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Square Root Functional Equation on Positive Cones

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Abstract. A square root functional equation on positive cones of C^*-algebras is introduced and its solution and Hyers-Ulam-Rassias stability are investigated.

Key Words: Hyers-Ulamstability, fixed point, additive functional equation.

AMS Subject Classifications: 39B52, 47H10

1 Introduction

The stability theory of functional equation is originated from the well-known Ulam's problem [1] concerning the stability of homomorphisms in metric groups: Let $(G,*)$ be a group and $(X,\cdot)$ be a metric group. Does for every $\varepsilon>0$ there exist $\delta>0$ such that if $f:G\to X$ satisfies

$$d(f(x*y),f(x)\cdot f(y))<\delta \quad \text{for } x,y\in G,$$

then a homomorphism $h:G\to X$ exists with $d(f(x),h(x))<\varepsilon$ for $x\in G$? Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers's Theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1.1 (Th. M. Rassias)**.** *Consider two Banach spaces E_1, E_2, and let $f:E_1\to E_2$ be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exist $\theta\ge 0$ and $p\in[0,1)$, such that*

$$\frac{\|f(x+y)-f(x)-f(y)\|}{\|x\|^p+\|y\|^p}\le\theta \quad \textit{for any } x,y\in E_1.$$

Then there exists a unique linear mapping $T:E_1\to E_2$ such that

$$\frac{\|f(x)-T(x)\|}{\|x\|^p}\le\frac{2\theta}{2-2^p} \quad \textit{for any } x\in E_1.$$

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The paper of Th. M. Rassias [4] has provided a lot of influence in the development of what we call generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Gavruta [5] by using a general controb function in place of the unbounded Cauchy difference in the spirit of Th. M. Rassias's approach. Following the innovative approach of the Th. M. Rassias theorem [4], J. M. Rassias [6] replaced the factor $\|x\|^p+\|y\|^p$ by $\|x\|^p\|y\|^q$ for $p,q\in\mathbb{R}$ with $p+q=1$. The stability problem of several functional equations has been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7–10]). Let $\mathcal{A}$ be a C^*-algebra and $a\in\mathcal{A}$ be a self-adjoint element, i.e., $a=a^*$. Then a is said to be positive if it is of the form $a=bb^*$ for some $a\in\mathcal{A}$. The set of positive elements of $\mathcal{A}$ is denoted by $\mathcal{A}^+$. Note that $\mathcal{A}^+$ is a closed convex cone (see [11]). Moreover, It is well-known that for a positive element a and a positive integer n there exists a unique positive element $x\in\mathcal{A}^+$ such that $a=x^n$. In this case, we denote x by $\sqrt[n]{a}$. In the following some preliminary properties of $\mathcal{A}^+$ are listed [11]:

Theorem 1.2. *Suppose that $\mathcal{A}$ is a C^*-algebra.*

(i) $\mathcal{A}^+$ *is closed in* $\mathcal{A}$,

(ii) $ax\in\mathcal{A}^+$ *if* $x\in\mathcal{A}^+$ *and* $a\geq 0$,

(iii) $x+y\in\mathcal{A}^+$ *if* $x,y\in\mathcal{A}^+$,

(iv) $xy\in\mathcal{A}^+$ *if* $x,y\in\mathcal{A}^+$ *and* $xy=yx$,

(v) $x\in\mathcal{A}^+$ *and* $-x\in\mathcal{A}^+$, *then* $x=0$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two C^*-algebra and $\mathcal{A}^+$ and $\mathcal{B}^+$ be the corresponding positive cones. We introduce the following pair of functional equations

$$\begin{cases} f(x)f(y)=f(y)f(x), \\ f(ax+by)=a^2f(x)+2ab\sqrt{f(x)f(y)}+b^2f(y), \end{cases} \tag{1.1}$$

for every $x,y\in\mathcal{A}^+$ and $f:\mathcal{A}^+\to\mathcal{B}^+$. Here a, b are two nonnegative real scalars that $a+b\neq 0$. By part (iv) of Theorem 1.2, the first equation of condition (1.1) is needed for the second equation of (1.1) to be well-defined. Note that the function $f:\mathcal{A}^+\to\mathcal{B}^+$ by $f(x)=cx^2$, $c\geq 0$, is a solution of the functional equation (1.1). Applying (1.1) for $x=y=0$, $x=0$, and $y=0$, separately, gives $f(0)=0$; $f(ax)=a^2f(x)$, and $f(by)=b^2f(y)$, respectively. Hence (1.1) can be modified by the following:

$$f(ax+by)=f(ax)+2\sqrt{f(ax)f(by)}+f(by)=\Big(\sqrt{f(ax)}+\sqrt{f(by)}\Big)^2,$$

and consequently,

$$\sqrt{f(ax+by)}=\sqrt{f(ax)}+\sqrt{f(by)}.$$

For this reason, the Eq. (1.1) is named by *square root functional equation*. Considering the solution of Cauchy additive mapping it is easy to see that if a function $f:\mathbb{R}^+\to\mathbb{R}^+$ satisfies the above equation it must be of the form $f(x)=cx^2$ for some scalar $c\geq 0$.

In the present paper, using the direct method and fixed point method, we prove the Hyers-Ulam stability of the functional equation (1.1) in positive cones of C^*-algebras.

2 Hyers-Ulam-Rassias stability of (1.1): direct method

In what follows a,b are two nonnegative real scalars which $a+b\neq 0$, $\mathcal{A},\mathcal{B}$ are two C^*-algebras with corresponding positive cones $\mathcal{A}^+,\mathcal{B}^+$ respectively. Using the direct method, we prove the stability of the positive functional equation (1.1).

Theorem 2.1. *Let $h:\mathcal{A}^+\times\mathcal{A}^+\to[0,+\infty)$ be a function with this property that*

$$\gamma:=\sup\Big\{\frac{h\big((a+b)x,(a+b)y\big)}{h(x,y)}:\ x,y\in\mathcal{A}^+,\, h(x,y)\neq 0\Big\}<(a+b)^2 \tag{2.1}$$

and $f:\mathcal{A}^+\to\mathcal{B}^+$ be a function satisfying

$$\begin{cases} f(x)f(y)=f(y)f(x),\\ \|f(ax+by)-a^2f(x)-2ab\sqrt{f(x)f(y)}-b^2f(y)\|\leq h(x,y), \end{cases} \tag{2.2}$$

for all $x,y\in\mathcal{A}^+$. Then there exists a unique mapping $T:\mathcal{A}^+\to\mathcal{B}^+$ such that

$$\|f(x)-T(x)\|\leq\frac{h(x,x)}{(a+b)^2-\gamma},\quad x\in\mathcal{A}^+, \tag{2.3}$$

and

$$\begin{cases} T(x)T(y)=T(y)T(x),\\ T(ax+by)=a^2T(x)+2ab\sqrt{T(x)T(y)}+b^2T(y). \end{cases} \tag{2.4}$$

for every $x,y\in\mathcal{A}^+$.

Proof. Let $\beta=a+b$, then by (2.1), $\gamma<\beta^2$ and

$$h(\beta x,\beta y)\leq\gamma h(x,y),\quad x,y\in\mathcal{A}^+.$$

Applying the second equation of (2.2) for $y=x$, we observe that f satisfies the following functional inequality

$$\|f(\beta x)-\beta^2f(x)\|\leq h(x,x),\quad x\in\mathcal{A}^+.$$

Let $q_n(x):=\beta^{-2n}f(\beta^nx)$ for every $n\geq 1$ and $x\in\mathcal{A}^+$. Then replacing $x\in\mathcal{A}$ by β^nx in the above inequality to see

$$\|f(\beta^{n+1}x)-\beta^2f(\beta^nx)\|\leq h(\beta^nx,\beta^nx),\quad x\in\mathcal{A}^+.$$

Thus

$$\begin{aligned}\|q_{n+1}(x)-q_n(x)\| &= \|\beta^{-2n-2}f(\beta^{n+1}x)-\beta^{-2n}f(\beta^n x)\|\\ &= \beta^{-2n-2}\|f(\beta^{n+1}x)-\beta^2 f(\beta^n x)\|\\ &\le \beta^{-2n-2}h(\beta^n x,\beta^n x),\end{aligned}$$

which in the last inequality, the contractive property of h is used. Hence,

$$\|q_{n+1}(x)-q_n(x)\| \le \beta^{-2}(\gamma\beta^{-2})^n h(x,x), \quad x\in\mathcal{A}^+, \tag{2.5}$$

and $\gamma\beta^{-2}<1$, so the sequence $\{q_n(x)\}$ is a Cauchy sequence for each $x\in\mathcal{A}^+$. Since $\mathcal{B}^+$ is closed in the complete metric space $\mathcal{B}$, there exists a limit function $T(x):=\lim_{n\to\infty}q_n(x)$ in $\mathcal{B}^+$. Now by induction on n, we prove that

$$\|q_n(x)-f(x)\| \le \sum_{i=0}^{n-1}\beta^{-2}(\gamma\beta^{-2})^i h(x,x), \tag{2.6}$$

for any $n\in\mathbb{N}$ and $x\in X$. Fix $x\in\mathcal{A}^+$, note that

$$\begin{aligned}\|q_1(x)-f(x)\| &= \|\beta^{-2}f(\beta x)-f(x)\|\\ &\le \beta^{-2}\|f(\beta x)-\beta^2 f(x)\| \le \beta^{-2}h(x,x).\end{aligned}$$

Now suppose (2.6) holds, then by (2.5) and (2.6), we obtain

$$\begin{aligned}\|q_{n+1}(x)-f(x)\| &\le \|q_{n+1}(x)-q_n(x)\|+\|q_n(x)-f(x)\|\\ &\le \beta^{-2}(\gamma\beta^{-2})^n h(x,x)+\sum_{i=0}^{n-1}\beta^{-2}(\gamma\beta^{-2})^i h(x,x)\\ &= \sum_{i=0}^{n}\beta^{-2}(\gamma\beta^{-2})^i h(x,x).\end{aligned}$$

Lending $n\to+\infty$ in (2.6), we get

$$\|T(x)-f(x)\| \le \frac{\beta^{-2}h(x,x)}{1-(\gamma\beta^{-2})} = \frac{h(x,x)}{\beta^2-\gamma}, \quad x\in\mathcal{A}^+.$$

We prove T satisfies (2.4). To see this, replacing x,y in the first equation of (2.2) with $\beta^n x$, $\beta^n y$, gives

$$f(\beta^n x)f(\beta^n y)=f(\beta^n y)f(\beta^n x), \quad x,y\in\mathcal{A}^+,$$

and so

$$q_n(x)q_n(y)=q_n(y)q_n(x), \quad x,y\in\mathcal{A}^+.$$

Taking $n\to+\infty$, yields the second equation of (2.4). On the other hand, replacing x,y in the second equation of (2.2) with $\beta^n x$, $\beta^n y$, gives

$$\left\|f(\beta^n(ax+by))-a^2f(\beta^n x)-2ab\sqrt{f(\beta^n x)f(\beta^n y)}-b^2f(\beta^n y)\right\|\le h(\beta^n x,\beta^n y).$$

Hence by definition of q_n we obtain

$$\begin{aligned}
&\left\|q_n(ax+by)-a^2q_n(x)-2ab\sqrt{q_n(x)q_n(y)}-b^2q_n(y)\right\|\\
=&\left\|\beta^{-2n}f(\beta^n(ax+by))-a^2\beta^{-2n}f(\beta^n x)-2ab\sqrt{\beta^{-4n}f(\beta^n x)f(\beta^n y)}-b^2\beta^{-2n}f(\beta^n y)\right\|\\
\le&\beta^{-2n}\left\|f(\beta^n(ax+by))-a^2f(\beta^n x)-2ab\sqrt{f(\beta^n x)f(\beta^n y)}-b^2f(\beta^n y)\right\|\\
\le&\beta^{-2n}h(\beta^n x,\beta^n y)\le\beta^{-2n}\gamma^n h(x,x)=(\gamma\beta^{-2})^n h(x,x).
\end{aligned}$$

Therefore,

$$\left\|q_n(ax+by)-a^2q_n(x)-2ab\sqrt{q_n(x)q_n(y)}-b^2q_n(y)\right\|\le(\gamma\beta^{-2})^n h(x,x),$$

for every $x,y\in\mathcal{A}^+$ and $n\in\mathbb{N}$. Considering $0<\gamma\beta^{-2}<1$, and taking $n\to+\infty$ in the last inequality; implies the second equation of (2.4).

We now want to prove that T is a unique function satisfying (2.3) and (2.4). Assume that there exists another one, denoted by $T':\mathcal{A}^+\to\mathcal{B}^+$ satisfying (2.3) and (2.4). Use (2.4) for $y=x$ to get $T\beta x=\beta^2Tx$ and $T'\beta x=\beta^2T'x$ and more generality

$$T(\beta^n x)=\beta^{2n}Tx\quad\text{and}\quad T'(\beta^n x)=\beta^{2n}T'x.$$

Hence,

$$Tx=\beta^{-2n}T(\beta^n x)\quad\text{and}\quad T'x=\beta^{-2n}T'(\beta^n x),\tag{2.7}$$

for every $x\in\mathcal{A}^+$ and $n\in\mathbb{N}$. By the triangle inequality, (2.3) and (2.7), we obtain

$$\begin{aligned}
\|T(x)-T'(x)\|&=\|\beta^{-2n}T(\beta^n x)-\beta^{-2n}T'(\beta^n x)\|\le\beta^{-2n}\|T(\beta^n x)-T'(\beta^n x)\|\\
&\le\beta^{-2n}(\|T(\beta^n x)-f(\beta^n x)\|+\|f(\beta^n x)-T'(\beta^n x)\|)\\
&\le\beta^{-2n}\Big(2\frac{\beta^{-2}h(\beta^n x,\beta^n x)}{1-\gamma\beta^{-2}}\Big)\le2\beta^{-2}\Big(\frac{\beta^{-2n}h(\beta^n x,\beta^n x)}{1-\gamma\beta^{-2}}\Big)\\
&\le2\beta^{-2}\frac{(\gamma\beta^{-2})^n h(x,x)}{1-\gamma\beta^{-2}},
\end{aligned}$$

for every $x\in\mathcal{A}^+$ and $n\in\mathbb{N}$. Since, $0<\gamma\beta^{-2}<1$, letting $n\to+\infty$, we get $T(x)=T'(x)$ for all $x\in\mathcal{A}^+$. $\square$

Recall that if X is a vector space, a function $h:X\times X\to[0,+\infty)$ is called homogeneous of order p whenever $h(\lambda x,\lambda y)=|\lambda|^p h(x,y)$ for every scalar λ and every $x,y\in X$. In this case

$$\gamma:=\sup\Big\{\frac{h\big((a+b)x,(a+b)y\big)}{h(x,y)}\Big\}=|a+b|^p.$$

This led to the following corollary:

Corollary 2.1. Let $h:\mathcal{A}^+\times\mathcal{A}^+\to[0,+\infty)$ be a homogeneous function of order $p<2$. If $f:\mathcal{A}^+\to\mathcal{B}^+$ is a function satisfying (2.2) then there exists a unique mapping $T:\mathcal{A}^+\to\mathcal{B}^+$ satisfying (2.4) and

$$\|f(x)-T(x)\|\le\frac{h(x,x)}{|a+b|^2-|a+b|^p},\quad x\in\mathcal{A}^+.$$

Since the function $h(x,y)=\theta(\|x\|^p+\|y\|^p)$ is a homogenous of order p, we have the following corollary:

Corollary 2.2. Assume that $f:\mathcal{A}^+\to\mathcal{B}^+$ is a function satisfying

$$\begin{cases} f(x)f(y)=f(y)f(x),\\ \|f(ax+by)-a^2f(x)-2ab\sqrt{f(x)f(y)}-b^2f(y)\|\le\theta(\|x\|^p+\|y\|^p),\end{cases}$$

for some $\theta\ge0$ and $p<2$, and for every $x,y\in\mathcal{A}^+$. Then there exists a unique mapping $T:\mathcal{A}^+\to\mathcal{B}^+$ satisfying (2.4) and

$$\|f(x)-A(x)\|\le\frac{h(x,x)}{|a+b|^2-|a+b|^p},\quad x\in\mathcal{A}^+.$$

3 Hyers-Ulam-Rassias stability of (1.1): fixed point method

In 2003, V. Radu proved the Hyers-Ulam-Rassias stability of the additive Cauchy equation by using the fixed point method (see [12] or [13]). In this section, using some ideas from [12] or [13], we prove Theorem 2.1 by the fixed point method.

Before state it we need to introduce one of the fundamental results of the fixed point theory. For the proof, we refer to [14].

Theorem 3.1. *Let (Ω,d) be a generalized complete metric space. Assume that $\Lambda:\Omega\to\Omega$ is a strictly contractive operator with the Lipschitz constant $L<1$, i.e.,*

$$d(\Lambda g,\Lambda h)\le Ld(g,h)$$

for all $g,h\in\Omega$. If there exists a nonnegative integer n_0 such that $d(\Lambda^{n_0+1}f,\Lambda^{n_0}f)<+\infty$ for some $f\in\Omega$, then the following statements are true:

(i) *The sequence* $\{\Lambda^n f\}$ *converges to a fixed point* A *of* Λ.

(ii) A *is the unique fixed point of* Λ *in* $\Omega^* = \{g \in \Omega : d(\Lambda^{n_0} f, g) < +\infty\}$.

(iii) *If* $g \in X^*$, *then*

$$d(g,A) \leq \frac{1}{1-L} d(\Lambda g, g).$$

Now we are ready to prove Theorem 2.1 is by the fixed point method:

Theorem 3.2. *Under the assumptions of Theorem 2.1, let* $h: \mathcal{A}^+ \times \mathcal{A}^+ \to [0,+\infty)$ *be a function satisfying* (2.1), *and* $f: \mathcal{A}^+ \to \mathcal{B}^+$ *be a function satisfying* (2.2). *Then there exists a unique mapping* $T: \mathcal{A}^+ \to \mathcal{B}^+$ *satisfying* (2.3) *and* (2.4).

Proof. Letting $y = x$ in (2.2), we get

$$\|f(\beta x) - \beta^2 f(x)\| \leq h(x,x)$$

for every $x \in \mathcal{A}^+$. Consider the set

$$\Omega := \{g : \mathcal{A}^+ \to \mathcal{B}^+ : g(x)g(y) = g(y)g(x) \text{ for all } x, y \in \mathcal{A}^+\}$$

and define the generalized metric d on Ω by

$$\begin{aligned} d(g_1, g_2) &= \inf\{\mu \in (0,+\infty) : \|g_1(x) - g_2(x)\| \leq \mu h(x,x) \text{ for all } x \in \mathcal{A}^+\} \\ &= \sup\Big\{\frac{\|g_1(x) - g_2(x)\|}{h(x,x)} : x \in \mathcal{A}^+,\, h(x,x) \neq 0\Big\}. \end{aligned}$$

It is easy to show that (Ω, d) is a complete metric space (see [15]). Now we consider the linear mapping $\Lambda : \Omega \to \Omega$ by

$$\Lambda g(x) = \frac{1}{\beta^2} g(\beta x)$$

for all $x \in \mathcal{A}^+$. For given $g_1, g_2 \in \Omega$,

$$\begin{aligned} \|\Lambda g_1(x) - \Lambda g_2(x)\| &= \Big\|\frac{1}{\beta^2} g_1(\beta x) - \frac{1}{\beta^2} g_2(\beta x)\Big\| \\ &\leq \frac{1}{\beta^2} \|g_1(\beta x) - g_2(\beta x)\| \\ &\leq \frac{1}{\beta^2} d(g_1, g_2) h(x,x) \end{aligned}$$

for every $x \in \mathcal{A}^+$. By definition of d,

$$d(\Lambda g_1, \Lambda g_2) \leq \frac{\gamma}{\beta^2} d(g_1, g_2).$$

Hence Λ is indeed a contraction. Note that

$$\begin{aligned}\|f(x)-\Lambda f(x)\| &= \Big\|f(x)-\frac{1}{\beta^2}f(\beta x)\Big\| \\ &\leq \frac{1}{\beta^2}\|\beta^2 f(x)-f(\beta x)\| \leq \frac{1}{\beta^2}h(x,x)\end{aligned}$$

for all $x\in\mathcal{A}^+$ whence $d(\Lambda f,f)\leq\beta^{-2}<+\infty$. By the preceding theorem, there exists a mapping $T:\mathcal{A}^+\to\mathcal{B}^+$ satisfying the following conditions:

(i) $T(x)T(y)=T(y)T(x)$ for all $x,y\in\mathcal{A}^+$.

(ii) T is a fixed point of Λ, i.e.,

$$\frac{1}{\beta^2}T(\beta x)=(\Lambda T)(x)=T(x),$$

whence $T(\beta x)=\beta^2 T(x)$ for all $x\in\mathcal{A}^+$. Moreover, T is a unique fixed point of Λ in the set

$$\Omega^*:=\{g\in\Omega: d(f,g)<+\infty\},$$

which implies that

$$\|f(x)-T(x)\|\leq d(f,T)h(x,x),\quad x\in\mathcal{A}^+.$$

(iii) $d(\Lambda^n f,T)\to 0$ as $n\to+\infty$, deducing that $T(x)=\lim_n\beta^{-n}f(\beta^n x)$.

(iv) By (iii) of the preceding theorem we conclude that

$$d(f,A)\leq\frac{1}{1-\gamma\beta^{-2}}d(f,\Lambda f)<\frac{1}{1-\gamma\beta^{-2}}\beta^{-2}=\frac{1}{\beta^2-\gamma},$$

deducing the inequality

$$\|f(x)-T(x)\|\leq d(f,T)h(x,x)\leq\frac{h(x,x)}{\beta^2-\gamma},\quad x\in\mathcal{A}^+.$$

From here we proceed exactly as in the proof of Theorem 2.1 to show that $T:\mathcal{A}^+\to\mathcal{B}^+$ is indeed a unique mapping satisfying (2.4). $\square$

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