[
= \frac{a}{1-c}\|x_m - x_n\| + \left( 1 + \frac{c}{1-c} \right) \|x_n - T(x_m)\|
\]
\[
+ \left( 1 + \frac{b}{1-c} \right) \|x_n - T(x_n)\|.
\]
Thus
\[
\|x_m - x_n\| \leq \frac{1}{1 - (a+c)} \left[ \|x_m - T(x_m)\| + (1 + b)\|x_n - T(x_n)\| \right].
\]
Taking the limit as \(n,m \to \infty\) proves that \(\{x_n\}_{n \in \mathbb{N}}\) is Cauchy sequence in \(C\). Since \(C\) is weakly Cauchy subset of \(X\), the sequence \(\{x_n\}_{n \in \mathbb{N}}\) has subsequence converging weakly to some point \(y_0 \in X\), since \(C\) is closed convex, using Lemma 3.1, we see that \(\{x_n\}_{n \in \mathbb{N}}\) has subsequence converging strongly to \(y_0\) and \(y \in C\)
\[
\|T(y_0) - y_0\| \leq \|T(y_0) - T(x_n)\| + \|T(x_n) - x_n\| + \|x_n - y_0\|
\]
\[
\leq \frac{1}{1-c} [a\|x_n - y_0\| + b\|x_n - T(x_n)\| + c\|y_0 - T(y_0)\|]
\]
\[
+ \|T(x_n) - x_n\| + \|x_n - y_0\|
\]
\[
\leq \left( \frac{a}{1-c} + 1 \right) \|x_n - y_0\| + \left( \frac{b}{1-c} + 1 \right) \|x_n - T(x_n)\|
\]
\[
+ \frac{c}{1-c} \|y_0 - T(y_0)\|. \]
Therefore,
\[ \| T(y_0) - y_0 \| \leq \left( \frac{1 + a - c}{1 - 2c} \right) \| x_n - y_0 \| + \left( \frac{1 + b - c}{1 - 2c} \right) \| x_n - T(x_n) \|. \]

Taking the limit as \( n \to \infty \) yields \( T(y_0) = y_0 \). Finally to prove that such a fixed point is unique, let \( y \) and \( z \) be two distinct fixed points of \( T \), then the following strict inequality gives an obvious contradiction: \( \| y - z \| = \| T(y) - T(z) \| \leq a \| y - z \| + b \| y - T(y) \| + c \max \{ \| y - T(y) \|, \| y - T(z) \| \} = (a + c) \| y - z \| < \| y - z \| . \)

Corollary 3.1. Let \( C \) be a closed convex and weakly Cauchy subset of a normed space \( X \), \( T \) be a mapping from \( C \) into \( C \) that satisfies \( \| T(x) - T(y) \| \leq a \| x - y \| + b \| y - T(y) \| + c \| x - T(x) \| + e \| x - T(y) \| \) for all \( x, y \in C \) and for some real numbers \( a, b, e \in [0, 1] \), \( 0 \leq a + e < 1/2 \), \( a + b + c + e = 1 \), and \( 0 < a < 1 \). Then \( T \) has a unique fixed point.

Proof. Let \( T \) satisfy \( \| T(x) - T(y) \| \leq a \| x - y \| + b \| y - T(y) \| + c \| x - T(x) \| + e \| x - T(y) \| \) for all \( x, y \in C \) and for some real numbers \( a, b, c \) and \( e \in [0, 1] \), \( a + b + c + e = 1 \). Then \( \| T(x) - T(y) \| \leq a \| x - y \| + b \| y - T(y) \| + (c + e) \max \{ \| x - T(x) \|, \| x - T(y) \| \} \) for all \( x, y \in C \), since \( a + b + (c + e) = 1 \), \( T \) is \( \{ a, b, c + e \} \)-type mapping, using Theorem 3.1 insures that \( T \) has a unique fixed point.

Theorem 3.2. Let \( C \) be a closed convex and weakly Cauchy subset of a normed space \( X \), \( T \) be \( \{ a, b, c \} \)-type mapping from \( C \) into \( C \), then \( T \) has a unique fixed point. Moreover, for any \( x \in C \) the sequence of iterates \( \{ T^n(x) \} \) has a subsequence strongly convergent to the unique fixed point of \( T \).

Proof. Using Lemma 3.3 with the fact that \( (a + b)/(1 - c) < 1 \), the inequalities (3.3) insure that for every \( m, n \in N \), \( n \leq m \), we have
\[ \| T^m(x) - T^n(x) \| \leq \left( \frac{a + b}{1 - (\frac{a + b}{1 - c})} \right)^n \| T(x) - x \|. \]

Taking the limit as \( n \to \infty \) proves that the sequence of iterates is a Cauchy sequence in \( C \), since \( C \) is weakly Cauchy, the sequence \( \{ T^n(x) \} \) converging weakly to some point \( y \in X \), since \( C \) is closed convex, the sequence \( \{ T^n(x) \} \) is strongly convergent to \( y \) and \( y \in C \). Taking the limit of each side of the inequality (3.3) as \( n \to \infty \), and using the fact that \( (a + b)/(1 - c) < 1 \), we prove that \( \lim_{n \to \infty} \| T^{n+1}(x) - T^n(x) \| = 0 \), hence
\[ \lim_{n \to \infty} \| T^{n+1}(x) - T^n(x) \| = 0, \quad (3.8) \]
on the other side, the inequalities
\[
\begin{align*}
\| T(y) - T^{i+1}(x) \| &\leq a \| y - T^{i}(x) \| + b \| T^{i}(x) - T^{i+1}(x) \| \\
&\quad + c \max \{ \| y - T(y) \|, \| y - T^{i+1}(x) \| \} \\
&\leq a \| y - T^{i}(x) \| + b \| T^{i}(x) - T^{i+1}(x) \| \\
&\quad + c \max \{ \| y - T(y) \|, \| y - T^{i+1}(x) \| \}
\end{align*}
\]

where...
show that

\[(1-c)\|y-T(y)-T^{i+1}(x)\| \leq a\|y-T^i(x)\| + b\|T^i(x)-T^{i+1}(x)\| + c\|y-T(y)\|\]

accordingly, we have

\[
\|T(y)-y\| \leq \|T(y)-T^{i+1}(x)\| + \|T^{i+1}(x)-T^i(x)\| + \|T^i(x)-y\|
\]

\[
\leq \left(\frac{1}{1-c}\right)[a\|y-T^i(x)\| + b\|T^i(x)-T^{i+1}(x)\| + c\|y-T(y)\|]
\]

\[
+ \|T^{i+1}(x)-T^i\| + \|T^i-x\|
\]

\[
\leq \left(\frac{1+a-c}{1-c}\right)\|y-T^i(x)\| + \left(\frac{1+b-c}{1-c}\right)\|T^i(x)-T^{i+1}(x)\|
\]

\[
+ \left(\frac{c}{1-c}\right)\|y-T(y)\|.
\]

Therefore,

\[(1-2c)\|T(y)-y\| \leq (1+a-c)\|y-T^i(x)\| + (1+b-c)\|T^i(x)-T^{i+1}(x)\|.
\]

Taking the limit as \(n \to \infty\) proves that \(T(y)=y\). The uniqueness fixed point proof is similar to the proof of last part of Theorem 3.1.

**Corollary 3.2.** Let \(C\) be a closed convex weakly Cauchy subset of a normed space \(X\), \(T\) be a mapping from \(C\) into \(C\) that satisfies \(\|T(x)-T(y)\| \leq a\|x-y\| + b\|T^i(x)-T^{i+1}(x)\| + c\|x-T(x)\| + e\|x-T(y)\|\) for all \(x,y \in C\) and for some real numbers \(a,b,c\), \(0 \leq a + e < 1/2\), and \(e \in [0,1]\) and \(a + b + c + e < 1\), then \(T\) has a unique fixed point.

**Proof.** Let \(T\) satisfy \(\|T(x)-T(y)\| \leq a\|x-y\| + b\|T^i(x)-T^{i+1}(x)\| + c\|x-T(x)\| + e\|x-T(y)\|\) for all \(x,y \in C\) and for some real numbers \(a,b,c\) and \(e \in [0,1]\), \(a + b + c + e < 1\). Then \(\|T(x)-T(y)\| \leq a\|x-y\| + b\|y-T(y)\| + (c+e)\max\\{\|x-T(x)\|,\|x-T(y)\|\}\) for all \(x,y \in C\), since \(a+b+(c+e) < 1\), \(T\) is \(\{a,b,c+e\}\)-type mapping, using Theorem 3.2 proves that \(T\) has unique fixed point.

4 Conclusions

This paper suggests two new Theorems 3.1 and 3.2, these Theorems prove the existence of a unique fixed point of novel defined \(\{a,b,c\}\)-type contraction and nonexpansive mappings, the mappings are defined on closed convex and weakly Cauchy subset of a normed space, hence their Corollaries 3.1 and 3.2 generalizes some of the results given in [2,4,7] and [13]. Based on the proof, the given normed space \(X\) is not necessarily complete and the new \(\{a,b,c\}\)-type conditions are valid only on a closed convex weakly Cauchy subset of a normed space \(X\) and \(C\) is not necessarily bounded.
References


