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Fixed Point Curve for Weakly Inward Contractions and Approximate Fixed Point Property

P. Riyas[1,*] and K. T. Ravindran[2]

[1] *Department of Mathematics, Krishna Menon Memorial Government Women's College, Kannur, India-670004, India*
[2] *Payyanur College, Department of Mathematics, Payyannur College, Kannur, India-670627, India*

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Abstract. In this paper, we discuss the concept of fixed point curve for linear interpolations of weakly inward contractions and establish necessary condition for a nonexpansive mapping to have approximate fixed point property.

Key Words: Nonexpansive mapping, contraction, weakly inward contraction, approximate fixed point property.

AMS Subject Classifications: 46B20, 47H07, 47H10, 54E40, 54H99

1 Introduction

The most well known result in the theory of fixed point is Banach contraction principle. It has been used to develop much of the rest of fixed point theory. Another key result in the field is a theorem due to Browder, Ghde, and Kirk involving Hilbert spaces and nonexpansive mappings. In 1988, Sam B. Nadler and K. Ushijima introduced the concept of fixed point curves for linear interpolations of contraction mappings using Banach contraction principle. Here we shall establish the concept of fixed point curves for linear interpolations of weakly inward contractions using the result [2,4], which is an extension of the Banach contraction principle for non-self contraction mappings.

Before introduce fixed point curve and approximate fixed point property for weakly inward contractions, we will give some preliminary definitions and theorems.

*Corresponding author. *Email addresses:* `riyasmankadavu@gmail.com` (P. Riyas), `ktravindran@rediffmail.com` (K. T. Ravindran)

2 Preliminaries

Definition 2.1 (see [1]). Let K be a closed convex subset of a Banach space X and let $C \subset K$. Then a mapping $T: C \to X$ is said to be *inward mapping (respectively, weakly inward)* on C relative to K if $Tx \in I_K(x)$ (repectively, $Tx \in \overline{I_K(x)}$) for $x \in C$, where

$$I_K(x) = \{(1-t)x + ty : y \in K \text{ and } t \geq 0\},$$

and $\overline{I_K(x)}$ is a closure of the $I_K(x)$ in X.

Lemma 2.1 (see [1]). *$\overline{I_K(x)}$ is a closed convex set containing K for each $x \in K$.*

Definition 2.2 (see [1]). Let K be a closed convex set. A map $T: C \subseteq K \to X$ is said to be *generalized inward* on C relative to K if the following condition is satisfied

$$d(Tx, K) < \|x - Tx\|, \quad \forall x \in C \quad \text{with } Tx \notin K,$$

where $d(y,K) = \inf\{\|y-u\| : u \in K\}$.

Theorem 2.1 (see [3]). *Let C be a non empty closed convex subset of a Banach space X and $T: C \to X$ a weakly inward contraction mapping. Then T has a unique fixed point in C.*

Proposition 2.1 (see [3]). Let C be a non empty closed convex subset of a Banach space X and $T: C \to X$ a non expansive mapping that is weakly inward. Then for $u \in C$ and $t \in (0,1)$ there exist one point $x_t \in C$ such that $x_t = (1-t)u + tTx_t$. If C is bounded then $x_t - Tx_t \to 0$ as $t \to 1$.

Corollary 2.1 (see [3]). Let C be a non-empty closed convex (or star shaped) and bounded subset of a Banach space X and $T: C \to C$ a non expansive mapping. Then there exist a sequence $\{x_n\}$ in C, such that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Definition 2.3 (see [2]). Let (X,d) be a metric space and $f: X \to X$. Then f has the *approximate fixed point property (a.f.p.p)* if $\forall \epsilon > 0$, $F_\epsilon(f) \neq \phi$.

Definition 2.4 (see [2]). Let (X,d) be a metric space. A mapping $f: X \to X$ is said to be asymptotically regular if $d(f^n(x), f^{n+1}(x)) \to 0$ as $n \to \infty$ and $\forall x \in X$.

Lemma 2.2 (see [2]). *Let (X,d) be a metric space and $f: X \to X$ be such that f is asymptotically regular. Then f has approximate fixed point property.*

3 Main results

Here first of all we shall discuss a necessary condition for a family of weakly inward contraction to have a fixed point and using this result we define fixed point curve.

Theorem 3.1. *Let X be a Banach space, K be a non-empty closed and convex subset of X and let $T_1,T_2\colon K\to X$, be two weakly inward contractions. Then there exists a family of weakly inward contraction maps $\{\phi_t\colon K\to X\colon t\in[0,1]\}$, such that, $\forall t\in[0,1]$, ϕ_t has a unique fixed point x_t in K.*

Proof. Define $\phi\colon K\times[0,1]\to X$ by $\phi(x,t)=(1-t)T_1(x)+tT_2(x)$. Then,

$$\begin{aligned}\|\phi(x,t)-\phi(y,t)\| &= \|(1-t)(T_1(x)-T_1(y))+t(T_2(x)-T_2(y)\| \\ &\le (1-t)\|T_1(x)-T_1(y)\|+t\|T_2(x)-T_2(y)\| \\ &\le (1-t)L_1\|x-y\|+tL_2\|x-y\|, \quad \text{where } 0<L_1,L_2<1, \\ &\le L((1-t)\|x-y\|+t\|x-y\|) \\ &= L\|x-y\|, \quad \forall x,y\in K \quad \text{and} \quad t\in[0,1], \end{aligned} \tag{3.1}$$

where $L=\max\{L_1,L_2\}$. Hence ϕ is a contraction uniformly over $[0,1]$. As T_1 and T_2 are weakly inward, $T_1(x)$ and $T_2(x)\in\overline{I_K(x)}$, $\forall x\in K$. Then by Lemma 2.1, $(1-t)T_1(x)+tT_2(x)\in(1-t)\overline{I_K(x)}+t\overline{I_K(x)}=\overline{I_K(x)}$, i.e.,

$$\phi(x,t)\in\overline{I_K(x)}, \quad \forall x\in K \quad \text{and} \quad t\in[0,1].$$

This shows that ϕ is a weakly inward contraction over $[0,1]$. Now define $\phi_t\colon K\to X$ by $\phi_t(x)=\phi(x,t)$ for $t\in[0,1]$ and so by (3.1), we have,

$$\|\phi_t(x)-\phi_t(y)\|\le L\|x-y\|, \quad \forall x,y\in K \quad \text{and} \quad t\in[0,1].$$

That is for each $t\in[0,1]$, ϕ_t is a weakly inward contractiion on a closed convex subset K of X into X. Hence by Theorem 2.1, ϕ_t has a unique fixed point x_t in K for each $t\in[0,1]$. ☐

Definition 3.1 (Fixed point curve)**.** Let T_1 and T_2 be two weakly inward contraction on a closed, convex and bounded subset K of X and $F_W(T_1,T_2)$ denote the set of all fixed points of ϕ_t, $t\in[0,1]$. Then the map $G\colon[0,1]\to F_W(T_1,T_2)$ defined by $G(t)=x_t$ is called the *fixed point curve* for T_1 and T_2.

Lemma 3.1. *Let $T_1,T_2\colon K\to X$ be two weakly inward contractions on a closed, convex and bounded subset K of X. Suppose that T_1 and T_2 have no common fixed points. Then the map $G\colon[0,1]\to F_W(T_1,T_2)$ defined by $G(t)=x_t$ is one to one.*

Proof. Suppose that G is not one to one. Then there exist some t_1 and t_2 $(t_1\neq t_2)$ such that $G(t_1)=G(t_2)$, i.e.,

$$x_{t_1}=x_{t_2}=p.$$

Then,

$$(1-t_1)T_1(p)+t_1T_2(p)=(1-t_2)T_1(p)+t_2T_2(p)\Longrightarrow T_1(p)=T_2(p).$$

Hence,

$$p=x_{t_1}=\phi_{t_1}(p)=(1-t_1)T_1(p)+t_1T_2(p)=T_1(p)=T_2(p),$$

which is a contradiction. □

Corollary 3.1. The fixed point curve $G\colon [0,1]\to F_W(T_1,T_2)$ is continuous.

Corollary 3.2. $F_W(T_1,T_2)$ is a closed set.

Now we discuss approximate fixed point property for a non expansive map defined in a real Hilbert space.

Theorem 3.2. *Let H be a real Hilbert space, C be a closed, convex subset of H and $f\colon C\to C$ be a non expansive mapping such that set of all $\epsilon-$ fixed points $F_\epsilon(f)$ of f is non empty. Then the mapping f_t defined by $f_t(x)=tx+(1-t)f(x)$ is asymptotically regular for all $t\in(0,1)$.*

Proof. Let $x\in C$. Consider a sequence (x_n) defined by the relation $x_{n+1}=f_t(x_n)$, $n=0,1,2,\cdots$, and $x_0=x$. Let y be an $\epsilon-$ fixed point of f. Then

$$\begin{aligned} x_{n+1}-y&=tx_n+(1-t)f(x_n)-y\\ &=t(x_n-y)+(1-t)(f(x_n)-y). \end{aligned} \tag{3.2}$$

For any $\theta(n)$,

$$\theta(n)(x_n-f(x_n))=\theta(n)(x_n-y)-\theta(n)(f(x_n)-y). \tag{3.3}$$

Also,

$$\begin{aligned} &\|x_{n+1}-y\|^2\\ =&\langle t(x_n-y)+(1-t)(f(x_n)-y),t(x_n-y)+(1-t)(f(x_n)-y)\rangle\\ =&\langle t(x_n-y),t(x_n-y)\rangle+\langle t(x_n-y),(1-t)(f(x_n)-y)\rangle\\ &+\langle(1-t)(f(x_n)-y),t(x_n-y)\rangle+\langle(1-t)(f(x_n)-y),(1-t)(f(x_n)-y)\rangle\\ =&t^2\|x_n-y\|^2+t(1-t)\langle(x_n-y),(f(x_n)-y)\rangle\\ &+t(1-t)\langle(f(x_n)-y),(x_n-y)\rangle+(1-t)^2\|f(x_n)-y\|^2\\ =&t^2\|x_n-y\|^2+2t(1-t)\langle(x_n-y),(f(x_n)-y)\rangle+(1-t)^2\|f(x_n)-y\|^2, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &[\theta(n)]^2\|x_n-f(x_n)\|^2\\ =&\langle\theta(n)(x_n-f(x_n)),\theta(n)(x_n-f(x_n))\rangle\\ =&\langle\theta(n)(x_n-y)-\theta(n)(f(x_n)-y),\theta(n)(x_n-y)-\theta(n)(f(x_n)-y)\rangle\\ =&[\theta(n)]]^2\|x_n-y\|^2-[\theta(n)]^2\langle x_n-y,f(x_n)-y\rangle\\ &-[\theta(n)]^2\langle f(x_n)-y,x_n-y\rangle+[\theta(n)]^2\|f(x_n)-y\|^2. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5),

$$\begin{aligned}
&\|x_{n+1}-y\|^2+[\theta(n)]^2\|x_n-f(x_n)\|^2\\
=&(t^2+[\theta(n)]^2)\|x_n-y\|^2+(2t(1-t)-2[\theta(n)]^2)\langle x_n-y,f(x_n)-y\rangle\\
&+((1-t)^2+[\theta(n)]^2)\|f(x_n)-y\|^2\\
\le&(t^2+[\theta(n)]^2)\|x_n-y\|^2+(2t(1-t)-2[\theta(n)]^2)\|x_n-y\|\|f(x_n)-y\|\\
&+((1-t)^2+[\theta(n)]^2)\|f(x_n)-y\|^2\\
\le&(t^2+[\theta(n)]^2)\|x_n-y\|^2+(2t(1-t)-2[\theta(n)]^2)\|x_n-y\|(\|f(x_n)-f(y)\|+\|f(y)-y\|)\\
&+((1-t)^2+[\theta(n)]^2)(\|f(x_n)-f(y)\|+\|f(y)-y\|)^2\\
<&(t^2+[\theta(n)]^2)\|x_n-y\|^2+(2t(1-t)-2[\theta(n)]^2)\|x_n-y\|(\|x_n-y\|+\epsilon)\\
<&(t^2+[\theta(n)]^2)\|x_n-y\|^2+(2t(1-t)-2[\theta(n)]^2)\|x_n-y\|(\|x_n-y\|+\epsilon)\\
&+((1-t)^2+[\theta(n)]^2)(\|x_n-y\|+\epsilon)^2\\
<&(\|x_n-y\|+\epsilon)^2,\\
\Longrightarrow&[\theta(n)]^2\|x_n-f(x_n)\|^2<(\|x_n-y\|+\epsilon)^2-\|x_{n+1}-y\|^2. \qquad (3.6)
\end{aligned}$$

If we take $\theta(n)=n\sqrt{(\|x_n-y\|+\epsilon)^2-\|x_{n+1}-y\|^2}$, then

$$\|x_n-f(x_n)\|^2<\frac{1}{n^2},\quad \forall n,$$

and therefore,

$$\begin{aligned}
\|x_n-f_t(x_n)\|&=\|x_n-tx_n-(1-t)f(x_n)\|\\
&=\|(1-t)x_n-(1-t)f(x_n)\|\\
&=(1-t)\|x_n-f(x_n)\|,\quad \forall n,\\
&\Longrightarrow\|x_n-x_{n+1}\|^2<\frac{1}{n^2},\quad \forall n.\\
&\Longrightarrow\sum_{n=1}^{\infty}\|x_n-x_{n+1}\|^2<\infty.
\end{aligned}$$

This shows that f_t is asymptotically regular. □

Corollary 3.3. Let H be a Hilbert space, C be a closed convex subset of H and $f\colon C\to C$ be a non expansive mapping such that set of all $\epsilon-$ fixed points $F_\epsilon(f)$ of f is non empty. Then the mapping f_t defined by $f_t(x)=tx+(1-t)f(x)$ has the approximate fixed point property.

Theorem 3.3. *Let H be a Hilbert space, C be a closed convex subset of H and $f\colon C\to C$ be a non expansive mapping, such that $F_\epsilon(f)\neq\phi$ for some ϵ. Then f has the approximate fixed point property.*

Proof. For each $t\in(0,1)$, define $f_t(x)=tx+(1-t)f(x)$. Then by the above corollary f_t has the approximate fixed point property for each t. Therefore, for any $\epsilon>0$ and $t\in(0,1)$, $F_\epsilon(f_t)\neq\phi$, i.e.,

$$\begin{aligned} &\|f_t(x)-x\|<\epsilon, && \forall t \text{ and } \epsilon,\\ \Longrightarrow &\|t(x-f(x)+(f(x)-x)\|<\epsilon, && \forall t \text{ and } \epsilon. \end{aligned}$$

Letting $t\to 0$, we thus obtain $\|f(x)-x\|<\epsilon$, $\forall\epsilon>0$. Hence f has the approximate fixed point property. □

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