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Uncertainty Principles for the Generalized Fourier Transform Associated with Spherical Mean Operator

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Abstract. The aim of this paper is to establish an extension of qualitative and quantitative uncertainty principles for the Fourier transform connected with the spherical mean operator.

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1 Introduction

Classical uncertainty principles give us information about a function and its Fourier transform. If we try to limit the behavior of one we lose control of the other. Uncertainty principles have implications in two main areas: quantum physics and signal analysis. In quantum physics they tell us that a particle speed and position cannot both be measured with infinite precision. In signal analysis they tell us that if we observe a signal only for a finite period of time, we will lose information about the frequencies the signal consisted of. The mathematical equivalence is that a function and its Fourier transform cannot both be arbitrarily localized. There is two categories of uncertainty principles: Quantitative uncertainty principles and Qualitative uncertainty principles.

Quantitative uncertainty principle is just another name for some special inequalities. These inequalities give us information about how a function and its Fourier transform relates. They are called uncertainty principles since they are similar to the classical Heisenberg Uncertainty Principle, which has had a big part to play in the development and

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understanding of quantum physics. For example: Benedicks [2], Slepian and Pollak [26], Landau and Pollak [17], and Donoho and Stark [9] paid attention to the supports of functions and gave qualitative uncertainty principles for the Fourier transforms.

Qualitative uncertainty principles are not inequalities, but are theorems that tell us how a function (and its Fourier transform) behave under certain circumstances. For example: Hardy [13], Morgan [21], Cowling and Price [7], Beurling [3], Miyachi [20] theorems enter within the framework of the quantitative uncertainty principles.

The quantitative and qualitative uncertainty principles has been studied by many authors for various Fourier transforms, for examples (cf. [5,6,11,12,18,19,27]).

Our aim here is to consider quantitative and qualitative uncertainty principles when the transform under consideration is the Fourier transform connected with the spherical mean operator. The spherical mean operator play an important role and have many applications, for example; in the image processing of so-called synthetic aperture radar (SAR) data [14,15], or in the linearized inverse scattering problem in acoustics [10]. These operators have been studied by many authors from many points of view [1,10,22,24].

The remaining part of the paper is organized as follows. In Section 2, we recall the main results about the spherical mean operator. Section 3 is devoted to generalize Cowling-Price's theorem for the generalized Fourier transform $\mathcal{F}$. In Section 4 we generalize Miyachi's theorem and in Section 5 Beurling's theorem for $\mathcal{F}$. Section 6 is devoted to Donoho-Stark's uncertainty principle and variants of Heisenberg's inequalities for $\mathcal{F}$.

Throughout this paper, the letter C indicates a positive constant not necessarily the same in each occurrence.

2 Spherical mean operator

In this section, we define and recall some properties of the spherical mean operator. For more details see [22,24]. We denote by

- $C_*(\mathbb{R}^{d+1})$ the space of continuous functions on $\mathbb{R}^{d+1}=\mathbb{R}\times\mathbb{R}^d$, even with respect to the first variable.
- $C_{*,c}(\mathbb{R}^{d+1})$ the subspace of $C_*(\mathbb{R}^{d+1})$ formed by functions with compact support.
- $\mathcal{E}_*(\mathbb{R}^{d+1})$ the space of infinitely differentiable functions on $\mathbb{R}^{d+1}$, even with respect to the first variable.
- S^d the unit sphere in $\mathbb{R}^{d+1}$,
$$S^d=\big\{(\eta,\xi)\in\mathbb{R}^{d+1}:\eta^2+\|\xi\|^2=1\big\},$$
where for $\xi=(\xi_1,\cdots,\xi_d)$, we have $\|\xi\|^2=\xi_1^2+\cdots+\xi_d^2$.
- $d\sigma_d$ the normalized surface measure on S^d.
- $\mathbb{R}^{d+1}_+=\big\{(r,x)\in\mathbb{R}^{d+1}:\ r>0\big\}$.

Definition 2.1. The spherical mean operator is defined on $C_*(\mathbb{R}^{d+1})$ by

$$\forall (r,x)\in\mathbb{R}_+^{d+1},\quad \mathcal{R}f(r,x)=\int_{S^d} f(r\eta,x+r\xi)d\sigma_d(\eta,\xi).$$

The spherical mean kernel is the function $\varphi_{\mu,\lambda}$, $(\mu,\lambda)\in\mathbb{C}^{d+1}=\mathbb{C}\times\mathbb{C}^d$, defined by

$$\forall (r,x)\in\mathbb{R}_+^{d+1},\quad \varphi_{\mu,\lambda}(r,x)=\mathcal{R}(\cos(\mu.)e^{-i\langle\lambda,\cdot\rangle})(r,x).$$

We have

$$\varphi_{\mu,\lambda}(r,x)=j_{\frac{d-1}{2}}\Big(r\sqrt{\mu^2+\lambda^2}\Big)e^{-i\langle\lambda,x\rangle},$$

where

- $\lambda^2=\lambda_1^2+\cdots+\lambda_d^2$, if $\lambda=(\lambda_1,\cdots,\lambda_d)\in\mathbb{C}^d$.
- $\langle\lambda,x\rangle=\lambda_1x_1+\cdots+\lambda_dx_d$, if $x=(x_1,\cdots,x_d)\in\mathbb{R}^d$ and $\lambda=(\lambda_1,\cdots,\lambda_d)\in\mathbb{C}^d$.
- $j_{\frac{d-1}{2}}$ is the normalized Bessel function defined by

$$j_{(d-1)/2}(x)=\Gamma((d+1)/2)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma((2k+1+d)/2)}(z/2)^{2k}.$$

Remark 2.1. For all $\nu\in\mathbb{N}^{d+1}$, $(r,x)\in\mathbb{R}^{d+1}$ and $z=(\mu,\lambda)\in\mathbb{C}^{d+1}$,

$$|D_z^\nu\varphi_{\mu,\lambda}(r,x)|\le\|(r,x)\|^{|\nu|}\exp(\|(r,x)\|\|\mathrm{Im} z\|), \tag{2.1}$$

where

$$D_z^\nu=\frac{\partial^{|\nu|}}{\partial z_1^{\nu_1}\cdots\partial z_{d+1}^{\nu_{d+1}}}\quad\text{and}\quad |\nu|=\nu_1+\cdots+\nu_{d+1}.$$

Now let Γ be the set

$$\Gamma=\mathbb{R}^{d+1}\cup\big\{(it,x);\ (t,x)\in\mathbb{R}^{d+1},\ |t|\le\|x\|\big\}.$$

Γ_+ the subset of Γ, given by

$$\Gamma_+=\mathbb{R}^{d+1}\cup\big\{(it,x);\ (t,x)\in\mathbb{R}^{d+1},\ 0\le t\le\|x\|\big\}.$$

We have for all $(\mu,\lambda)\in\Gamma$,

$$\sup_{(r,x)\in\mathbb{R}^{d+1}}|\varphi_{\mu,\lambda}(r,x)|=1.$$

In the following, we denote by

- $d\nu(r,x)$ the measure defined on $\mathbb{R}_+^{d+1}$ by
$$d\nu(r,x)=k_d r^d dr\otimes dx,$$
with
$$k_d=\frac{1}{2^{(d-1)/2}\Gamma((d+1)/2)(2\pi)^{d/2}}.$$
- $L^p(d\nu)$, $1\le p\le\infty$, the space of measurable functions on $\mathbb{R}_+^{d+1}$, satisfying
$$\|f\|_{L^p(d\nu)}=\Big(\int_{\mathbb{R}_+^{d+1}}|f(r,x)|^p d\nu(r,x)\Big)^{1/p}<\infty, \qquad 1\le p<\infty,$$
$$\|f\|_{L^\infty(d\nu)}=\text{esssup}_{(r,x)\in\mathbb{R}_+^{d+1}}|f(r,x)|<\infty, \qquad p=\infty.$$
- $\mathcal{B}_{\Gamma_+}$ the σ-algebra defined on Γ_+ by
$$\mathcal{B}_{\Gamma_+}=\{\theta^{-1}(B):\ B\in\mathcal{B}_{Bor}(\mathbb{R}_+^{d+1})\},$$
where θ defined on the set Γ_+ by $\theta(\lambda,\mu)=(\sqrt{\mu^2+\|\lambda\|^2},\lambda)$.
- $d\gamma$ the measure defined on $\mathcal{B}_{\Gamma_+}$ by
$$\forall A\subset\mathcal{B}_{\Gamma_+},\quad \gamma(A)=\nu(\theta(A)).$$
- $L^p(d\gamma)$, $1\le p\le\infty$, the space of measurable functions on Γ_+, satisfying
$$\|f\|_{L^p(d\gamma)}=\Big(\int_{\Gamma_+}|f(\mu,\lambda)|^p d\gamma(\mu,\lambda)\Big)^{1/p}<\infty, \qquad 1\le p<\infty,$$
$$\|f\|_{L^\infty(d\gamma)}=\text{esssup}_{(\mu,\lambda)\in\Gamma_+}|f(\mu,\lambda)|<\infty, \qquad p=\infty.$$

We have the following properties:

Proposition 2.1. i) For every nonnegative measurable function f on Γ_+, we have
$$\int_{\Gamma_+}f(\mu,\lambda)d\gamma(\mu,\lambda)=k_d\Big[\int_{\mathbb{R}_+^{d+1}}f(\mu,\lambda)(\mu^2+\|\lambda\|^2)^{(d-1)/2}\mu d\mu d\lambda$$
$$+\int_{\mathbb{R}^d}\int_0^{\|\lambda\|}f(i\mu,\lambda)(\|\lambda\|^2-\mu^2)^{(d-1)/2}\mu d\mu d\lambda\Big].$$

ii) For every nonnegative measurable function f on $\mathbb{R}_+^{d+1}$ (resp. integrable on $\mathbb{R}_+^{d+1}$ with respect to the measure $d\nu$), $f\circ\theta$ is a measurable nonnegative function on Γ_+, (resp. integrable on Γ_+ with respect to the measure $d\gamma$) and we have
$$\int_{\Gamma_+}f\circ\theta(\mu,\lambda)d\gamma(\mu,\lambda)=\int_{\mathbb{R}_+^{d+1}}f(r,x)d\nu(r,x). \tag{2.2}$$

In the following we recall some results on the dual of the spherical mean operator $\mathcal{R}$.

Definition 2.2. The dual ${}^t\mathcal{R}$ of the spherical mean operator $\mathcal{R}$ is defined by: $\forall(s,y)\in\mathbb{R}^{d+1}$,

$$ {}^t\mathcal{R}(f)(s,y)=\frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}\int_{\mathbb{R}^d} f\Big(\sqrt{s^2+\|y-z\|^2},z\Big)dz, \quad f\in C_{*,c}(\mathbb{R}^{d+1}). \tag{2.3}$$

Example 2.1. Let $p\in[1,\infty)$. For all $a>0$, $\beta>0$ we have

$$\forall(s,y)\in\mathbb{R}^{d+1}, \quad {}^t\mathcal{R}(E_a^p)(s,y)=C(a,\beta,p)E^p_{\frac{a\beta}{1+\beta},1+\beta}(s,y), \tag{2.4}$$

with $E_{a,\beta}$ is the Gauss kernel associated with the spherical mean operator $\mathcal{R}$ defined by

$$\forall(r,x)\in\mathbb{C}^{d+1}, \quad E_{a,\beta}(r,x)=k(a,\beta)e^{-a(\beta r^2+\|x\|^2)}, \tag{2.5}$$

where

$$k(a,\beta)=\frac{2\sqrt{\pi}a^{d+\frac{1}{2}}}{\Gamma(\frac{d+1}{2})}\Big(\frac{\beta}{\pi}\Big)^{\frac{d+1}{2}} \quad \text{and} \quad C(a,\beta,p)=\frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}}\Big[\frac{(1+\beta)^{p-1}}{a\beta^p p}\Big]^{\frac{d}{2}}.$$

Proposition 2.2. The function ${}^t\mathcal{R}(f)$ defined almost everywhere on $\mathbb{R}^{d+1}_+$ by

$${}^t\mathcal{R}(f)(s,y)=\frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}\int_{\mathbb{R}^d} f\Big(\sqrt{s^2+\|y-x\|^2},x\Big)dx$$

is Lebesgue integrable on $\mathbb{R}^{d+1}_+$. Moreover for all bounded function $g\in C_*(\mathbb{R}^{d+1})$, we have the formula

$$\int^t_{\mathbb{R}^{d+1}_+}\mathcal{R}(f)(s,y)g(s,y)dsdy=\int_{\mathbb{R}^{d+1}_+}{}^t\mathcal{R}(f)(s,y)g(s,y)dsdy. \tag{2.6}$$

Remark 2.2. Let f be in $L^1(d\nu)$. By taking $g\equiv 1$ in the relation (2.6) we deduce that

$$\int^t_{\mathbb{R}^{d+1}_+}\mathcal{R}(f)(s,y)dsdy=\int_{\mathbb{R}^{d+1}_+}{}^t\mathcal{R}(f)(s,y)dsdy, \tag{2.7}$$

where

$$C(d):=\int_{S^d}d\sigma_d(\eta,\xi).$$

We consider the generalized Fourier transform $\mathcal{F}$ associated with the spherical mean operator $\mathcal{R}$ and we recall its main properties.

Definition 2.3. The Fourier transform associated with the spherical mean operator is defined on $L^1(d\nu)$ by

$$\forall(\mu,\lambda)\in\Gamma, \quad \mathcal{F}f(\mu,\lambda)=\int_{\mathbb{R}^{d+1}_+}f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu(r,x). \tag{2.8}$$

Example 2.2. Let $a>0$, $\beta>0$. The Fourier transform of Gauss kernel associated with spherical mean operator is given by

$$\forall(\mu,\lambda)\in\Gamma,\quad \mathcal{F}(E_{a,\beta})(\mu,\lambda)=C(a,\beta,d)E_{\frac{1+\beta}{4a\beta}a,\frac{1}{1+\beta}}(\mu,\lambda),$$

where

$$C(a,\beta,d)=2^{2d}\Gamma\Big(\frac{d+1}{2}\Big)(a\beta)^{d+\frac{1}{2}}\Big(\frac{\pi}{1+\beta}\Big)^{\frac{d}{2}}.$$

Proposition 2.3. For all f in $L^1(d\nu)$, we have the relation

$$\forall(\mu,\lambda)\in\Gamma,\quad \mathcal{F}f(\mu,\lambda)=\mathcal{F}_0\circ{}^t\mathcal{R}(f)(\mu,\lambda), \tag{2.9}$$

where $\mathcal{F}_0$ is the Fourier-cosine transform on $\mathbb{R}^{d+1}$ defined for f in $\mathcal{S}_*(\mathbb{R}^{d+1})$ by

$$\forall(\mu,\lambda)\in\mathbb{R}^{d+1},\quad \mathcal{F}_0(f)(\mu,\lambda)=\int_{\mathbb{R}_+^{d+1}}f(r,x)e^{-i\langle\lambda,x\rangle}\cos(r\mu)drdx.$$

In the follow we recall some properties on the Fourier transform $\mathcal{F}$.

For all $f\in L^1(d\nu)$,

$$\|\mathcal{F}(f)\|_{L^\infty(d\gamma)}\le\|f\|_{L^1(d\nu)}. \tag{2.10}$$

For $f\in L^1(d\nu)$ such that $\mathcal{F}f\in L^1(d\gamma)$, we have the inversion formula for $\mathcal{F}$: for almost every $(r,x)\in\mathbb{R}_+^{d+1}$,

$$f(r,x)=\int_{\Gamma_+}\mathcal{F}f(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma(\mu,\lambda). \tag{2.11}$$

Theorem 2.1 (Plancherel formula)**.** *For every f in $\mathcal{S}_*(\mathbb{R}^{d+1})$, we have*

$$\int_{\Gamma_+}|\mathcal{F}(f)(\lambda,\mu)|^2d\gamma(\lambda,\mu)=\int_{\mathbb{R}_+^{d+1}}|f(r,x)|^2d\nu(r,x). \tag{2.12}$$

In particular, the Fourier transform $\mathcal{F}$ can be extended to an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$.

For $(r,x)\in\mathbb{R}_+^{d+1}$, $s>0$, we note $N_s(r,x)$, by

$$N_s(r,x):=e^{-\frac{\|(r,x)\|^2}{4s}}. \tag{2.13}$$

We have

$$\mathcal{F}(N_s(r,x))(t,y)=C(s)e^{-s\|(t,y)\|^2}.$$

We define the following functions $W_l^s(r,x)$, $l\in\mathbb{N}^{d+1}$, $s>0$ by

$$W_l^s(r,x)=r^{2k}x^\alpha e^{-s(r^2+\|x\|^2)},\quad l=(k,\alpha). \tag{2.14}$$

We denote by $\mathcal{P}_m(\mathbb{R}^{d+1})$ the set of homogeneous polynomials of degree m.

Proposition 2.4 (see [6])**.** Let $\psi\in\mathcal{P}_m(\mathbb{R}^{d+1})$ be homogeneous. Then for all $\delta>0$, there exists a homogeneous $Q\in\mathcal{P}_m(\mathbb{R}^{d+1})$, such that

$$\mathcal{F}(\psi(\cdot)e^{-\delta\|\cdot\|^2})(r,x)=Q(r,x)e^{-\frac{1}{4\delta}\|(r,x)\|^2}. \tag{2.15}$$

3 Generalized Cowling-Price theorem for the generalized Fourier transform

Theorem 3.1. *Let f be a measurable function on $\mathbb{R}_+^{d+1}$ such that*

$$\int_{\mathbb{R}_+^{d+1}} \frac{e^{ap\|(r,x)\|^2}|f(r,x)|^p}{(1+\|(r,x)\|)^n} d\nu(r,x) < \infty \tag{3.1}$$

and

$$\int_{\mathbb{R}_+^{d+1}} \frac{e^{bq\|(\alpha,\xi)\|^2}|\mathcal{F}(f)(\alpha,\xi)|^q}{(1+\|(\alpha,\xi)\|)^m} d\alpha d\xi < \infty, \tag{3.2}$$

for some constants $a>0$, $b>0$, $1\le p,q<\infty$, and for any $n\in(d+1,d+p+1]$ and $m\in(d+1,d+q+1]$. Then

(i) If $ab>1/4$, we have $f=0$ almost everywhere.

(ii) If $ab=1/4$, we have $f=CN_b$.

(iii) If $ab<1/4$, for all $\delta\in[b,1/(4a)]$, the functions of the form $f(r,x)=N_\delta(r,x)P(r,x)$, where $P\in\mathcal{P}$, satisfy (3.1) *and* (3.2).

Proof. We shall show that $\mathcal{F}(f)(z)$ exists and is an entire function in $z\in\mathbb{C}^{d+1}$ and

$$|\mathcal{F}(f)(z)| \le Ce^{\frac{1}{4a}\|Imz\|^2}(1+\|Imz\|)^s \quad \text{for all } z\in\mathbb{C}^{d+1}, \quad \text{for some } s>0. \tag{3.3}$$

The first assertion follows from the hypothesis on the function f and Hölder's inequality using (3.1) and the derivation theorem under the integral sign. We want to prove (3.3). Actually, it follows from (2.8) and (2.1) that for all $z=(\alpha+i\lambda,\xi+i\eta)\in\mathbb{C}^{d+1}$,

$$\begin{aligned}
&|\mathcal{F}(f)(\alpha+i\lambda,\xi+i\eta)| \\
\le& \int_{\mathbb{R}_+^{d+1}} |f(r,x)||\varphi_{(\alpha+i\lambda,\xi+i\eta)}(r,x)| d\nu(r,x) \\
\le& e^{\frac{\|(\lambda,\eta)\|^2}{4a}} \int_{\mathbb{R}_+^{d+1}} \frac{e^{a\|(r,x)\|^2}|f(r,x)|}{(1+\|(r,x)\|)^{\frac{n}{p}}} (1+\|(r,x)\|)^{\frac{n}{p}} e^{-a(\|(r,x)\|-\|\frac{(\lambda,\eta)}{2a}\|)^2} d\nu(r,x).
\end{aligned}$$

Then by using the Hölder inequality, (3.1) we can obtain that

$$\begin{aligned}
|\mathcal{F}(f)(\alpha+i\lambda,\xi+i\eta)| &\le Ce^{\frac{\lambda^2+\|\eta\|^2}{4a}} \Big(\int_{\mathbb{R}_+^{d+1}} (1+\|(r,x)\|)^{\frac{np'}{p}} e^{-ap'(\|(r,x)\|-\|\frac{(\lambda,\eta)}{2a}\|)^2} d\nu(r,x) \Big)^{\frac{1}{p'}} \\
&\le Ce^{\frac{\lambda^2+\|\eta\|^2}{4a}} \Big(\int_0^\infty (1+t)^{\frac{np'}{p}+d} e^{-ap'(t-\|\frac{(\lambda,\eta)}{2a}\|)^2} dt \Big)^{\frac{1}{p'}} \\
&\le Ce^{\frac{\|(\lambda,\eta)\|^2}{4a}} (1+\|(\lambda,\eta)\|)^{\frac{n}{p}+\frac{d}{p'}} = Ce^{\frac{1}{4a}\|Imz\|^2}(1+\|Imz\|)^{\frac{n}{p}+\frac{d}{p'}}.
\end{aligned}$$

Thus (3.3) is proved.

If $ab=1/4$, then

$$|\mathcal{F}(f)(\alpha+i\lambda,\xi+i\eta)|\leq Ce^{b\|Imz\|^2}(1+\|Imz\|)^{\frac{n}{p}+\frac{d}{p'}}.$$

Therefore, if we let $g(z)=e^{bz^2}\mathcal{F}(f)(z)$, then

$$|g(z)|\leq Ce^{b(\mathrm{Re}z)^2}(1+\|\mathrm{Im}z\|)^{\frac{n}{p}+\frac{d}{p'}}.$$

Hence it follows from (3.2) that

$$\int_{\mathbb{R}_+^{d+1}}\frac{|g(\alpha,\xi)|^q}{(1+\|(\alpha,\xi)\|)^m}d\alpha d\xi<\infty.$$

Here we use the following lemma:

Lemma 3.1 (see [25]). *Let h be an entire function on $\mathbb{C}^{d+1}$ such that*

$$|h(z)|\leq Ce^{a\|\mathrm{Re}z\|^2}(1+\|\mathrm{Im}z\|)^m$$

for some $m>0$, $a>0$ and

$$\int_{\mathbb{R}_+^{d+1}}\frac{|h(x)|^q}{(1+\|x\|)^s}|Q(x)|dx<\infty,$$

for some $q\geq 1$, $s>1$ and $Q\in\mathcal{P}_M(\mathbb{R}^{d+1})$. Then h is a polynomial with

$$\deg h\leq\min\Big\{m,\frac{s-M-d-1}{q}\Big\}$$

and, if $s\leq q+M+d+1$, then h is a constant.

Hence by this lemma g is a polynomial, we say P_b, with

$$\deg P_b\leq\min\Big\{\frac{n}{p}+\frac{d}{p'},\frac{m-d-1}{q}\Big\}.$$

Then

$$\mathcal{F}(f)(r,x)=P_b(r,x)e^{-b\|(r,x)\|^2}$$

and thus,

$$f(r,x)=Q_b(r,x)e^{-a\|(r,x)\|^2}\quad\text{for all }\ (r,x)\in\mathbb{R}^{d+1},$$

where Q_b is a polynomial with $\deg Q_b=\deg P_b$. Therefore, nonzero f satisfies (3.1) provided that

$$n>d+1+p\min\Big\{\frac{n}{p}+\frac{d}{p'},\frac{m-d-1}{q}\Big\}.$$

Furthermore, if $m\leq d+q+1$, then g is a constant by the Lemma 3.1 and thus

$$\mathcal{F}(f)(r,x)=Ce^{-b\|(r,x)\|^2} \quad \text{and} \quad f(r,x)=C_b e^{-a\|(r,x)\|^2}.$$

When $n>d+1$ and $m>d+1$, these functions satisfy (3.2) and (3.1) respectively. This proves (ii).

If $ab>1/4$, then we can choose positive constants, a_1, b_1 such that

$$a>a_1=\frac{1}{4b_1}>\frac{1}{4b}.$$

Then f and $\mathcal{F}(f)$ also satisfy (3.1) and (3.2) with a and b replaced by a_1 and b_1 respectively. Therefore, it follows that

$$\mathcal{F}(f)(r,x)=P_{b_1}(r,x)e^{-b_1\|(r,x)\|^2}.$$

But then $\mathcal{F}(f)$ cannot satisfy (3.2) unless $P_{b_1}\equiv 0$, which implies $f\equiv 0$. This proves (i).

If $ab<1/4$, then for all $\delta\in(b,1/(4a))$, the functions of the form $f(r,x)=P(r,x)N_\delta(r,x)$, where $P\in\mathcal{P}$, satisfy (3.1) and (3.2). This proves (iii). □

The following is an immediate consequence of Theorem 3.1.

Corollary 3.1. Let f be a measurable function on $\mathbb{R}_+^{d+1}$ such that

$$|f(r,x)|\leq Me^{-a\|(r,x)\|^2}(1+\|(r,x)\|)^m \ \text{ a.e.} \tag{3.4}$$

and for all $(\alpha,\xi)\in\mathbb{R}_+^{d+1}$,

$$|\mathcal{F}(f)(\alpha,\xi)|\leq Me^{-b\|(\alpha,\xi)\|^2} \tag{3.5}$$

for some constants $a,b>0$, $r\geq 0$ and $M>0$.

- (i) If $ab>1/4$, then $f=0$ almost everywhere.
- (ii) If $ab=1/4$, then f is of the form $f(r,x)=CN_b(r,x)$.
- (iii) If $ab<1/4$, then there are infinity many nonzero f satisfying (3.4) and (3.5).

4 Miyachi's theorem for the Generalized Fourier transform

Theorem 4.1. *Let f be a measurable function on $\mathbb{R}_+^{d+1}$ even with respect to the first variable such that*

$$E_{a,\beta}^{-1}f\in L^p(d\nu)+L^q(d\nu) \tag{4.1}$$

and

$$\int_{\mathbb{R}^{d+1}}\log^+\frac{E^{-1}_{\frac{b(1+\beta)}{\beta},\frac{1}{1+\beta}}(\alpha,\xi)|\mathcal{F}(f)(\alpha,\xi)|}{\lambda}d\alpha d\xi<\infty, \tag{4.2}$$

for some constants $a>0$, $b>0$ $\lambda>0$, $1\leq p,q\leq\infty$. Then *(i) If $ab>1/4$, we have $f=0$ almost everywhere.*

(ii) If $ab=1/4$, we have $f=CE_{b,\beta}$ with $|C|\leq\lambda$.

(iii) If $ab<1/4$, for all $\delta\in(b,1/(4a))$, the functions of the form $f(x)=CE_{\delta,\beta}$, satisfy (4.1) *and* (4.2).

To prove this result we need the following lemmas.

Lemma 4.1 (see [19]). *Let h be an entire on $\mathbb{C}^{d+1}$ function such that*

$$|h(z)|\leq Ae^{B\|Rez\|^2} \quad \text{and} \quad \int_{\mathbb{R}^{d+1}}\log^+|h(y)|dy<\infty, \tag{4.3}$$

for some positive constants A, B. Then h is a constant on $\mathbb{C}^{d+1}$.

Lemma 4.2. *Let r be in $[1,\infty]$. We consider a function g in $L^r(d\nu)$. Then there exists a positive constant C such that:*

$$\Big\|E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}\,{}^t\mathcal{R}(E_{a,\beta}g)\Big\|_{L^r(\mathbb{R}^{d+1}_+)}\leq C\|g\|_{L^r(d\nu)},$$

where $\|\cdot\|_{L^r(\mathbb{R}^{d+1}_+)}$ is the norm of the usual Lebesgue space $L^r(\mathbb{R}^{d+1}_+)$ and $a>0$.

Proof. From the hypothesis it follows that $E_{a,\beta}g$ belongs to $L^1(d\nu)$. Then by Proposition 2.2, the function ${}^t\mathcal{R}(E^{-1}_{a,\beta}g)$ is defined almost everywhere on $\mathbb{R}^{d+1}$. Now we consider two cases.

i) If $r\in[1,\infty)$, we have

$$\begin{aligned}&\Big\|E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}\,{}^t\mathcal{R}(E_{a,\beta}g)\Big\|^r_{L^r(\mathbb{R}^{d+1}_+)}\\ \leq& C\int_{\mathbb{R}^{d+1}_+}E^{-r}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)\times\Big(\int_{\mathbb{R}^d}E_{a,\beta}\Big(\sqrt{s^2+\|y-z\|^2},z\Big)\Big|g\Big(\sqrt{s^2+\|y-z\|^2},z\Big)\Big|dz\Big)^r dsdy.\end{aligned}$$

By applying Hölder's inequality in the middle integral, we obtain

$$\begin{aligned}&\Big\|E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}\,{}^t\mathcal{R}(E_{a,\beta}g)\Big\|^r_{L^r(\mathbb{R}^{d+1}_+)}\\ \leq&\int_{\mathbb{R}^{d+1}_+}E^{-r}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)\Big(\int_{\mathbb{R}^d}\Big|g\Big(\sqrt{s^2+\|y-z\|^2},z\Big)\Big|^r dz\Big)\times\Big(\int_{\mathbb{R}^d}E^{r'}_{a,\beta}\Big(\sqrt{s^2+\|y-z\|^2},z\Big)dz\Big)^{r/r'}dyds,\end{aligned}$$

where r' is the conjugate exponent of r. But from (2.4) we deduce that

$$\Big\|E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}\,{}^t\mathcal{R}(E_{a,\beta}g)\Big\|^r_{L^r(\mathbb{R}^{d+1}_+)}\leq C\int_{\mathbb{R}^{d+1}_+}{}^t\mathcal{R}(|g|^r)(s,y)dsdy.$$

Thus using the relation (2.7) we obtain

$$\Big\|E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}\,{}^t\mathcal{R}(E_{a,\beta}g)\Big\|^r_{L^r(\mathbb{R}^{d+1}_+)}\leq C\int_{\mathbb{R}^{d+1}_+}|g(s,y)|^r d\nu(s,y)<\infty.$$

ii) If $r=\infty$, we have

$$\Big|E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)\,{}^t\mathcal{R}(E_{a,\beta}g)(s,y)\Big|\le E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)\,{}^t\mathcal{R}(E_{a,\beta})(s,y)\|g\|_{L^\infty(d\nu)},$$

and from (2.4) we deduce that

$$\Big|E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)\,{}^tR(E_{a,\beta}g)(s,y)\Big|\le C\|g\|_{L^\infty(d\nu)}<\infty.$$

This completes the proof. □

Lemma 4.3. *Let p,q in $[1,\infty]$ and f a measurable function on $\mathbb{R}^{d+1}_+$ such that*

$$E^{-1}_{a,\beta}f\in L^p(d\nu)+L^q(d\nu), \tag{4.4}$$

for some $a>0$, $\beta>0$. Then the function defined on $\mathbb{C}^{d+1}$ by

$$\mathcal{F}(f)(\mu,\lambda)=\int_{\mathbb{R}^{d+1}_+}f(r,x)\varphi_{(\mu,\lambda)}(r,x)d\nu(r,x), \tag{4.5}$$

is well defined and entire on $\mathbb{C}^{d+1}$. Moreover there exists a positive constant C such that for all ξ, η in $\mathbb{R}^d$ and $\alpha,\theta\in\mathbb{R}$ we have

$$|\mathcal{F}(f)(\alpha+i\theta,\xi+i\eta)|\le Ce^{\frac{(1+\beta)\|\eta\|^2+\theta^2}{4a\beta}}. \tag{4.6}$$

Proof. The first assertion follows from the hypothesis on the function f and Hölder's inequality using (4.4) and the derivation theorem under the integral sign. We want to prove (4.6).

The condition (4.4) implies that the function f belongs to $L^1(d\nu)$. Hence we deduce from (2.9) that for all ξ, η in $\mathbb{R}^d$ and $\alpha,\theta\in\mathbb{R}$, we have

$$\begin{aligned}\big|\mathcal{F}(f)(\alpha+i\theta,\xi+i\eta)\big|&=\Big|\int_{\mathbb{R}^{d+1}_+}{}^t\mathcal{R}(f)(s,y)e^{-i\langle y,\xi+i\eta\rangle}\cos(s(\alpha+i\theta))\Big|\\&\le\int_{\mathbb{R}^{d+1}_+}\Big|{}^t\mathcal{R}(f)(s,y)\Big|e^{\langle y,\eta\rangle}e^{|\theta|s}dsdy.\end{aligned}$$

The integral of the second member can also be written in the form

$$c_0E^{-1}_{\frac{1+\beta}{4a\beta},\frac{1}{1+\beta}}(\theta,\eta)\int_{\mathbb{R}^{d+1}_+}E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)\,{}^t\mathcal{R}(|f|)(s,y)E_{\frac{a\beta}{1+\beta},1+\beta}\Big(s-\frac{|\theta|}{2a\beta},y-\frac{1+\beta}{2a\beta}\eta\Big)dsdy,$$

where c_0 is a positive constant. On the follow we will to estimate

$$\int_{\mathbb{R}^{d+1}_+}E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)\,{}^t\mathcal{R}(|f|)(s,y)E_{\frac{a\beta}{1+\beta},1+\beta}\Big(s-\frac{|\theta|}{2a\beta},y-\frac{1+\beta}{2a\beta}\eta\Big)dsdy.$$

Indeed from (4.4) there exists u in $L^p(d\nu)$ and v in $L^q(d\nu)$ such that

$$f=E_{a,\beta}(u+v).$$

Thus using the Lemma 4.2 and Hölder inequality we obtain

$$\begin{aligned}&\int_{\mathbb{R}_+^{d+1}} E^{-1}_{\frac{a\beta}{1+\beta},1+\beta}(s,y)^t\mathcal{R}(|f|)(s,y)E_{\frac{a\beta}{1+\beta},1+\beta}\Big(s-\frac{|\theta|}{2a\beta},y-\frac{1+\beta}{2a\beta}\eta\Big)dsdy\\ \leq& C(\|u\|_{L^p(d\nu)}+\|v\|_{L^q(d\nu)})<\infty.\end{aligned}$$

Hence there exists a positive constant C such that

$$|\mathcal{F}(f)(\xi+i\eta)|\leq Ce^{\frac{(1+\beta)\|\eta\|^2+\theta^2}{4a\beta}}.$$

Thus, we complete the proof of the lemma. □

Proof of Theorem 4.1. We will divide the proof in several cases.
1st case. $ab>1/4$. Consider the function h defined on $\mathbb{C}^{d+1}$ by

$$h(\mu,\lambda)=E^{-1}_{\frac{1+\beta}{4a\beta},\frac{1}{1+\beta}}(\mu,\lambda)\mathcal{F}(f)(\mu,\lambda), \tag{4.7}$$

with $\mu=\alpha+i\theta\in\mathbb{C}$ and $\lambda=\xi+i\eta\in\mathbb{C}^d$. This function is entire on $\mathbb{C}^{d+1}$ and using (4.6) we obtain:

$$|h(\mu,\lambda)|\leq CE^{-1}_{\frac{1+\beta}{4a\beta},\frac{1}{1+\beta}}(\alpha,\xi), \tag{4.8}$$

for all $\lambda\in\mathbb{C}^d$ and $\mu\in\mathbb{C}$. On the other hand we have

$$\begin{aligned}&\int_{\mathbb{R}_+^{d+1}}\log^+|h(\alpha,\xi)|d\alpha d\xi\\ =&\int_{\mathbb{R}_+^{d+1}}\log^+\left|E^{-1}_{\frac{1+\beta}{4a\beta},\frac{1}{1+\beta}}(\alpha,\xi)\mathcal{F}(f)(\alpha,\xi)\right|d\alpha d\xi\\ =&\int_{\mathbb{R}_+^{d+1}}\log^+\left[\frac{E^{-1}_{\frac{b(1+\beta)}{\beta},\frac{1}{1+\beta}}(\alpha,\xi)|\mathcal{F}(f)(\alpha,\xi)|}{\lambda}\right]\lambda E_{\frac{(1+\beta)(4ab-1)}{4a\beta},\frac{1}{1+\beta}}(\alpha,\xi)d\alpha d\xi\\ \leq&\int_{\mathbb{R}_+^{d+1}}\log^+\left[\frac{E^{-1}_{\frac{b(1+\beta)}{\beta},\frac{1}{1+\beta}}(\alpha,\xi)|\mathcal{F}(f)(\alpha,\xi)|}{\lambda}\right]d\alpha d\xi+\int_{\mathbb{R}_+^{d+1}}\lambda E_{\frac{(1+\beta)(4ab-1)}{4a\beta},\frac{1}{1+\beta}}(\alpha,\xi)d\alpha d\xi,\end{aligned}$$

because $\log^+(cd)\leq\log^+(c)+d$ for all $c,d>0$. Since $ab>1/4$, (4.2) implies that

$$\int_{\mathbb{R}_+^{d+1}}\log^+|h(\alpha,\xi)|d\alpha d\xi<\infty. \tag{4.9}$$

From the relations (4.8) and (4.9), it follows from Lemma 4.1 that there exists a constant C such that

$$h(\mu,\lambda)=C, \quad (\mu,\lambda)\in\mathbb{C}^{d+1}.$$

Thus

$$\mathcal{F}(f)=CE_{\frac{1+\beta}{4a\beta},\frac{1}{1+\beta}}.$$

Using now the condition (4.2) and that $ab>1/4$, we deduce that $C=0$ and hence we obtain

$$\forall(\mu,\lambda)\in\Gamma, \quad \mathcal{F}(f)(\mu,\lambda)=0.$$

Then the injectivity of $\mathcal{F}$ implies the result of the theorem.

2nd case. $ab=1/4$. The same proof as for the the first step give that

$$\mathcal{F}(f)=CE_{\frac{1+\beta}{4a\beta},\frac{1}{1+\beta}},$$

with $|C|\le\lambda$. Thus

$$f=CE_{\frac{b}{4a\beta},\frac{1}{1+\beta}}.$$

3rd case. $ab<1/4$. In the sequel we will construct a family of nonzero functions which satisfy the conditions (4.1) and (4.2). By considering the family of functions $cE_{\delta,\beta}$, we see that

$$\mathcal{F}(f)=cE_{\frac{1+\beta}{4\delta\beta},\frac{1}{1+\beta}}.$$

These functions clearly satisfy the conditions (4.1), (4.2) for all $\delta\in(b,1/(4a))$. The proof of the Theorem is complete. □

The following is an immediate corollary of Theorem 4.1.

Corollary 4.1. Let f be a measurable function on $\mathbb{R}_+^{d+1}$ such that

$$E_{a,\beta}^{-1}f\in L^p(d\nu)+L^q(d\nu) \tag{4.10}$$

and

$$\int_{\mathbb{R}_+^{d+1}}E^{-r}_{\frac{b(1+\beta)}{\beta},\frac{1}{1+\beta}}(\alpha,\xi)|\mathcal{F}(f)(\alpha,\xi)|d\alpha d\xi<\infty, \tag{4.11}$$

for some constants $a>0$, $b>0$, $1\le p,q\le\infty$, $0<r\le\infty$. Then

(i) If $ab\ge1/4$, we have $f=0$ almost everywhere.

(ii) If $ab<1/4$, for all $\delta\in(b,1/(4a))$, the functions of the form $CE_{\delta,\beta}$ satisfy (4.10) and (4.11).

5 Beurling's theorem for the generalized Fourier transform

Beurling's theorem and Bonami, Demange, and Jaming's extension are generalized for the generalized Fourier transform as follows.

Theorem 5.1. *Let* $N\in\mathbb{N}$, $\delta>0$ *and* $f\in L^2(d\nu)$ *satisfy*

$$\int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}}\frac{|f(r,x)||\mathcal{F}(f)(t,y)||R(t,y)|^\delta}{(1+\|(r,x)\|+\|(t,y)\|)^N}e^{\|(r,x)\|\|(t,y)\|}d\nu(r,x)dtdy<\infty, \tag{5.1}$$

where R *is a polynomial of degree* m. *If* $N\geq d+m\delta+3$, *then*

$$f(r,x)=\sum_{|l|<\frac{N-d-m\delta-1}{2}}a_l^s W_l^s(r,x)\ \ a.e., \tag{5.2}$$

where $s>0$, $a_l^s\in\mathbb{C}$ *and* W_l^s *is given by* (2.14). *Otherwise,* $f(r,x)=0$ *almost everywhere.*

Proof. We start the following lemma.

Lemma 5.1. *We suppose that* $f\in L^2(d\nu)$ *satisfies* (5.1). *Then* $f\in L^1(d\nu)$.

Proof. We may suppose that f is not negligible. (5.1) and the Fubini theorem imply that for almost every $(t,y)\in\mathbb{R}_+^{d+1}$,

$$\frac{|\mathcal{F}(f)(y)||R(t,y)|^\delta}{(1+\|(t,y)\|)^N}\int_{\mathbb{R}_+^{d+1}}\frac{|f(r,x)|}{(1+\|(r,x)\|)^N}e^{\|(r,x)\|\|(t,y)\|}d\nu(r,x)<\infty.$$

Since f and thus, $\mathcal{F}(f)$ are not negligible, there exist $(t_0,y_0)\in\mathbb{R}_+^{d+1}$, $(t_0,y_0)\neq(0,0)$, such that $\mathcal{F}(f)(t_0,y_0)R(t_0,y_0)\neq 0$. Therefore,

$$\int_{\mathbb{R}_+^{d+1}}\frac{|f(r,x)|}{(1+\|(r,x)\|)^N}e^{\|(r,x)\|\|(t_0,y_0)\|}d\nu(r,x)<\infty.$$

Since

$$\frac{e^{\|(r,x)\|\|(t_0,y_0)\|}}{(1+\|x\|)^N}\geq 1$$

for large $\|(r,x)\|$, it follows that

$$\int_{\mathbb{R}_+^{d+1}}|f(r,x)|d\nu(r,x)<\infty.$$

So, the lemma is proved. ∎

This lemma and Proposition 2.2 imply that ${}^t\mathcal{R}(f)$ is well-defined almost everywhere on $\mathbb{R}^{d+1}_+$. By the same techniques used in [18], we can deduce that

$$\int_{\mathbb{R}^{d+1}_+}\int_{\mathbb{R}^{d+1}_+}\frac{e^{\|(r,x)\|\|(t,y)\|}|{}^t\mathcal{R}(f)(r,x)\|\mathcal{F}_0({}^t\mathcal{R})(f)(t,y)||R(t,y)|^\delta}{(1+\|(r,x)\|+\|(t,y)\|)^N}d\nu(r,x)dtdy<\infty.$$

According to Theorem 2.3 in [23], we conclude that for all $(r,x)\in\mathbb{R}^{d+1}_+$,

$${}^t\mathcal{R}(f)(r,x)=P(r,x)e^{-\frac{\|(r,x)\|^2}{4s}},$$

where $s>0$ and P a polynomial of degree strictly lower than $(N-d-m\delta-1)/2$. Then by (2.9),

$$\mathcal{F}(f)(t,y)=\mathcal{F}_0\circ{}^t\mathcal{R}(f)(t,y)=\mathcal{F}_0\Big(P(x)e^{-\frac{\|(r,x)\|^2}{4s}}\Big)(t,y)=Q(t,y)e^{-s\|(t,y)\|^2},$$

where Q is a polynomial of degree $\deg P$. Then by using (2.15), we can find constants a_l^s such that

$$\mathcal{F}(f)(t,y)=\mathcal{F}\left(\sum_{|l|<\frac{N-d-m\delta-1}{2}}a_l^sW_l^s\right)(t,y).$$

By the injectivity of $\mathcal{F}$ the desired result follows. □

As an application of Theorem 5.1, by using the same techniques in [18], we can deduce the following Gelfand-Shilov type theorem for the generalized Fourier transform.

Corollary 5.1. Let $N,m\in\mathbb{N}$, $\delta>0$, $a,b>0$ with $ab\geq 1/4$, and $1<p,q<\infty$ with $1/p+1/q=1$. Let $f\in L^2(d\nu)$ satisfy

$$\int_{\mathbb{R}^{d+1}_+}\frac{|f(r,x)|e^{\frac{(2a)^p}{p}\|(r,x)\|^p}}{(1+\|(r,x)\|)^N}d\nu(r,x)<\infty \tag{5.3}$$

and

$$\int_{\mathbb{R}^{d+1}_+}\frac{|\mathcal{F}(f)(t,y)|e^{\frac{(2b)^q}{q}\|(t,y)\|^q}|R(t,y)|^\delta}{(1+\|(t,y)\|)^N}dtdy<\infty \tag{5.4}$$

for some $R\in\mathcal{P}_m$.

(i) If $ab>1/4$ or $(p,q)\neq(2,2)$, then $f(r,x)=0$ almost everywhere.

(ii) If $ab=1/4$ and $(p,q)=(2,2)$, then f is of the form (5.2) whenever $N\geq(d+m\delta+3)/2$ and $r=2b^2$. Otherwise, $f(x)=0$ almost everywhere.

Proof. Since

$$4ab\|(r,x)\|\|(t,y)\| \le \frac{(2a)^p}{p}\|(r,x)\|^p + \frac{(2b)^q}{q}\|(t,y)\|^q,$$

it follows from (5.3) and (5.4) that

$$\int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}} \frac{|f(r,x)||\mathcal{F}(f)(t,y)\|R(t,y)|^\delta}{(1+\|(r,x)\|+\|(t,y)\|)^{2N}} e^{4ab\|(r,x)\|\|(t,y)\|} d\nu(r,x)dtdy < \infty.$$

Then (5.1) is satisfied, because $4ab \ge 1$. Therefore, according to the proof of Theorem 5.1, we can deduce that

$$\int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}} \frac{e^{4ab\|(r,x)\|\|(t,y)\|}|{}^t\mathcal{R}(f)(r,x)\|\mathcal{F}_0({}^t\mathcal{R})(f)(t,y)||R(t,y)|^\delta}{(1+\|(r,x)\|+\|(t,y)\|)^{2N}} d\nu(r,x)dtdy < \infty,$$

and ${}^t\mathcal{R}(f)$ and f are of the forms

$${}^t\mathcal{R}(f)(r,x) = P(r,x)e^{-\frac{\|(r,x)\|^2}{4s}} \quad \text{and} \quad \mathcal{F}(f)(t,y) = Q(t,y)e^{-s\|(t,y)\|^2},$$

where $s > 0$ and P, Q are polynomials of the same degree strictly lower than $(2N-d-m\delta-1)/2$. Therefore, substituting these from, we can deduce that

$$\int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}} \frac{e^{-(\sqrt{s}\|(t,y)\|-\frac{1}{2\sqrt{s}}\|(r,x)\|)^2} e^{(4ab-1)\|(r,x)\|\|(t,y)\|}|P(r,x)\|Q(r,x)\|R(t,y)|^\delta}{(1+\|(r,x)\|+\|(t,y)\|)^{2N}} d\nu(r,x)dtdy < \infty.$$

When $4ab > 1$, this integral is not finite unless $f = 0$ almost everywhere. Moreover, it follows from (5.3) and (5.4) that

$$\int_{\mathbb{R}_+^{d+1}} \frac{|P(r,x)|e^{-\frac{1}{4s}\|(r,x)\|^2} e^{\frac{(2a)^p}{p}\|(r,x)\|^p}}{(1+\|(r,x)\|)^N} d\nu(r,x) < \infty$$

and

$$\int_{\mathbb{R}_+^{d+1}} \frac{|Q(t,y)|e^{-s\|(t,y)\|^2} e^{\frac{(2b)^q}{q}\|(t,y)\|^q}|R(t,y)|^\delta}{(1+\|(t,y)\|)^N} dtdy < \infty.$$

Hence, one of these integrals is not finite unless $(p,q) = (2,2)$. When $4ab = 1$ and $(p,q) = (2,2)$, the finiteness of above integrals implies that $r = 2b^2$ and the rest follows from Theorem 5.1. □

6 Quantitative uncertainty principle for the generalized Fourier transform

We shall investigate the case where f and $\mathcal{F}(f)$ are close to zero outside measurable sets. Here the notion of "close to zero" is formulated as follows. If $f \in L^2(d\nu)$ is ε-concentrated on a measurable set $E\subset\mathbb{R}_+^{d+1}$ if there is a measurable function g vanishing outside E such that $\|f-g\|_{L^2(d\nu)}\leq\varepsilon\|f\|_{L^2(d\nu)}$. Therefore, if we introduce a projection operator P_E as

$$P_Ef(r,x)=\begin{cases} f(r,x), & \text{if } (r,x)\in E,\\ 0, & \text{if } (r,x)\notin E,\end{cases}$$

then f is ε-concentrated on E if and only if $\|f-P_Ef\|_{L^2(d\nu)}\leq\varepsilon\|f\|_{L^2(d\nu)}$. We define a projection operator Q_W as

$$Q_Wf(r,x)=\mathcal{F}^{-1}\big(P_W(\mathcal{F}(f))\big)(r,x).$$

Then $\mathcal{F}(f)$ is ε-concentrated on W if and only if $\|f-Q_Wf\|_{L^2(d\nu)}\leq\varepsilon\|f\|_{L^2(d\nu)}$. We note that, for measurable set $E\subset\mathbb{R}_+^{d+1}$ and $W\subset\Gamma$,

$$Q_WP_Ef(r,x)=\int_{\mathbb{R}_+^{d+1}}q(t,y;r,x)f(t,y)d\nu(t,y),$$

where

$$q(t,y;r,x)=\begin{cases} \int_W\varphi_{\mu,\lambda}(t,y)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma(\mu,\lambda), & \text{if } (t,y)\in E,\\ 0, & \text{if } (t,y)\notin E.\end{cases}$$

Indeed, by the Fubini's theorem we see that

$$\begin{aligned}Q_WP_Ef(r,x)&=\int_W\mathcal{F}(P_Ef)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma(\mu,\lambda)\\&=\int_W\Big(\int_Ef(t,y)\varphi_{\mu,\lambda}(t,y)d\nu(t,y)\Big)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma(\mu,\lambda)\\&=\int_Ef(t,y)\Big(\int_W\varphi_{\mu,\lambda}(t,y)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma(\mu,\lambda)\Big)d\nu(t,y).\end{aligned}$$

The Hilbert-Schmidt norm $\|Q_WP_E\|_{HS}$ is given by

$$\|Q_WP_E\|_{HS}=\Big(\int_{\mathbb{R}_+^{d+1}}\int_{\mathbb{R}_+^{d+1}}|q(t,y;r,x)|^2d\nu(t,y)d\nu(r,x)\Big)^{\frac{1}{2}}.$$

We denote by $\|T\|_2$ the operator norm on $L^2(d\nu)$. Since P_E and Q_W are projections, it is clear that $\|P_E\|_2=\|Q_W\|_2=1$. Moreover, it follows that

$$\|Q_WP_E\|_{HS}\geq\|Q_WP_E\|_2. \tag{6.1}$$

Lemma 6.1. *If E and W are sets of finite measure, then*

$$\|Q_WP_E\|_{HS}\leq\sqrt{mes_\nu(E)mes_\gamma(W)},$$

where

$$mes_\nu(E):=\int_Ed\nu(r,x),\quad mes_\gamma(W):=\int_Wd\gamma(\mu,\lambda).$$

Proof. For $(t,y)\in E$, let $g_{t,y}(r,x)=q(t,y;r,x)$. Eq. (2.11) implies that

$$\mathcal{F}(g_{t,y})(\mu,\lambda)=P_W(\varphi_{\mu,\lambda}(t,y)).$$

Then by Parseval's identity (2.12) and (2.1), it follows that

$$\begin{aligned}\int_{\mathbb{R}_+^{d+1}}|q(t,y;r,x)|^2d\nu(r,x)&=\int_{\mathbb{R}_+^{d+1}}|g_{t,y}(r,x)|^2d\nu(r,x)\\&=\int_\Gamma|\mathcal{F}(g_{t,y})(\mu,\lambda)|^2d\gamma(\mu,\lambda)\le mes_\gamma(W).\end{aligned}$$

Hence, integrating over $(t,y)\in E$, we see that $\|Q_WP_E\|_{HS}^2\le mes_\nu(E)mes_\gamma(W)$. □

Proposition 6.1. Let E and W be measurable sets and suppose that

$$\|f\|_{L^2(d\nu)}=\|\mathcal{F}(f)\|_{L^2(d\gamma)}=1.$$

Assume that $\varepsilon_E+\varepsilon_W<1$, f is ε_E-concentrated on E and $\mathcal{F}(f)$ is ε_W-concentrated on W. Then

$$mes_\nu(E)mes_\gamma(W)\ge(1-\varepsilon_E-\varepsilon_W)^2.$$

Proof. Since $\|f\|_{L^2(d\nu)}=\|\mathcal{F}(f)\|_{L^2(d\nu)}=1$ and $\varepsilon_E+\varepsilon_W<1$, the measures of E and W must both be non-zero. Indeed, if not, then the ε_E-concentration of f implies that

$$\|f-P_Ef\|_{L^2(d\nu)}=\|f\|_{L^2(d\nu)}=1\le\varepsilon_E,$$

which contradicts with $\varepsilon_E<1$, likewise for $\mathcal{F}(f)$. If at least one of $mes_\nu(E)$ and $mes_\gamma(W)$ is infinity, then the inequality is clear. Therefore, it is enough to consider the case where both E and W have finite positive measures. Since $\|Q_W\|_2=1$, it follows that

$$\begin{aligned}\|f-Q_WP_Ef\|_{L^2(d\nu)}\le&\|f-Q_Wf\|_{L^2(d\nu)}+\|Q_Wf-Q_WP_Ef\|_{L^2(d\nu)}\\\le&\varepsilon_W+\|Q_W\|_2\|f-P_Ef\|_{L^2(d\nu)}\\\le&\varepsilon_E+\varepsilon_W,\end{aligned}$$

and thus,

$$\|Q_WP_Ef\|_{L^2(d\nu)}\ge\|f\|_{L^2(d\nu)}-\|f-Q_WP_Ef\|_{L^2(d\nu)}\ge1-\varepsilon_E-\varepsilon_W.$$

Hence $\|Q_WP_E\|_2\ge1-\varepsilon_E-\varepsilon_W$. (6.1) and Lemma 6.1 yields the desired inequality. □

In the following we shall consider the case of $f\in L^1(d\nu)$. As in the $L^2(d\nu)$ case, we say that $f\in L^1(d\nu)$ is ε-concentrated to E if $\|f-P_Ef\|_{L^1(d\nu)}\le\varepsilon\|f\|_{L^1(d\nu)}$. Let $B_{L^1(d\nu)}(W)$ denote the subspace of $L^1(d\nu)$, which consists of all $g\in L^1(d\nu)$ such that $Q_Wg=g$. We say that f is ε-bandlimited to W if there is a $g\in B_{L^1(d\nu)}(W)$ with $\|f-g\|_{L^1(d\nu)}<\varepsilon\|f\|_{L^1(d\nu)}$. Here we denote by $\|P_E\|_1$ the operator norm of P_E on $L^1(d\nu)$ and by $\|P_E\|_{1,W}$ the operator norm of $P_E:B_{L^1(d\nu)}(W)\to L^1(d\nu)$. Corresponding to (6.1) and Lemma 6.1 in the $L^2(d\nu)$ case, we can obtain the following.

Lemma 6.2. $\|P_E\|_{1,W} \leq mes_\nu(E) mes_\gamma(W)$.

Proof. For $f \in B_{L^1(d\nu)}(W)$, we see that

$$\begin{aligned} f(t,y) &= \int_W \overline{\varphi_{\mu,\lambda}(t,y)} \mathcal{F}(f)(\mu,\lambda) d\gamma(\mu,\lambda) \\ &= \int_W \overline{\varphi_{\mu,\lambda}(t,y)} \Big(\int_{\mathbb{R}^d} f(r,x) \varphi_{\mu,\lambda}(r,x) d\nu(r,x) \Big) d\gamma(\mu,\lambda) \\ &= \int_{\mathbb{R}_+^{d+1}} f(r,x) \Big(\int_W \overline{\varphi_{\mu,\lambda}(t,y)} \varphi_{\mu,\lambda}(r,x) d\gamma(\mu,\lambda) \Big) d\nu(r,x). \end{aligned}$$

Therefore, $\|f\|_{L^\infty(d\nu)} \leq mes_\gamma(W) \|f\|_{L^1(d\nu)}$ by (2.1) and then,

$$\|P_E f\|_{L^1(d\nu)} = \int_E |f(r,x)| d\nu(r,x) \leq mes_\nu(E) \|f\|_{L^\infty(d\nu)} \leq mes_\nu(E) mes_\gamma(W) \|f\|_{L^1(d\nu)}.$$

Then, it follows that for $f \in B_{L^1(d\nu)}(W)$,

$$\frac{\|P_E f\|_{L^1(d\nu)}}{\|f\|_{L^1(d\nu)}} \leq \frac{mes_\nu(E) mes_\gamma(W) \|f\|_{L^1(d\nu)}}{\|f\|_{L^1(d\nu)}} = mes_\nu(E) mes_\gamma(W),$$

which implies the desired inequality. □

Proposition 6.2. Let $f \in L^1(d\nu)$. If f is ε_E-concentrated to E and ε_W-bandlimited to W, then

$$mes_\nu(E) mes_\gamma(W) \geq \frac{1 - \varepsilon_E - \varepsilon_W}{1 + \varepsilon_W}.$$

Proof. Without loss of generality, we may suppose that $\|f\|_{L^1(d\nu)} = 1$, Since f is ε_E-concentrated to E, it follows that $\|P_E f\|_{L^1(d\nu)} \geq \|f\|_{L^1(d\nu)} - \|f - P_E\|_{L^1(d\nu)} \geq 1 - \varepsilon_E$. Moreover, since f is ε_W-bandlimited, there is a $g \in B_{L^1(d\nu)}(W)$ with $\|g - f\|_{L^1(d\nu)} \leq \varepsilon_W$. Therefore, it follows that

$$\|P_E g\|_{L^1(d\nu)} \geq \|P_E f\|_{L^1(d\nu)} - \|P_E(g-f)\|_{L^1(d\nu)} \geq \|P_E f\|_{L^1(d\nu)} - \varepsilon_W \geq 1 - \varepsilon_E - \varepsilon_W$$

and $\|g\|_{L^1(d\nu)} \leq \|f\|_{L^1(d\nu)} + \varepsilon_W = 1 + \varepsilon_W$. Then, we see that

$$\frac{\|P_E g\|_{L^1(d\nu)}}{\|g\|_{L^1(d\nu)}} \geq \frac{1 - \varepsilon_E - \varepsilon_W}{1 + \varepsilon_W}.$$

Hence $\|P_E\|_{1,W} \geq (1 - \varepsilon_E - \varepsilon_W)/1 + \varepsilon_W$ and Lemma 6.2 yields the desired inequality. □

Proposition 6.3. Let $f \in L^2(d\nu) \cap L^1(d\nu)$ with $\|f\|_{L^2(d\nu)} = 1$. If f is ε_E-concentrated to E in L^1_ν-norm and $\mathcal{F}(f)$ is ε_W-concentrated to W in L^2_ν-norm, then

$$mes_\nu(E) \geq (1 - \varepsilon_E)^2 \|f\|^2_{L^1(d\nu)} \quad \text{and} \quad mes_\gamma(W) \|f\|^2_{L^1(d\nu)} \geq (1 - \varepsilon_W)^2.$$

In particular,

$$mes_\nu(E) mes_\gamma(W) \geq (1 - \varepsilon_E)^2 (1 - \varepsilon_W^2).$$

Proof. By the orthogonality of the projection operator P_W, $\|f\|_{L^2(d\nu)}=\|\mathcal{F}(f)\|_{L^2(d\gamma)}=1$ and f is ε_W-concentrated to W in L^2_ν-norm, it follows that

$$\|P_W(\mathcal{F}(f))\|^2_{L^2(d\nu)}=\|\mathcal{F}(f)\|^2_{L^2(d\gamma)}-\|\mathcal{F}(f)-P_W(\mathcal{F}(f))\|^2_{L^2(d\gamma)}\geq 1-\varepsilon_W^2,$$

and thus,

$$\begin{aligned}1-\varepsilon_W^2\leq&\int_W|\mathcal{F}(f)(\xi)|^2d\gamma(\mu,\lambda)\\ \leq& mes_\gamma(W)\|\mathcal{F}(f)\|^2_{L^\infty(d\gamma)}\leq mes_\gamma(W)\|f\|^2_{L^1(d\nu)}.\end{aligned}$$

Similarly, $\|f\|_{L^1(d\nu)}=1$ and f is ε_E-concentrated to E in L^1_ν-norm,

$$(1-\varepsilon_E)\|f\|_{L^1(d\nu)}\leq\int_E|f(r,x)|d\nu(r,x)\leq\sqrt{mes_\nu(E)}.$$

Here we used the Cauchy-Schwarz inequality and the fact that $\|f\|_{L^2(d\nu)}=1$. □

Proposition 6.4. Let $s>0$. Then there exists a constant $C_1(d,s)$ such that for all $f\in L^1(d\nu)\bigcap L^2(d\nu)$

$$\|f\|^{2+\frac{2s}{d}}_{L^2(d\nu)}\leq C_1(d,s)\|f\|^{\frac{2s}{d}}_{L^1(d\nu)}\|\|\theta(\lambda,\mu)\|^s\mathcal{F}(f)\|^2_{L^2(d\gamma)}, \tag{6.2}$$

where

$$C_1(d,s)=(2^d\Gamma(d)s)^{-\frac{s}{d}}\Big(\frac{s+d}{d}\Big)^{\frac{s+d}{d}}.$$

Proof. Let $A>0$. From Plancherel's theorem we have

$$\|f\|^2_{L^2(d\nu)}=\|\mathcal{F}(f)\|^2_{L^2(d\gamma)}=\|1_{\theta^{-1}(B(0,A))}\mathcal{F}(f)\|^2_{L^2(d\gamma)}+\|(1-1_{\theta^{-1}(B(0,A))})\mathcal{F}(f)\|^2_{L^2(d\gamma)}.$$

By (2.2) and (2.10),

$$\|1_{\theta^{-1}(B(0,A))}\mathcal{F}(f)\|^2_{L^2(d\gamma)}\leq\|f\|^2_{L^1(d\nu)}\int_{\mathbb{R}^{d+1}_+}1_{B(0,A)}(r,x)d\nu(r,x).$$

By a simple calculations we find

$$\|1_{\theta^{-1}(B(0,A))}\mathcal{F}(f)\|^2_{L^2(d\gamma)}\leq\frac{A^{2d}}{2^d\Gamma(d+1)}\|f\|^2_{L^1(d\nu)}.$$

On the other hand

$$\begin{aligned}&\|(1-1_{\theta^{-1}(B(0,A))})\mathcal{F}(f)\|^2_{L^2(d\gamma)}\\ \leq& A^{-2s}\|(1-1_{\theta^{-1}(B(0,A))})\|\theta(\lambda,\mu)\|^s\mathcal{F}(f)\|^2_{L^2(d\gamma)}\\ \leq& A^{-2s}\|\|\theta(\lambda,\mu)\|^s\mathcal{F}(f)\|^2_{L^2(d\gamma)}.\end{aligned}$$

It follows then

$$\|f\|^2_{L^2(d\nu)} \le \frac{A^{2d}}{2^d\Gamma(d+1)}\|f\|^2_{L^1(d\nu)} + A^{-2s}\|\|\theta(\lambda,\mu)\|^s\mathcal{F}(f)\|^2_{L^2(d\gamma)}.$$

Minimizing the right hand side of that inequality over $A>0$ gives

$$\|f\|^2_{L^2(d\nu)} \le \frac{d+s}{d}(2^d\Gamma(d)s)^{-\frac{s+d}{d}}\|f\|^{\frac{2s}{d+s}}_{L^1(d\nu)}\|\|\theta(\lambda,\mu)\|^s\mathcal{F}(f)\|^{\frac{2d}{s+d}}_{L^2(d\gamma)}. \tag{6.3}$$

The desired result follows immediately from (6.3). □

Proposition 6.5. Let $s>0$. Then there exists a constant $C_2(d,s)$ such that for all $f\in L^1(d\nu)\bigcap L^2(d\nu)$,

$$\|f\|^{2+\frac{2s}{d}}_{L^1(d\nu)} \le C_2(d,s)\|f\|^{\frac{2s}{d}}_{L^2(d\nu)}\|\|(r,x)\|^s f\|^2_{L^1(d\nu)}, \tag{6.4}$$

where

$$C_2(d,s) = (2^{d+2}\Gamma(d)s^2)^{-\frac{s}{d}}\Big(\frac{2s+d}{\sqrt{d}}\Big)^{\frac{2s+d}{d}}.$$

Proof. Let $A>0$. We have

$$\|f\|_{L^1(d\nu)} \le \|1_{B(0,A)}f\|_{L^1(d\nu)} + \|(1-1_{B(0,A)})f\|_{L^1(d\nu)}.$$

By Cauchy-Schwarz inequality, we obtain

$$\|1_{B(0,A)}f\|_{L^1(d\nu)} \le \frac{A^d}{\sqrt{2^d\Gamma(d+1)}}\|f\|_{L^2(d\nu)}.$$

On the other hand

$$\|(1-1_{B(0,A)})f\|_{L^1(d\nu)} \le A^{-2s}\|\|(r,x)\|^{2s}(1-1_{B(0,A)})f\|_{L^1(d\nu)}.$$

It follows then

$$\|f\|_{L^1(d\nu)} \le \frac{A^d}{\sqrt{2^d\Gamma(d+1)}}\|f\|_{L^2(d\nu)} + A^{-2s}\|\|(r,x)\|^{2s}f\|_{L^1(d\nu)}.$$

Minimizing the right hand side of that inequality over $A>0$ gives

$$\|f\|_{L^1(d\nu)} \le \frac{d+2s}{\sqrt{d}}(2^{d+2}\Gamma(d)s^2)^{\frac{2s+d}{s}}\|f\|^{\frac{2s}{d+2s}}_{L^2(d\nu)}\|\|(r,x)\|^{2s}f\|^{\frac{d}{2s+d}}_{L^1(d\nu)}. \tag{6.5}$$

The desired result follows immediately from (6.5). □

From the previous results we deduce the following variation on Heisenberg's uncertainty inequality for the generalized Fourier transform.

Theorem 6.1. *Let $s>0$. Then for all $f\in L^1(d\nu)\bigcap L^2(d\nu)$,*

$$\|f\|^2_{L^2(d\nu)}\|f\|_{L^1(d\nu)}\leq C_1(d,s)C_2(d,s)\|\|(r,x)\|^s f\|^2_{L^1(d\nu)}\|\|\theta(\lambda,\mu)\|^s\mathcal{F}(f)\|^2_{L^2(d\gamma)}. \tag{6.6}$$

Proof. The result follow immediately by multiplying inequality (6.2) by (6.4). □

Proposition 6.6. Let $s>0$ and let W a measurable subset of Γ with $0<mes_\gamma(W)<\infty$. Then there exists a constant $C(d,s)$ such that for all $f\in L^1(d\nu)\bigcap L^2(d\nu)$,

$$\|1_W\mathcal{F}(f)\|_{L^2(d\gamma)}\leq C(d,s)\sqrt{mes_\gamma(W)}\|f\|^{\frac{2s}{2s+d}}_{L^2(d\nu)}\|\|(r,x)\|^{2s}f\|^{\frac{d}{2s+d}}_{L^1(d\nu)}, \tag{6.7}$$

where

$$C(d,s)=(2^{d+2}\Gamma(d)s^2)^{-\frac{s}{2s+d}}\Big(\frac{2s+d}{\sqrt{d}}\Big).$$

Proof. We have

$$\|1_W\mathcal{F}(f)\|_{L^2(d\gamma)}\leq\sqrt{mes_\gamma(W)}\|\mathcal{F}(f)\|_{L^\infty(d\gamma)}\leq\sqrt{mes_\gamma(W)}\|f\|_{L^1(d\nu)}.$$

The desired result follows from Carlson Inequality (6.5). □

We adapt the method of Ghorbal-Jaming [12], we obtain.

Theorem 6.2. *Let E, W be a pair of measurable subsets with finite measure $0<mes_\nu(E),mes_\gamma(W)<\infty$. Then the following uncertainty principles hold,*

(1) Local uncertainty principle of $\mathcal{F}$.

(i) For $0<s<d$, there exists a constant $C(d,s)$ such that for all $f\in L^2(d\nu)$,

$$\|1_W\mathcal{F}(f)\|_{L^2(d\gamma)}\leq C(d,s)(mes_\gamma(W))^{\frac{s}{2d}}\|\|(r,x)\|^s f\|_{L^2(d\nu)}. \tag{6.8}$$

(ii) For $s>d$, there exists a constant $C(d,s)$ such that for all $f\in L^2(d\nu)$,

$$\|1_W\mathcal{F}(f)\|_{L^2(d\gamma)}\leq C(d,s)\sqrt{mes_\gamma(W)}\|\|(r,x)\|^s f\|^{\frac{d}{s}}_{L^2(d\nu)}\|f\|^{1-\frac{d}{s}}_{L^2(d\nu)}. \tag{6.9}$$

(2) Global uncertainty principle of $\mathcal{F}$.

For $s,t>0$, there exists a constant $C(d,s)$ such that for all $f\in L^2(d\nu)$

$$\|\|(r,x)\|^s f\|^{\frac{2t}{s+t}}_{L^2(d\nu)}\|\|\theta(\lambda,\mu)\|^t\mathcal{F}(f)\|^{\frac{2s}{s+t}}_{L^2(d\gamma)}. \tag{6.10}$$

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