On Approximation of Function Classes in Lorentz Spaces with Anisotropic Norm

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Abstract. In this paper, Lorentz space of functions of several variables and Besov’s class are considered. We establish an exact approximation order of Besov’s class by partial sums of Fourier’s series for multiple trigonometric system.

Key Words: Lorentz space, Besov’s class, approximation.

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1 Introduction

Let \( \mathbf{x} = (x_1, \cdots, x_m) \in I^m = [0, 2\pi)^m \) and \( \theta_j, q_j \in [1, +\infty), j = 1, \cdots, m \). We shall denote by \( L^*_{\vec{\theta}, \vec{q}}(I^m) \) the Lorentz space with anisotropic norm of Lebesgue-measurable functions \( f(\mathbf{x}) \) of period \( 2\pi \) in each variable, such that the quantity

\[
\|f\|_{\vec{q}, \vec{\theta}} = \left[ \int_0^{2\pi} \left( \int_0^{2\pi} (f^*_{1, \ldots, m}(t_1, \ldots, t_m))^{\theta_1 t_1^{-1}} dt_1 \right)^{q_1 \theta_1^{-1}} \cdots \left( \int_0^{2\pi} (f^*_{1, \ldots, m}(t_1, \ldots, t_m))^{\theta_m t_m^{-1}} dt_m \right)^{q_m \theta_m^{-1}} dt_m \right]^{1/m}
\]

is finite, where \( f^*_{1, \ldots, m}(t_1, \ldots, t_m) \) is the non-increasing rearrangement of the function \( |f(\mathbf{x})| \) in each variable \( x_j \) whereas the other variables are fixed (see [7, 19]).

Let \( L^*_{\vec{\theta}, \vec{q}}(I^m) \) be the set of functions \( f \in L^*_{\vec{\theta}, \vec{q}}(I^m) \), such that

\[
\int_0^{2\pi} f(\mathbf{x}) dx_j = 0, \quad \forall j = 1, \cdots, m,
\]

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and let $a_\pi(f)$ be the Fourier coefficients of $f \in L_1(I^m)$ with respect to the multiple trigonometric system. Then we set

$$
\delta_\pi(f, \varpi) = \sum_{\pi \in \varpi} a_\pi(f) e^{i\langle \pi, \varpi \rangle},
$$

where $\langle g, \tilde{x} \rangle = \sum_{j=1}^m y_j x_j$, $\rho(\tilde{s}) = \{ \tilde{k} = (k_1, \ldots, k_m) \in \mathbb{Z}^m : 2^{k_j - 1} \leq |k_j| < 2^{k_j}, j = 1, \ldots, m \}$, $\tilde{s} = (s_1, \ldots, s_m) \in \mathbb{Z}_+^m$.

A number sequence $\{ a_\pi \}_{\pi \in \mathbb{Z}^m}$ belongs to $l_\varpi$ if

$$
\| \{ a_\pi \}_{\pi \in \mathbb{Z}^m} \|_{l_\varpi} = \left\{ \sum_{n_{\pi} = -\infty}^{\infty} \left[ \left( \sum_{j=-\infty}^{\infty} |a_\pi|^n \right)^{\frac{1}{n}} \right]^{\frac{n}{m}} \right\}^{\frac{1}{m}} < +\infty,
$$

where $\varpi = (p_1, \ldots, p_m)$, $1 \leq p_j < +\infty$, $j = 1, 2, \ldots, m$.

By analogy with [15] and [3] consider the Besov class

$$
S^\varphi_{p, \tau, \varpi} B = \left\{ f \in L^\varphi_{p, \tau}(I^m) : \| \varphi(f) \|_{L^\varphi_{p, \tau}} \leq 1 \right\},
$$

where $\varphi = (p_1, \ldots, p_m)$, $\varphi = (\theta_1, \ldots, \theta_m)$, $\tau = (\tau_1, \ldots, \tau_m)$, $1 < p_j < +\infty$, $1 < \theta_j$, $\tau_j < +\infty$, $j = 1, \ldots, m$.

For a fixed vector $\tilde{\tau} = (\gamma_1, \ldots, \gamma_m)$, with $\gamma_j > 0$, $j = 1, \ldots, m$, set

$$
Q_{\tilde{\tau}} = \bigcup_{(k_j) \in \varphi} Q_{\tilde{\tau}}(k) = \left\{ t(\varpi) = \sum_{k \in Q^\varphi_{\tilde{\tau}}} b_k e^{i\langle \tilde{k}, \varpi \rangle} \right\},
$$

$E_n^\varphi(f)_{p, \tau}$ is the best approximation of a function $f \in L^\varphi_{p, \tau}(I^m)$ by polynomials in $T(Q_{\tilde{\tau}})$, and $S^\varphi_{p, \tau}(f, \varpi) = \sum_{k \in Q^\varphi_{\tilde{\tau}}} a_k(f) e^{i\langle \tilde{k}, \varpi \rangle}$ is a partial sum of the Fourier series of $f$.

As pointed out in [28], the difficulty in the theory of approximation of functions of several variables is the choice of the harmonics of the approximating polynomials. The first author suggesting approximation of functions of several variables by polynomials with harmonics in hyperbolic crosses was K. I. Babenko [4]. After that, approximation of various classes of smooth functions by this method was considered by S. A. Telyakovskii [25], B. S. Mityagin [16], Ya. S. Bugrov [8], N. S. Nikol’skaya [18], E. M. Galeev [12], V. N. Temlyakov [26, 27], Din Dung [11], N. N. Pustovoitov [20], E. S. Belinskii [6], B. S. Kashin and V. N. Temlyakov [14], A. R. DeVore, P. Petrushev and V. N. Temlyakov [9], A. S. Romanyuk [21, 22].

In particular, A. S. Romanyuk [21] obtained the following result for the Besov classes in Lebesgue spaces with isotropic norm.

**Theorem 1.1** (A. S. Romanyuk). Assume that $1 \leq p < q < +\infty$, $1 \leq \tau < +\infty$, $1/p - 1/q < r_1 = \cdots = r_v < r_{v+1} \leq \cdots \leq r_m$. Then

$$
\sup_{f \in S^\varphi_{p, \tau} B} E_n^\varphi(f)^{\gamma}_q = E_n^\varphi(S^\varphi_{p, \tau} B)^{\gamma}_q \asymp 2^{-n(r_1 + \frac{1}{2})} n^{(v-1)(\frac{1}{2} - \frac{1}{q})},
$$

where

$$
\sup_{f \in S^\varphi_{p, \tau} B}
$$

and

$$
E_n^\varphi(S^\varphi_{p, \tau} B)^{\gamma}_q \asymp 2^{-n(r_1 + \frac{1}{2})} n^{(v-1)(\frac{1}{2} - \frac{1}{q})}.
$$
where \( a_+ = \max \{0, a\} \).

There is corresponding survey in [28].

The main aim of the present paper is to estimate the order of the quantity

\[
S^n(\bar{\gamma}P) = \sup_{f \in \bar{\gamma}P} \| f - \bar{\gamma}P(f) \|_{\bar{\gamma}P}^*
\]

This paper is organized as follows. In the second section we give auxiliary results. In third section we establish an estimate of order approximation of Besov classes in a Lorentz space with anisotropic norm.

Let

\[
Y^m(\gamma, n) = \{ s = (s_1, \ldots, s_m) \in \mathbb{Z}^m_+ : \sum_{j=1}^{m} s_j \gamma_j \geq n \}.
\]

We shall denote by \( C(p, q, y, \cdots) \) positive quantities which depend only on the parameter in the parentheses and not necessarily the same in distinct formulae. The notation \( A(y) \asymp B(y) \) means that there exist positive constants \( C_1, C_2 \) such that

\[
C_1 A(y) \leq B(y) \leq C_2 A(y).
\]

2 Auxiliary results

We now give several auxiliary results.

**Lemma 2.1** (see [13]). Assume that \( 0 < \theta < +\infty \), and \( a_k, b_k, k = 1, 2, \ldots \), are positive numbers.

a) If \( \sum_{k=1}^{n} a_k \leq C \cdot a_n \), then

\[
\sum_{n=1}^{\infty} a_n \left( \sum_{k=n}^{\infty} b_k \right)^{\theta} \leq C \cdot \sum_{n=1}^{\infty} a_n b_n^{\theta}.
\]

b) If \( \sum_{k=n}^{\infty} a_k \leq C \cdot a_n \), then

\[
\sum_{n=1}^{\infty} a_n \left( \sum_{k=1}^{n} b_k \right)^{\theta} \leq C \cdot \sum_{n=1}^{\infty} a_n b_n^{\theta}.
\]

**Lemma 2.2.** Let \( \nu \) be a positive, \( \nu \leq m \) and let \( \gamma = (\gamma_1, \cdots, \gamma_m) \), \( \gamma' = (\gamma_1', \cdots, \gamma_m') \), where \( 1 = \gamma_1 = \cdots = \gamma_{\nu} \leq \cdots \leq \gamma_m \), \( \gamma_j' = \gamma_{j'}, j = 1, \cdots, \nu, 1 < \gamma_j' < \gamma_{j'}, j = \nu + 1, \cdots, m \) and let \( \alpha \in (0, +\infty) \), \( \theta_j \in [1, +\infty), j = 1, \cdots, m \), and \( \bar{\theta} = (\theta_1, \cdots, \theta_m) \). Then

\[
\left\| \left\{ 2^{-\alpha(\nu, \gamma)} \right\}_{\gamma \in Y^m(\gamma, n)} \right\|_{\bar{\theta}} \leq C(\alpha, \theta, \gamma, m) 2^{-n \alpha} \sum_{j=1}^{\nu} \frac{\theta_j}{\gamma_j}.
\]

**Remark 2.1.** Note that for \( \theta_1 = \cdots = \theta_m \), Lemma 2.2 was proved before by V. N. Temlyakov [27].
In what follows we denote by \( \chi_\mathcal{X}(n) \) the characteristic function of the set \( \mathcal{X}(n) = \{ s = (s_1, \cdots, s_m) \in \mathbb{Z}_+^m : \langle \bar{s}, \gamma \rangle = \gamma \} \).

**Lemma 2.3.** Let \( \bar{\tau} = (\tau_1, \cdots, \tau_m) \), \( 1 \leq \tau_j < +\infty \), \( j = 1, \cdots, m \). Then the following relation holds:

\[
\left\| \left\{ \chi_\mathcal{X}(n) \right\}_{\bar{s} \in \mathcal{X}(n)} \right\|_{L^\tau} \leq \frac{C}{n^j}.
\]

**Lemma 2.4.** Let \( \bar{\gamma} = (\gamma_1, \cdots, \gamma_m) \), \( \bar{\gamma}' = (\gamma_1', \cdots, \gamma_m') \), \( \gamma_j = \gamma_j' \), \( j = 1, \cdots, n \), \( 1 < \gamma_j < \gamma_j' \), \( j = n+1, \cdots, m \), and let \( \bar{\tau} = (\tau_1, \cdots, \tau_m) \) be with \( 1 \leq \tau_j < +\infty \), \( j = 1, \cdots, m \), and \( \alpha > 0 \). Then the following inequality holds:

\[
I_n = \left\| \left\{ 2^{-\alpha \langle \bar{s}, \gamma \rangle} \right\}_{\bar{s} \in \mathcal{X}(n)} \right\|_{L^\tau} \geq C(\tau, \gamma, m) 2^{-\alpha} \cdot n^{-j}.
\]

Lemmas 2.2-2.4 are proved in [1, 2].

**Lemma 2.5.** Let \( 1 \leq p, \theta, \bar{\gamma} < \infty \), \( \alpha \), \( j = 1, \cdots, m \), and \( E_j \subset [0, 2\pi) \), \( j = 1, \cdots, m \), be measurable sets. Then for any trigonometric polynomial

\[
T_n(\bar{x}) = \sum_{k \in \mathbb{R}} a_k e^{i\langle \bar{k}, \bar{x} \rangle},
\]

the following inequality holds:

\[
\int_{E_1} \cdots \int_{E_m} |T_n(\bar{x})| dx_1 \cdots dx_m \leq C(\alpha, \theta, m) \cdot \prod_{j \in \mathbb{R}} P_{\mathcal{N}}^{\xi_j} \cdot \prod_{\bar{j} \in \mathcal{N}} |E_j| \cdot \prod_{\bar{k} \in \mathcal{N}} |E_{\bar{k}}| \cdot \left( |\mathcal{N}| \right)^{1 - \frac{1}{\theta}} \cdot \| T_n \|_{p, \theta}^\alpha.
\]

where \( e \subset \{1, \cdots, m\} \), \( |E_j| \) - Lebesgue measure of \( E_j \).

**Proof.** Let \( e \) be an arbitrary subset of \( \{1, \cdots, m\} \). Also let \( E_e = \prod_{j \in e} E_j \), \( d^e \bar{x} = \prod_{j \in e} dx_j \).

It is known that for any trigonometric polynomial \( T_n(\bar{x}) \) for fixed \( x_j, j \in e \), the following formula holds:

\[
T_n(\bar{x}) = \frac{1}{(2\pi)^{|e|}} \int_{[0, 2\pi]} T_n(\bar{y}, \bar{x}) \cdot \prod_{j \in e} D_{\eta_j}(x_j - y_j) d\bar{y},
\]

where \( D_{\eta}(t) \) - is the Dirichlet kernel of the trigonometric system, \( \bar{y} \) - the vector with the coordinates \( y_j \) if \( j \in e \), and \( \bar{x} \) - the vector with the coordinates \( x_j \) if \( j \in \bar{e} \) (\( \bar{e} \) is the set-theoretic completion of \( e \)) and \( |e| \) is the number of elements of \( e \). By the property of Lebesgue integral, we have

\[
\int_{E_e} |T_n(\bar{x})| d^{e} \bar{x} \leq \frac{1}{(2\pi)^{|e|}} \int_{E_e} \left\| T_n(\bar{y}, \bar{x}) \cdot \prod_{j \in e} D_{\eta_j}(x_j - y_j) d\bar{y} \right\|_{L^\tau} d^e \bar{x}
\]

\[
= \frac{1}{(2\pi)^{|e|}} \int_{E_e} \int_{[0, 2\pi]} |T_n(\bar{y}, \bar{x})| \cdot \prod_{j \in e} |D_{\eta_j}(x_j - y_j)| d\bar{y} d^e \bar{x}
\]

\[
= \frac{1}{(2\pi)^{|e|}} \int_{E_e} \int_{[0, 2\pi]} T_n(\bar{y}, \bar{x}) \cdot \prod_{j \in e} \chi_{E_j}(x_j) d\bar{y} d^e \bar{x},
\]
where $\chi_E$ is the characteristic function of the set $E$.

Apply the Holder inequality to the integrals in the right side of the last expression. Then

$$
\int_{E_\varepsilon} |T_\varepsilon(x)| \, dx \leq \frac{1}{(2\pi)^{|\varepsilon|}} \| T_\varepsilon \|_{p,\beta}^* \prod_{j \in \varepsilon} \| D_{n_j} (x_j - \bullet) \|_{p_j', \beta_j'} \prod_{j \in \varepsilon} \| \chi_{E_j} \|_{p_j', \beta_j'}^*
$$

where $p_j' = p_j/(p_j - 1)$, $\theta_j' = \theta_j/\theta_j - 1$, $j = 1, \ldots, m$. Since

$$
\| \chi_{E_j} \|_{p_j', \beta_j'}^* = \left\{ \int_0^{\| E_j \|_{p_j', \beta_j'}^{-1}} \| t_j \|_{p_j'}^\beta \cdot | E_j | \cdot \prod_{j \in \varepsilon} \| D_{n_j} (x_j - \bullet) \|_{p_j', \beta_j'}^* \right\} \left( \frac{p_j'}{\theta_j'} \right) \cdot | E_j |^{\theta_j'} \cdot \prod_{j \in \varepsilon} \| D_{n_j} (x_j - \bullet) \|_{p_j', \beta_j'}^*
$$

it follows from (2.1) that

$$
\int_{E_\varepsilon} |T_\varepsilon(x)| \, dx \leq \frac{1}{(2\pi)^{|\varepsilon|}} \| T_\varepsilon \|_{p,\beta}^* \prod_{j \in \varepsilon} \| p_j' \cdot \| E_j | \cdot \prod_{j \in \varepsilon} \sup_{x_j} \| D_{n_j} (x_j - \bullet) \|_{p_j', \beta_j'}^*.
$$

From this by taking into account the estimation

$$
\sup_{x_j} \| D_{n_j} (x_j - \bullet) \|_{p_j', \beta_j'}^* \leq C(p, \theta, m) \cdot n_j^{1 - \frac{1}{p_j'}},
$$

we have

$$
\int_{E_\varepsilon} |T_\varepsilon(x)| \, dx \leq C(p, m, \theta) \| T_\varepsilon \|_{p,\beta}^* \prod_{j \in \varepsilon} \| E_j | \cdot \prod_{j \in \varepsilon} n_j^{1 - \frac{1}{p_j'}}
$$

for fixed $x_j$, $j \in \varepsilon$. By integrating by the variables $x_j$, $j \in \varepsilon$, the both sides of this inequality, we get

$$
\int_{E_\varepsilon} \int_{E_\varepsilon} |T_\varepsilon(x)| \, dx \, dx \leq C(p, m, \theta) \| T_\varepsilon \|_{p,\beta}^* \prod_{j \in \varepsilon} \| E_j | \cdot \prod_{j \in \varepsilon} \| E_j | \cdot \prod_{j \in \varepsilon} n_j^{1 - \frac{1}{p_j'}}.
$$

The proof is finished. \qed

3 An estimate of the order of approximation of the Besov classes by partial sums of Fourier’s series

We now prove the main results. We set

$$
G_c(n) = \{ s = (s_1, \ldots, s_m) \in \mathbb{N}^m : s_j \leq n_j, j \in \varepsilon; s_j > n_j, j \notin \varepsilon \}.
$$
where \( e \subset \{1, \ldots, m\} \) and
\[
U_{\bar{h}}(f, \bar{x}) = \sum_{e \subset \{1, \ldots, m\}} \sum_{s \in G_e(\bar{h})} \delta_s(f, \bar{x}).
\]
Let
\[
\tilde{f}(I) = \sup_{|E_w| \geq t_w} \frac{1}{|E_w|} \int_{E_w} \ldots \sup_{|E_1| \geq n_1} \frac{1}{|E_1|} \int_{E_1} |f(x_1, \ldots, x_m)| dx_1 \ldots dx_m,
\]
where \( |E_j| \) is the Lebesgue measure of the subset \( E_j \subset [0, 2\pi] \).

**Theorem 3.1.** Assume that \( 1 \leq p_j, \theta_j < +\infty, j = 1, \ldots, m \). Then for each function \( f \in L^*_{p, \theta}([1]^m) \) the next inequality holds
\[
\tilde{f}(I) \leq C(p, m) \left\{ \prod_{j=1}^m t_j^{-\frac{1}{p_j}} \|f - U_{\bar{h}}(f)\|_{p, \theta}^* + \sum_{e \subset \{1, \ldots, m\}} \prod_{j \not \in e} t_j^{-\frac{1}{p_j}} \prod_{s \in G_e(\bar{h})} \sum_{j \in e} ^2 \|\delta_s(f)\|_{p, \theta}^* \right\},
\]
for \( t_j \in (2\pi 2^{-n_j-1}, 2\pi 2^{-n_j}], n_j = 0, 1, \ldots, j = 1, \ldots, m \).

**Proof.** Let \( E_j \subset [0, 2\pi] \) be a Lebesgue measurable subset. Then, by the properties of the integral,
\[
\int_{E_w} \ldots \int_{E_1} |f(x_1, \ldots, x_m)| dx_1 \ldots dx_m \\
\leq \int_{E_w} \ldots \int_{E_1} |f(\bar{x}) - U_{\bar{h}}(f, \bar{x})| dx_1 \ldots dx_m + \int_{E_w} \ldots \int_{E_1} |U_{\bar{h}}(f, \bar{x})| d\bar{x}.
\]
(3.1)

Using Holder’s integral inequality, we obtain (see [7])
\[
\int_{E_w} \ldots \int_{E_1} |f(\bar{x}) - U_{\bar{h}}(f, \bar{x})| d\bar{x} \leq \prod_{j=1}^m |E_j|^{-\frac{1}{p_j}} \|f - U_{\bar{h}}(f)\|_{p, \theta}^*.
\]
(3.2)

where \( p_j' = p_j / (p_j - 1), j = 1, \ldots, m \).

Let \( e \subset \{1, \ldots, m\} \). Applying Lemma 2.5, we obtain
\[
\int_{E_w} \ldots \int_{E_1} \left| \sum_{s \in G_e(\bar{h})} \delta_s(f, \bar{x}) \right| d\bar{x} \leq \int_{E_w} \ldots \int_{E_1} \sum_{s \in G_e(\bar{h})} |\delta_s(f, \bar{x})| d\bar{x} \\
\leq C \prod_{j \not \in e} |E_j| \prod_{j \in e} |E_j|^{-\frac{1}{p_j}} \sum_{s \in G_e(\bar{h})} \sum_{j \in e} ^2 \|\delta_s(f)\|_{p, \theta}^*.
\]
(3.3)

In the definition of \( \tilde{f} \), \( |E_j| \geq t_j, j = 1, \ldots, m \). Hence, by inequalities (3.1)-(3.3), we obtain
\[
\prod_{j=1}^m \frac{1}{|E_j|} \int_{E_w} \ldots \int_{E_1} |f(\bar{x})| d\bar{x} \\
\leq C \left\{ \prod_{j=1}^m t_j^{-\frac{1}{p_j}} \|f - U_{\bar{h}}(f)\|_{p, \theta}^* + \sum_{e \subset \{1, \ldots, m\}} \prod_{j \not \in e} t_j^{-\frac{1}{p_j}} \sum_{s \in G_e(\bar{h})} \sum_{j \in e} ^2 \|\delta_s(f)\|_{p, \theta}^* \right\}.
\]
Now, by the definition of the least upper bound of a set, we obtain the theorem. \( \square \)
Theorem 3.2. Let $\tilde{p} = (p_1, \cdots, p_m)$, $\tilde{q} = (q_1, \cdots, q_m)$, $\tilde{\theta}(1) = (\theta_1^{(1)}, \cdots, \theta_m^{(1)})$, $\tilde{\theta}(2) = (\theta_1^{(2)}, \cdots, \theta_m^{(2)})$. Assume that $1 < p_i < q_i < +\infty$, $1 \leq \theta_i^{(1)}, \theta_i^{(2)} < +\infty$, $j = 1, \cdots, m$. If $f \in L_{\tilde{p}, \tilde{\theta}(1)}^+(1^m)$ and the quantity

$$
\sigma(f) \equiv \left\{ \sum_{s_m=1}^{\infty} 2^{s_m\theta_m^{(2)}} \left( \frac{1}{p_m} \right)^{\theta_m^{(1)}} \cdots \left( \frac{1}{q_1} \right)^{\theta_1^{(1)}} \left( \| \delta_j(f) \|_{\tilde{p}, \tilde{\theta}(1)}^{\theta_j^{(2)}} \right)^{\theta_j^{(2)}} \right\}^{\frac{1}{\sum_{j=1}^{m} \theta_j^{(2)}}} \frac{1}{\sigma_m^{(2)}}
$$

is finite, then $f \in L_{\tilde{q}, \tilde{\theta}(2)}^+(1^m)$ and

$$
\| f \|_{\tilde{q}, \tilde{\theta}(2)}^+ \leq C(p, q, \theta) \cdot \sigma(f).
$$

Proof. Blozinski (see [7], pp. 161) proved, that

$$
\| f \|_{\tilde{q}, \tilde{\theta}(2)}^+ \times \| \tilde{f} \|_{\tilde{q}, \tilde{\theta}(2)}^+ \leq C(p, q, \theta) \cdot \sigma(f).
$$

By this relation, we obtain

$$
\| f \|_{\tilde{q}, \tilde{\theta}(2)}^+ \leq C \left\{ \sum_{n_m=0}^{\infty} \int_{2^n \pi^{2-n} - 1}^{2^{n+1} \pi^{2-n}} t^m \left( \begin{array}{c} \sum_{s_m=1}^{\infty} 2^{s_m\theta_m^{(2)}} \left( \frac{1}{p_m} \right)^{\theta_m^{(1)}} \cdots \left( \frac{1}{q_1} \right)^{\theta_1^{(1)}} \left( \| f - U_n(f) \|_{\tilde{p}, \tilde{\theta}(1)}^{\theta_j^{(2)}} \right)^{\theta_j^{(2)}} \right) \frac{1}{\sigma_m^{(2)}} \right\}^{\frac{1}{\sum_{j=1}^{m} \theta_j^{(2)}}} \frac{1}{\sigma_m^{(2)}}. (3.4)
$$

Using now Theorem 3.1 and taking into account the relation

$$
\int_{2^n \pi^{2-n} - 1}^{2^{n+1} \pi^{2-n}} t^\beta dt \propto 2^{-n\beta}, \quad \beta > 0,
$$

(3.5)

it follows from (3.4) that

$$
\| f \|_{\tilde{q}, \tilde{\theta}(2)}^+ \leq \left\{ \sum_{n_m=0}^{\infty} \int_{2^n \pi^{2-n} - 1}^{2^{n+1} \pi^{2-n}} t^m \left( \sum_{s_m=1}^{\infty} 2^{s_m\theta_m^{(2)}} \left( \frac{1}{p_m} \right)^{\theta_m^{(1)}} \cdots \left( \frac{1}{q_1} \right)^{\theta_1^{(1)}} \left( \| f - U_n(f) \|_{\tilde{p}, \tilde{\theta}(1)}^{\theta_j^{(2)}} \right)^{\theta_j^{(2)}} \right) \frac{1}{\sigma_m^{(2)}} \right\}^{\frac{1}{\sum_{j=1}^{m} \theta_j^{(2)}}} \frac{1}{\sigma_m^{(2)}}
$$

$$
+ \left\{ \sum_{n_m=0}^{\infty} \int_{2^n \pi^{2-n} - 1}^{2^{n+1} \pi^{2-n}} t^m \left( \sum_{s_m=1}^{\infty} 2^{s_m\theta_m^{(2)}} \left( \frac{1}{p_m} \right)^{\theta_m^{(1)}} \cdots \left( \frac{1}{q_1} \right)^{\theta_1^{(1)}} \left( \| f \|_{\tilde{p}, \tilde{\theta}(1)}^{\theta_j^{(2)}} \right)^{\theta_j^{(2)}} \right) \frac{1}{\sigma_m^{(2)}} \right\}^{\frac{1}{\sum_{j=1}^{m} \theta_j^{(2)}}} \frac{1}{\sigma_m^{(2)}}.
$$

$$
\times \left( \sum_{\sigma \subseteq \{1, \cdots, m\}} \prod_{j \notin \sigma} \frac{1}{\tilde{q}_j} \sum_{s \in \mathcal{Q}_n(\sigma)} \prod_{j \notin \sigma} 2^{\theta_j^{(2)} \left( \| \delta_j(f) \|_{\tilde{p}, \tilde{\theta}(1)}^{\theta_j^{(2)}} \right)^{\theta_j^{(2)}} \frac{1}{\sigma_m^{(2)}} \right\}^{\frac{1}{\sum_{j=1}^{m} \theta_j^{(2)}}} \frac{1}{\sigma_m^{(2)}} \right]. (3.6)
$$
Let \( e = \{1, \cdots, i\}, i \leq m \). Then using the relation (3.5) and applying repeatedly the triangle inequality, we obtain

\[
J(f) = \left\{ \sum_{n \in \mathbb{N}} \int_{\mathbb{T}^m} \frac{\phi^{(2)}(t)}{t_{m+1}^{n_0}} \left[ \cdots \left[ \sum_{n_{l+1} \in \mathbb{N}} \int_{\mathbb{T}^m} \frac{\phi^{(2)}(t)}{t_{m+1}^{n_0}} \right] \cdots \right] \frac{1}{\phi_{m+1}^{(2)}} dt \right\} \leq C \left\{ \sum_{n \in \mathbb{N}} 2^{|e|} \phi^{(2)}(\eta_{m+1}) \left[ \cdots \left[ \sum_{n_{l+1} \in \mathbb{N}} \phi^{(2)}(\eta_{m+1}) \right] \cdots \right] \frac{1}{\phi_{m+1}^{(2)}} \right\}.
\]

We apply the part (a) of Lemma 2.1 to the sum with respect to \( n_j, j = 1, \cdots, i \), and the part (b) to the sum with respect to \( n_j, j = i+1, \cdots, m \). Then

\[
J(f) \leq C \left\{ \sum_{n \in \mathbb{N}} 2^{|e|} \phi^{(2)}(\eta_{m+1}) \left[ \cdots \left[ \sum_{n_{l+1} \in \mathbb{N}} \phi^{(2)}(\eta_{m+1}) \right] \cdots \right] \frac{1}{\phi_{m+1}^{(2)}} \right\} \leq C \left\{ \sum_{n \in \mathbb{N}} 2^{|e|} \phi^{(2)}(\eta_{m+1}) \left[ \cdots \left[ \sum_{n_{l+1} \in \mathbb{N}} \phi^{(2)}(\eta_{m+1}) \right] \cdots \right] \frac{1}{\phi_{m+1}^{(2)}} \right\}.
\]
It follows from the properties of the norm and Lemma 2.1 that

\[ \|f\|_{q, \beta, m}^* \leq C \left\{ \sum_{n_m=0}^{\infty} 2^n \beta_0^n \left( \frac{1}{p_m - \frac{1}{q_m}} \right)^{2} \left[ \sum_{n_{m-1}=0}^{\infty} 2^n \beta_0^n \left( \frac{1}{p_{m-1} - \frac{1}{q_{m-1}}} \right)^{2} \left( \|f - U_n(f)\|_{p, \beta, m}^* \right) \right]^{\frac{2}{q_m}} \right\}^{\frac{1}{q_m}} \]

Since

\[ f(\bar{x}) - U_n(f, \bar{x}) = \sum_{s_m=n_m+1}^{\infty} \cdots \sum_{s_1=n_1+1}^{\infty} \delta(s, f, \bar{x}), \]

it follows from the properties of the norm and Lemma 2.1 that

\[ \left\{ \sum_{n_m=0}^{\infty} 2^n \beta_0^n \left( \frac{1}{p_m - \frac{1}{q_m}} \right)^{2} \left[ \sum_{n_{m-1}=0}^{\infty} 2^n \beta_0^n \left( \frac{1}{p_{m-1} - \frac{1}{q_{m-1}}} \right)^{2} \left( \|\delta(s, f, \bar{x})\|_{p, \beta, m}^* \right) \right]^{\frac{2}{q_m}} \right\}^{\frac{1}{q_m}} \leq C \left\{ \sum_{n_m=0}^{\infty} 2^n \beta_0^n \left( \frac{1}{p_m - \frac{1}{q_m}} \right)^{2} \left[ \sum_{n_{m-1}=0}^{\infty} 2^n \beta_0^n \left( \frac{1}{p_{m-1} - \frac{1}{q_{m-1}}} \right)^{2} \left( \|\delta(s, f, \bar{x})\|_{p, \beta, m}^* \right) \right]^{\frac{2}{q_m}} \right\}^{\frac{1}{q_m}}. \]  

(3.9)

The inequalities (3.8) and (3.9) now yield the required result.

**Theorem 3.3.** Let \( \bar{q} = (q_1, \ldots, q_m), \bar{\theta} = (\theta_1, \ldots, \theta_m), \lambda = (\lambda_1, \ldots, \lambda_m) \). Assume that \( 1 < q_j < \lambda_j < +\infty, 1 < \theta_j < +\infty, j = 1, \ldots, m \). If \( f \in L_{q, \beta}^*(\Lambda^n) \) and

\[ f(\bar{x}) \sim \sum_{\bar{s} \in Z^n} b_{\bar{s}} \sum_{k \in \mathbb{Z}^n} e^{i(k, \bar{x})}, \]

Then

\[ \|f\|_{q, \beta, m}^* \geq C(q, \theta, \lambda, m) \left\{ \sum_{n_m=1}^{\infty} 2^n \beta_0^n \left( \frac{1}{p_m - \frac{1}{q_m}} \right)^{2} \left[ \sum_{n_{m-1}=0}^{\infty} 2^n \beta_0^n \left( \frac{1}{p_{m-1} - \frac{1}{q_{m-1}}} \right)^{2} \left( \|\delta(s, f, \bar{x})\|_{\lambda, \beta}^* \right) \right]^{\frac{2}{q_m}} \right\}^{\frac{1}{q_m}}. \]

**Proof.** Similarly to the proof of Theorem 3 in [1].

**Theorem 3.4.** Let \( \bar{p} = (p_1, \ldots, p_m), \bar{q} = (q_1, \ldots, q_m), \bar{\theta} = (\theta_1, \ldots, \theta_m), \lambda = (\lambda_1, \ldots, \lambda_m) \). Assume that \( 1 < p_j < q_j < +\infty, 1 \leq \theta_j^1, \theta_j^2 < +\infty, \lambda_j > 0, j = 1, \ldots, m \), and \( r_j > 1/p_j - 1/q_j, j = 1, \ldots, m \).
1) If \( 1 \leq \theta_j(2) < \tau_j < +\infty, j = 1, \ldots, m, \) then
\[
S^n_H \left( S^n_{\tau(1), \tau(2)} B \right)_{\tau(2)} \leq C(\theta, q, m, r) \cdot \left\{ \prod_{j=1}^{m} 2^{-s_j} 2^{s_j \left( \frac{1}{r_j} - \frac{1}{\tau_j} \right)} \right\}_{\tau(1), \tau(2)}
\]
where \( \tau = (\epsilon, \ldots, \epsilon_m), \epsilon_j = \theta_j(2), \beta_j = \tau_j / \theta_j(2), \) \( 1 / \beta_j + 1 / \beta_j' = 1, j = 1, \ldots, m. \)

2) If \( \tau_j \leq \theta_j(2) < +\infty, j = 1, \ldots, m, \) then
\[
S^n_H \left( S^n_{\tau(1), \tau(2)} B \right)_{\tau(2)} \leq C(\theta, r, m, r) \cdot \sup_{\tau \in Y^n(\tau, \nu)} \left\{ \prod_{j=1}^{m} 2^{-s_j} 2^{s_j \left( \frac{1}{r_j} - \frac{1}{\tau_j} \right)} \right\}_{\tau(1), \tau(2)}
\]

Proof. Let \( f \in S^n_{\tau(1), \tau(2)} B. \) Since \( \delta_\tau(f - S^n_H(f)) = 0, \) \( \tau \notin Y^n(\tau, \nu), \) and \( \delta_\tau(f - S^n_H(f)) = \delta_\tau(f), \)
\( \tau \in Y^n(\tau, \nu), \) it follows from Theorem 3.2 that
\[
\left\| f - S^n_H(f) \right\|_{\tau(2)} \leq C(\theta, m) \left\{ \prod_{j=1}^{m} 2^{-s_j} 2^{s_j \left( \frac{1}{r_j} - \frac{1}{\tau_j} \right)} \right\}_{\tau(1), \tau(2)}
\]
For the proof of the first part it remains to apply Holder’s inequality with exponents \( \beta_j = \tau_j / \theta_j(2), \) \( 1 / \beta_j + 1 / \beta_j' = 1, j = 1, \ldots, m, \) to the right side of (3.10). So
\[
\left\| \left\{ \prod_{j=1}^{m} 2^{-s_j} 2^{s_j \left( \frac{1}{r_j} - \frac{1}{\tau_j} \right)} \right\}_{\tau(1), \tau(2)} \left\| \begin{array}{c} \tau(1), \tau(2) \end{array} \right\|_{\tau(1), \tau(2)}
\]
\[
\leq \left\| \left\{ \prod_{j=1}^{m} 2^{-s_j} 2^{s_j \left( \frac{1}{r_j} - \frac{1}{\tau_j} \right)} \right\}_{\tau(1), \tau(2)} \left\| \begin{array}{c} \tau(1), \tau(2) \end{array} \right\|_{\tau(1), \tau(2)}
\]
For the proof of the second part use Jensen’s inequality (see [18], Lemma 3.3.3). So
\[
\left\| \left\{ \prod_{j=1}^{m} 2^{-s_j} 2^{s_j \left( \frac{1}{r_j} - \frac{1}{\tau_j} \right)} \right\}_{\tau(1), \tau(2)} \left\| \begin{array}{c} \tau(1), \tau(2) \end{array} \right\|_{\tau(1), \tau(2)}
\]
\[
\leq \left\{ \sup_{\tau(1), \tau(2)} \left\{ \prod_{j=1}^{m} 2^{-s_j} 2^{s_j \left( \frac{1}{r_j} - \frac{1}{\tau_j} \right)} \right\}_{\tau(1), \tau(2)} \right\} \times \left\{ \sup_{\tau(1), \tau(2)} \left\{ \prod_{j=1}^{m} 2^{-s_j} 2^{s_j \left( \frac{1}{r_j} - \frac{1}{\tau_j} \right)} \right\}_{\tau(1), \tau(2)} \right\}
\]
The inequalities (3.10), (3.11) and (3.12) prove the theorem.

\[\Box\]

**Theorem 3.5.** Let \( \tilde{\theta}^{(1)} = (\theta_1^{(1)}, \ldots, \theta_m^{(1)}), \tilde{\theta}^{(2)} = (\theta_1^{(2)}, \ldots, \theta_m^{(2)}), \tilde{\tau} = (\tau_1, \ldots, \tau_m), \tilde{\mu} = (\mu_1, \ldots, \mu_m), \)
\( \tilde{q} = (q_1, \ldots, q_m), \tilde{r} = (r_1, \ldots, r_m), \tilde{\gamma} = (\gamma_1, \ldots, \gamma_m), \gamma_j = \gamma_j^{(1)} / \gamma_j^{(2)}, \)
\( j = 1, \ldots, m, \) and \( \gamma_j^{(1)} < \gamma_j^{(2)}, j = v + 1, \ldots, m. \) Assume that \( 1 \leq \theta_j^{(1)}, \theta_j^{(2)}, \tau_j < +\infty, 1 < p_j < q_j < +\infty, \)
for each function \( f \), 0 < r_1 + q_1 - 1/p_1 = \cdots = r_v + q_v - 1/p_v < r_{v+1} + q_{v+1} - 1/p_{v+1} \leq \cdots \leq r_m + q_m - 1/p_m. \) Then the following relation holds:

\[
S^\gamma_n (S^p_{\rho, \beta(1), \tau} B)_{\tilde{q}, \tilde{\theta}(2)} \approx \begin{cases} 
2^{-n(r_1 + \frac{1}{p_1} - \frac{1}{p_f}) \cdot n^{\gamma_j}}, & \theta_j^{(2)} < \tau_j, \ j = 1, \ldots, m, \\
2^{-n(r_1 + \frac{1}{p_1} - \frac{1}{p_f})}, & \tau_j \leq \theta_j^{(2)}, \ j = 1, \ldots, m.
\end{cases}
\]

**Proof.** Assume that \( \theta_j^{(2)} < \tau_j, \ j = 1, \ldots, m. \) Then by Theorem 3.4, we obtain

\[
S^\gamma_n (S^p_{\rho, \beta(1), \tau} B)_{\tilde{q}, \tilde{\theta}(2)} \leq C(\theta, m, q) \cdot \left\{ \left\{ \sum_{j=1}^{m} 2^{-s_j(r_1 + \frac{1}{p_1} - \frac{1}{p_f})} \right\}_{\tilde{s} \in Y = (\tilde{\gamma}, \tilde{\nu})} \right\}_{l_1} = C(\theta, m, q) \cdot \left\{ \left\{ \sum_{j=1}^{m} 2^{-(r_1 + \frac{1}{p_1} - \frac{1}{p_f}) s_j} \right\}_{\tilde{s} \in Y = (\tilde{\gamma}, \tilde{\nu})} \right\}_{l_1},
\]

where \( \tilde{\epsilon} = (\epsilon_1, \ldots, \epsilon_m), \epsilon_j = \theta_j^{(2)} \cdot \beta'_j, \beta'_j = \beta_j / \beta_j - 1, \beta_j = \tau_j / \theta_j^{(2)}, \ j = 1, \ldots, m. \)

Setting in Lemma 2.2, \( \alpha = r_1 + q_1 - 1/p_1, \) and bearing in mind that \( 1/\epsilon_j = 1/\theta_j^{(2)} - 1/\tau_j, \ j = 1, \ldots, m, \) by (13.3), we obtain

\[
S^\gamma_n (S^p_{\rho, \beta(1), \tau} B)_{\tilde{q}, \tilde{\theta}(2)} \leq C(\theta, m, q, \rho) \cdot 2^{-n(r_1 + \frac{1}{p_1} - \frac{1}{p_f}) \cdot n^{\gamma_j}} \sum_{j=1}^{m} 1 = C(\theta, m, q, \rho) \cdot 2^{-n(r_1 + \frac{1}{p_1} - \frac{1}{p_f}) \cdot n^{\gamma_j}},
\]

for each function \( f \in S^p_{\rho, \beta(1), \tau} B, \theta_j^{(2)} < \tau_j, \ j = 1, \ldots, m. \)

Consider now the case of \( \tau_j \leq \theta_j^{(2)}, \ j = 1, \ldots, m. \) By the second part of Theorem 3.4 and using the conditions \( r_1 + q_1 - 1/p_1 > 0, \gamma'_j \leq \gamma_j, \ j = 1, \ldots, m, \) we get

\[
S^\gamma_n (S^p_{\rho, \beta(1), \tau} B)_{\tilde{q}, \tilde{\theta}(2)} \leq C(\theta, m, q) \cdot \sup_{\tilde{z} \in Z^n_{s, \tilde{\gamma}}} 2^{-\sum_{j=1}^{m} s_j(r_1 + \frac{1}{p_1} - \frac{1}{p_f})} \leq C(\theta, r, m, q) \cdot 2^{-n(r_1 + \frac{1}{p_1} - \frac{1}{p_f})}.
\]

The proof of the upper estimate is thus complete.

We now proved the lower estimate. Let \( \chi_v(n) = \{ \tilde{s}_v = (s_1, \ldots, s_v, 1, \ldots, 1) \in Z^n_{s, \tilde{\gamma}} : (s_1, \tilde{\gamma}_j) = n \} \) and \( \chi_A \) be the characteristic function of the set \( A. \)

Assume that \( \theta_j^{(2)} < \tau_j, \ j = 1, \ldots, m. \) Consider the function

\[
f_0(\tilde{s}) = n^{-\frac{\tilde{k}}{\tilde{j}+\frac{1}{2}}} \sum_{(\tilde{s}_v, \tilde{\gamma}) = n}^m \prod_{l=1}^{m} 2^{-s_l(r_1 + \frac{1}{p_1})} \sum_{k \in p(\tilde{s}_v)} e^{i(\tilde{k}, \tilde{s})}.
\]
Since \( f_0 \) is continuous, it follows that \( f_0 \in L^p_{\rho, \theta(1)}(I^m) \). By the well-known relation
\[
\left\| \sum_{k \in \rho(s)} e^{i(k, x)} \right\|_{\rho, \theta}^* \lesssim \prod_{j=1}^m 2^{s_j(1 - \frac{1}{p_j})}, \quad 1 < p_j < +\infty,
\]  
(3.14)
and by Lemma 2.3, we obtain
\[
I_4(n) = \left\| \left\{ \prod_{j=1}^m 2^{s_j(r_j)} \| \delta_s(f_0) \|_{\rho, \theta(1)}^* \right\}_{s \in \mathcal{K}(n)} \right\|_{L^1} 
= n^{-\frac{\sum_{j=1}^m s_j}{2}} \left\| \left\{ \prod_{j=1}^m 2^{s_j(r_j)} \prod_{j=1}^m 2^{-s_j(r_j + 1 - \frac{1}{p_j})} \sum_{k \in \rho(s)} e^{i(k, x)} \right\}_{s \in \mathcal{K}(n)} \right\|_{L^1} 
\leq C(p, \theta) \cdot n^{-\frac{\sum_{j=1}^m s_j}{2}} \left\| \left\{ \chi_{\mathcal{K}_s(n)}(s) \right\}_{s \in \mathcal{K}(n)} \right\|_{L^1} \leq C_0(\tau, q, p, m, r).
\]
Hence \( C_0 \cdot f_0 \in S^p_{\rho, \theta(1), \tau, B} \). Now, we have \( S^\gamma_n(f_0, x) = 0, \ x \in I^m \), and therefore
\[
\left\| f_0 - S^\gamma_n(f_0) \right\|_{q, \theta(2)} = \left\| f_0 \right\|_{q, \theta(2)}.
\]  
(3.15)
We shall select \( \lambda_j \) such that \( q_j < \lambda_j, j = 1, \ldots, m \). Then, by Theorem 3.3 in view of relation (3.14) and Lemma 2.4, it follows by (3.15) that
\[
\left\| f_0 - S^\gamma_n(f_0) \right\|_{q, \theta(2)} \geq C(q, \theta, m) \cdot \left\| \left\{ \prod_{j=1}^m 2^{s_j(r_j + 1 - \frac{1}{p_j})} \| \delta_s(f_0) \|_{\lambda, \theta(1)}^* \right\}_{s \in \mathcal{K}(n)} \right\|_{L^1(\gamma)} 
\geq C(q, \theta, m) \cdot n^{-\frac{\sum_{j=1}^m s_j}{2}} \left\| \left\{ \prod_{j=1}^m 2^{-s_j(r_j + 1 - \frac{1}{p_j})} \right\}_{s \in \mathcal{K}(n)} \right\|_{L^1(\gamma)} 
\geq C(r, \theta, q, p, \tau) \cdot 2^{-n(r_1 + 1 - \frac{1}{p_1}) \cdot \sum_{j=1}^m \gamma_j \cdot \frac{1}{p_j}} \cdot n^{-\frac{\sum_{j=1}^m \gamma_j}{p}} 
\geq C(r, \theta, q, p, \tau) \cdot 2^{-n(r_1 + 1 - \frac{1}{p_1}) \cdot \sum_{j=1}^m \gamma_j \cdot \frac{1}{p}} \cdot n^{-\frac{\sum_{j=1}^m \gamma_j}{p}}.
\]  
(3.16)
for \( n > 2\sum_{j=1}^m \gamma_j \). This proves the lower estimate for \( \theta_j^{(2)} < \tau_j, j = 1, \ldots, m \).

Assume now that \( \tau_j < \theta_j^{(2)}, j = 1, \ldots, m \) and let \( \bar{s} = (\bar{s}_1, \ldots, \bar{s}_v, 1, \ldots, 1) \in \mathbb{Z}_+^m \) be a vector such that \( \langle \bar{s}, \gamma' \rangle = n \). Consider the function
\[
f_1(\bar{x}) = 2^{-\sum_{j=1}^m \bar{s}_j(r_j + 1 - \frac{1}{p_j})} \sum_{k \in \rho(\bar{s})} e^{i(k, \bar{x})}.
\]
Then \( f_1 \in L^*_p(1^m) \) and by relation (3.14), we obtain

\[
\left\| \sum_{j=1}^{m} 2^{i_j} \| \delta_j(f_1) \|_{p, q\theta(1)}^* \right\|_{L^1} = \sum_{j=1}^{m} 2^{i_j} \| \delta_j(f_1) \|_{p, q\theta(1)}^* \\
= \prod_{j=1}^{m} 2^{i_j} \cdot 2^{-\sum_{j=1}^{m} \bar{e}_j(r_j+1-\frac{1}{r})} \| \sum_{k \in \mathcal{S}(\delta)} \varepsilon_j(k, \bar{x}) \|_{p, q\theta(1)}^* \leq C_1(p, q, \theta, r).
\]

Hence \( C_1 \cdot f_1 \in S^p_{p, \theta(\cdot), \pi} B \). By the definition of the partial sum and the function \( f_1 \) and relation (3.14), it follows that

\[
\left\| f_1 - S_{\theta}\left(f_1\right) \right\|_{p, q\theta(2)}^* = 2^{\sum_{j=1}^{m} \bar{e}_j(r_j+1-\frac{1}{r})} \cdot 2^{-\sum_{j=1}^{m} \bar{e}_j(r_j+1-\frac{1}{r})} \\
\geq C(q, m, \theta) \cdot \prod_{j=1}^{m} 2^{\bar{e}_j(r_j+1-\frac{1}{r})} \cdot 2^{-\sum_{j=1}^{m} \bar{e}_j(r_j+1-\frac{1}{r})} \\
= C(q, m, \theta) \cdot 2^{-(r_1+\frac{1}{p_1} - \frac{1}{p_1}) \sum_{j=1}^{m} \bar{e}_j(r_j+1-\frac{1}{r})}.
\]

We have thus proved the lower estimate for \( \tau_j < \theta_j^{(2)}, j = 1, \cdots, m \).

Hence inequalities (3.16) and (3.17) yield the lower estimates of Theorem 3.5. This completes the proof of this theorem.

**Remark 3.1.** The cases \( \theta_j^{(1)} = \theta_j^{(2)} = \theta_j, j = 1, \cdots, m \), of Theorems 3.2, 3.4 and 3.5 were proved in [1]. In [1], Theorem 3.1 is proved using the inequalities of different metrics for trigonometric polynomials. There Theorem 3.5 is proved with the additional conditions \( 1/p_1 - 1/q_1 = \cdots = 1/p_v - 1/q_v, r_1(1/p_j - 1/q_j) < r_j(1/p_1 - 1/q_1), j = v+1, \cdots, m, \gamma_j' = r_j/r_1, j = 1, \cdots, m \).

**References**


