Approximation Properties by $q$-Durrmeyer-Stancu Operators

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Abstract. In this paper, we are dealing with $q$-Bernstein-Durrmeyer-Stancu operators. Firstly, we have estimated moments of these operators. Then we have discussed some approximation properties and asymptotic formulas. We have obtained better estimations by using King type approach and given statistical convergence for the operators.

Key Words: $q$-Durrmeyer operator, $q$-Jackson integral, $q$-Beta function.

AMS Subject Classifications: 41A25, 41A35

1 Introduction

We first mention some notations of $q$-calculus. Throughout the present article $q$ is a real number satisfying the inequality $0 < q \leq 1$. For $n \in \mathbb{N}$,

\[
[n]_q = [n] := \begin{cases} \frac{(1-q^n)}{(1-q)}, & q \neq 1, \\ n, & q = 1, \end{cases} \\
[n]_q! = [n]! := \begin{cases} [n][n-1]\cdots[1], & n \geq 1, \\ 1, & n = 0, \end{cases}
\]

and

\[
(1+x)_q^n := \begin{cases} \prod_{j=0}^{n-1} (1+q^jx), & n = 1,2,\cdots, \\ 1, & n = 0. \end{cases}
\]

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For the integers $n, k, n \geq k \geq 0$, the $q$-polynomial coefficients are defined by

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = \frac{n!}{k! (n-k)!}.$$

The $q$-analogue of integration, discovered by Thomae [16] and Jackson [10] in the interval $[0,a]$, is defined by

$$\int_0^a f(t)d_q t := a(1-q) \sum_{n=0}^{\infty} f(aq^n) q^n, \quad 0 < q \leq 1 \quad \text{and} \quad a > 0.$$

In [3], the two $q$-Gamma functions are defined as

$$\Gamma_q(x) = \int_0^{1/1-q} t^{x-1} E_q(-qt)d_q t \quad \text{and} \quad \gamma_q^A(x) = \int_0^{\infty/A(1-q)} t^{x-1} e_q(-t)d_q t. \quad (1.1)$$

There are two $q$-analogues of the exponential function $e^x$, see [11],

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!} = \frac{1}{(1-(1-q)x)^\infty_q}, \quad |x| < \frac{1}{1-q}, \quad |q| < 1,$$

and

$$E_q(x) = \sum_{k=0}^{\infty} q^{(k-1)/2} \frac{x^k}{[k]!} = (1+(1-q)x)^\infty_q, \quad |q| < 1.$$

By Thomae [16] and Jackson [10], it was shown that the $q$-Beta function defined by the usual formula

$$B_q(t,s) = \frac{\Gamma_q(s) \Gamma_q(t)}{\Gamma_q(s+t)}$$

has the $q$-integral representation, which is a $q$-analogue of Euler’s formula:

$$B_q(t,s) = \int_0^1 x^{t-1} (1-qx)^{s-1} d_q x, \quad t,s > 0. \quad (1.2)$$

Due to the importance of polynomials, a variety of their generalizations and related topics have been studied (see [5] and [15]). Recently, an intensive research has been conducted on operators based on $q$-integers. In 1997, G. M. Phillips [14] proposed the following $q$-analogue of the well-known Bernstein polynomials, for each positive integer $n$ and $f \in C[0,1]$, are defined as

$$B_{n,q}(f;x) := \sum_{k=0}^{n} f\left( \left[ \begin{array}{c} k \\ n \end{array} \right] \right) p_{n,k}(q;x), \quad (1.3)$$

where

$$p_{n,k}(q;x) = \left[ \begin{array}{c} n \\ k \end{array} \right] \frac{x^k}{\prod_{s=0}^{n-k-1} (1-q^s x)}. \quad (1.4)$$
While for $q = 1$ these polynomials coincide with the classical ones, for $q \neq 1$ we obtain new polynomials possessing interesting properties. T. Trif [17] introduced the $q$-Meyer-König and Zeller operators for each positive integer $n$ and $f \in C[0,1]$. Like the classical operators, the $q$-Bernstein operators and the $q$-Meyer-König and Zeller operators are shared some good properties such as the shape-preserving properties and monotonicity for convex function.

Very recently, V. Gupta (see [1,7] and [8]) introduced and studied approximation properties of $q$-Durrmeyer operators. V. Gupta and W. Heping [9] introduced the $q$-Durrmeyer type operators and studied estimation of the rate of convergence for continuous functions in terms of modulus of continuity. Z. Finta and V. Gupta [6] studied some direct local and global approximation theorems for the $q$-Durrmeyer operators $M_{n,q}$ for $0 < q < 1$. Some other analogues of the Bernstein-Durrmeyer operators related to the $q$-Bernstein operators, the $q$-König and Zeller operators for each positive integer $n$ and $f \in C[0,1]$, shared some good properties such as the shape-preserving properties and monotonicity for convex function.

In [13], N. I. Mahmudov and V. Gupta defined Bernstein-Durrmeyer-Stancu operators as

$$M_{n,q}(f, x) = \left[ n + 1 \right] \sum_{k=0}^{n} p_{n,k}(q;x) \int_{0}^{1} p_{n,k-1}(q;qt) f(t) dt + f(0) p_{n,0}(q;x),$$  \hspace{1cm} (1.5)

where $f \in C[0,1]$, $n = 1, 2, \cdots$, $0 < q < 1$ and $p_{n,k}(q;x)$ is given by (1.4).

They obtained the moments as

$$M_{n,q}(1, x) = 1, \hspace{0.5cm} M_{n,q}(t, x) = \frac{|n|}{|n+2|} x,$$  \hspace{1cm} (1.6a)

$$M_{n,q}(t^2, x) = \frac{(1+q)|n|}{|n+2|} x + \frac{q|n|(|n-1|)}{|n+2|} x^2.$$  \hspace{1cm} (1.6b)

In [13], N. I. Mahmudov and V. Gupta defined Bernstein-Durrmeyer-Stancu operators as

$$U_n^{(\alpha, \beta)}(f, x) = (n-1) \sum_{k=1}^{n-1} p_{n,k}(z) \int_{0}^{1} p_{n-2,k-1}(t) f\left( \frac{nt + \alpha}{n+\beta} \right) dt + p_{n,0}(z) f\left( \frac{\alpha}{n+\beta} \right) + p_{n,n}(z) f\left( \frac{n + \alpha}{n + \beta} \right).$$  \hspace{1cm} (1.7)

$\alpha$ and $\beta$ are two given real parameters satisfying the conditions $0 \leq \alpha \leq \beta$. $f : C[0,1] \to C[0,1]$ and $p_{n,k}(z) = \binom{n}{k} z^k (1-z)^{n-k}$ and $p_{-1,k}(z) = 0$. They obtained the moments as

$$U_n^{(\alpha, \beta)}(1, x) = 1, \hspace{0.5cm} U_n^{(\alpha, \beta)}(t, x) = \frac{n x + \alpha}{n + \beta},$$  \hspace{1cm} (1.8a)

$$U_n^{(\alpha, \beta)}(t^2, x) = \frac{n^2 (n-1)}{(n+\beta)^2 (n+1)} x^2 + \frac{1}{(n+\beta)^2} \left( \frac{2n^2}{n+1} + 2n \alpha \right) x + \frac{\alpha^2}{(n+\beta)^2}.$$  \hspace{1cm} (1.8b)
In this paper, we obtain approximation results for the $q$-Bernstein-Durrmeyer-Stancu operators defined by

$$U_{n,q}^{(\alpha,\beta)}(f,x) = [n+1] \sum_{k=1}^{n} q^{1-k} p_{n,k}(q;x) \int_{0}^{1} p_{n,k-1}(q;qt) f \left( \frac{[n]t+\alpha}{[n]+\beta} \right) d_q t$$

where $p_{n,k}(q;x)$ is given in (1.4).

$$+ p_{n,0}(q;x) f \left( \frac{\alpha}{[n]+\beta} \right), \quad (1.9)$$

where $p_{n,k}(q;x)$ is given in (1.4).

2 Moments of $U_{n,q}^{(\alpha,\beta)}$ operators

In this section, we will calculate $U_{n,q}^{(\alpha,\beta)}(t^i, x), i = 0, 1, 2$.

Lemma 2.1. For $x \in [0,1]$ and $n \in \mathbb{N}$, we have

$$U_{n,q}^{(\alpha,\beta)}(1,x) = 1, \quad U_{n,q}^{(\alpha,\beta)}(t,x) = \frac{[n]^2}{[n+2]([n]+\beta)} x + \frac{\alpha}{[n]+\beta} \quad (2.1)$$

and

$$U_{n,q}^{(\alpha,\beta)}(t^2, x) = \frac{q^2 [n] [n-1]}{[n+2]([n]+\beta)} x^2 + \frac{2 [n]^3 + 2 \alpha [n]^2 [n+3]}{[n+2]([n]+\beta)^2} x + \left( \frac{\alpha}{[n]+\beta} \right)^2. \quad (2.2)$$

Proof. For $i = 0$, we get

$$U_{n,q}^{(\alpha,\beta)}(1,x) = [n+1] \sum_{k=1}^{n} q^{1-k} p_{n,k}(q;x) \int_{0}^{1} p_{n,k-1}(q;qt) d_q t + p_{n,0}(q;0)$$

$$= M_{n,q}(1,x) = 1.$$

For $i = 1$, we have

$$U_{n,q}^{(\alpha,\beta)}(t,x) = [n+1] \sum_{k=1}^{n} q^{1-k} p_{n,k}(q;x) \int_{0}^{1} p_{n,k-1}(q;qt) \frac{[n]t+\alpha}{[n]+\beta} d_q t + p_{n,0}(q;0) \frac{\alpha}{[n]+\beta}$$

$$= \frac{[n]}{[n]+\beta} M_{n,q}(t,x) + \frac{\alpha}{[n]+\beta} M_{n,q}(1,x)$$

$$= \frac{[n]^2}{[n+2]([n]+\beta)} x + \frac{\alpha}{[n]+\beta}.$$
Finally, for $i = 2$, we see

$$
U_{n,q}^{(a,β)}(t^2,x) = \left[ n + 1 \right] \sum_{k=1}^{n} q^{1-k} p_{n,k}(q;x) \int_{0}^{1} p_{n,k-1}(q;qt) \left( \frac{|n|t + α}{|n| + β} \right) d_q t + p_{n,0}(q;0) \left( \frac{α}{|n| + β} \right)^2 
$$

$$
= \left( \frac{|n|}{|n| + β} \right)^2 M_{n,q}(t^2,x) + \frac{2α |n|}{(|n| + β)^2} M_{n,q}(t,x) + \left( \frac{α}{|n| + β} \right)^2 M_{n,q}(1,x) 
$$

$$
= \frac{q |n|^3 (|n| - 1)}{|n| + β)^2} x^2 + \frac{(1+q) |n|^3}{|n| + β)^2} x^2 
$$

$$
+ \frac{2α |n|^2}{|n| + β)^2} x + \left( \frac{α}{|n| + β} \right)^2 
$$

$$
= \frac{q |n|^3 (|n| - 1)}{|n| + β)^2} x^2 + \left[ \frac{2 |n|^3 + 2α |n|^2 |n| + 3}{|n| + β)^2} x + \left( \frac{α}{|n| + β} \right)^2 \right]. 
$$

This completes the proof of theorem. \( \square \)

**Lemma 2.2.** For $x \in [0,1]$ and $n \in \mathbb{N}$, we get

$$
\mu_{n,1}^{(a,β)}(x) = U_{n,q}^{(a,β)}(t - x,x) = \frac{|n|^2 - |n| + 2}{|n| + β} x + \frac{α}{|n| + β}, 
$$

$$
(2.3a)
$$

$$
\mu_{n,2}^{(a,β)}(x) = U_{n,q}^{(a,β)}((t - x)^2,x) = \left( \frac{q |n|^3 (|n| - 2 |n|^3 |n| + 3)}{|n| + β)^2} + 1 \right) x^2 
$$

$$
+ \left[ \frac{2 |n|^3 + 2α |n|^2 |n| + 3}{|n| + β)^2} x + \left( \frac{α}{|n| + β} \right)^2 \right]. 
$$

$$
(2.3b)
$$

**Remark 2.1.** For all $m,n \in \mathbb{N} \cup \{0\}$, $0 ≤ α ≤ β$ and $x \in [0,1]$, we have

$$
U_{n,q}^{(a,β)}(t^m,x) = \sum_{j=0}^{m} \binom{m}{j} \frac{|n|^j α^{m-j}}{(|n| + β)^m} M_{n,q}(t^j,x) 
$$

$$
3 \quad \text{Local approximation}
$$

For $η > 0$ and $C^2[0,1] = \{ g \in C[0,1] : g', g'' \in C[0,1] \}$, the Peetre’s $K$-functional is defined as

$$
K_2(f,η) = \inf \{ |f - g| + η |g''| : g \in C^2 \}, 
$$

$$
(3.1)
$$

where $| \cdot |$ is the uniform norm on $C[0,1]$. There exists a positive constant $L > 0$ such that

$$
K_2(f,η) ≤ Lω_2(f,√η), 
$$

$$
(3.2)
$$

where the second order modulus of smoothness for $f \in C[0,1]$ is defined as

$$
ω_2(f,√η) = \sup_{0 < h ≤ √η} \sup_{x + h \in [0,1]} |f(x + 2h) - 2f(x + h) + f(x)|. 
$$
We define the usual modulus of continuity for $f \in C[0,1]$ as
$$\omega(f,\eta) = \sup_{0<h \leq \eta} \sup_{x+h, x \in [0,1]} |f(x+h) - f(x)|.$$  

Now we state our next main result.

**Theorem 3.1.** Let $f \in C[0,1]$. Then for every $x \in [0,1]$, there exists a constant $M > 0$ such that
$$|U^{(\alpha,\beta)}_{n,q}(f,x) - f(x)| \leq M\omega_2 \left( f, \sqrt{\mu^{\alpha,\beta,q}_{n,2} + (\mu^{\alpha,\beta,q}_{n,1})^2} \right) + \omega(f,\mu^{\alpha,\beta,q}_{n,1}),$$

(3.3)

where $\mu^{\alpha,\beta,q}_{n,1}$ and $\mu^{\alpha,\beta,q}_{n,2}$ are given (2.3a) and (2.3b), respectively.

**Proof.** Let $g \in C^2[0,1]$ and $x,t \in [0,1]$. By Taylor’s expression, we have
$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.$$  

(3.4)

We are introducing the auxiliary operators $\tilde{U}^{(\alpha,\beta)}_{n,q}$ as follows
$$\tilde{U}^{(\alpha,\beta)}_{n,q}(f,x) = U^{(\alpha,\beta)}_{n,q}(f,x) + f(x) - f \left( x + \frac{n^2}{n+1} x + \frac{\alpha}{[n]+\beta} \right).$$

Now one can see
$$\tilde{U}^{(\alpha,\beta)}_{n,q}(t-x,x) = 0.$$

Applying $\tilde{U}^{(\alpha,\beta)}_{n,q}$ on both side of (3.4), we get
$$\tilde{U}^{(\alpha,\beta)}_{n,q}(g,x) - g(x) = g'(x)\tilde{U}^{(\alpha,\beta)}_{n,q}(t-x,x) + \tilde{U}^{(\alpha,\beta)}_{n,q} \left( \int_x^t (t-u)g''(u)du, x \right)$$
$$= U^{(\alpha,\beta)}_{n,q} \left( \int_x^t (t-u)g''(u)du, x \right) + \int_x^{A^{\alpha,\beta}_{n,q}(x)} (A^{\alpha,\beta}_{n,q}(x) - u)g''(u)du,$$

where
$$A^{\alpha,\beta}_{n,q}(x) = \frac{n^2}{n+1} x + \frac{\alpha}{[n]+\beta}.$$

On the other hand, we calculate
$$\left| \int_x^t (t-u)g''(u)du \right| \leq (t-x)^2 |g''|$$

and
$$\left| \int_x^{A^{\alpha,\beta}_{n,q}(x)} (A^{\alpha,\beta}_{n,q}(x) - u)g''(u)du \right| \leq \left( \frac{n^2 - [n]+\beta}{[n]+\beta} x + \frac{\alpha}{[n]+\beta} \right)^2 |g''|$$
$$= \left( \mu^{\alpha,\beta,q}_{n,1}(x) \right)^2 |g''|.$$
Notice that

\[
\left| \tilde{U}^{(a,b)}_{n,q}(g,x) - g(x) \right| \leq U^{(a,b)}_{n,q}(t-x)^2 \left| g'' \right| + (\mu_{n,1}^{a,bq}(x))^2 \left| g'' \right|
\]

\[
= \left| g'' \right| (\mu_{n,2}^{a,bq}(x) + (\mu_{n,1}^{a,bq}(x))^2).
\]

Observe that

\[
\left| U^{(a,b)}_{n,q}(f,g,x) - f(x) \right| \leq \left| U^{(a,b)}_{n,q}(f-g,g,x) - (f-g)(x) \right| + \left| U^{(a,b)}_{n,q}(g,x) - g(x) \right|
\]

\[
\leq |f-g| + \left| g'' \right| (\mu_{n,1}^{a,bq}(x) + (\mu_{n,1}^{a,bq}(x))^2) + (f,\mu_{n,1}^{a,bq}(x)).
\]

Now, taking infimum on the right-hand side over all \( g \in C^2[0,1] \) and using (3.2), we get

\[
\left| U^{(a,b)}_{n,q}(f,x) - f(x) \right| \leq 4K2 \left( f,\mu_{n,2}^{a,bq}(x) + (\mu_{n,1}^{a,bq}(x))^2 + (f,\mu_{n,1}^{a,bq}(x)) \right)
\]

\[
\leq 4M2 \left( f,\mu_{n,2}^{a,bq}(x) + (\mu_{n,1}^{a,bq}(x))^2 + (f,\mu_{n,1}^{a,bq}(x)) \right).
\]

This completes the proof of the theorem.

\[\square\]

4 Better estimation

To make the convergence faster J. P. King [12] proposed an approach to modify the classical Bernstein polynomials, so that this sequence preserves two test functions \( x \) and \( x^2 \). After this, several researchers have studied many approximating operators.

For this purpose we will modify the operators which preserve the constant as well as linear functions, as the modification of \( U^{(a,b)}_{n,q}(f,x) \) as follows:

\[
*U^{(a,b)}_{n,q}(f,x) = [n+1]^n \sum_{k=1}^n q^{1-k} p_{n,k}(q;r_n(x)) \int_0^1 p_{n,k-1}(q;qt) f\left( \frac{n t + q}{n + \beta} \right) dt
\]

\[
+ p_{n,0}(q;r_n(x)) f\left( \frac{\alpha}{n + \beta} \right), \tag{4.1}
\]

where

\[
r_n(x) = \frac{n+2}{n^2} \left( \frac{n}{n + \beta} \right) \left( x - \frac{\alpha}{n + \beta} \right) \quad \text{and} \quad x \in I_n = \left[ \frac{\alpha}{n + \beta}, \frac{n^2 + a n + 2}{n + 2} \right].
\]

Lemma 4.1. For each \( x \in I_n \), we have

\[
*U^{(a,b)}_{n,q}(1,x) = 1, \quad *U^{(a,b)}_{n,q}(t,x) = x, \tag{4.2a}
\]

\[
*U^{(a,b)}_{n,q}(t^2,x) = \frac{q^2 n^2 - n - 1}{n n + 3} x^2 + \frac{2 n^2 + 2 a q (n - 1 + n + 2) + 2 a}{n (n + 1) (n + \beta)} x
\]

\[
- \frac{2 a n^2 + a^2 q (n - 1 + n + 2) + a^2}{n (n + 3) (n + \beta)} x. \tag{4.2b}
\]
Lemma 4.2. For each \( x \in I_n \), the following equalities hold
\[
\begin{align*}
\mu_{n,1}^\alpha(x) &= U_{n,q}^{\alpha,\beta}(t-x,x) = 0, \\
\mu_{n,2}^\alpha(x) &= U_{n,q}^{\alpha,\beta}((t-x)^2,x) = -
\frac{q[n-1]+q[n+2]+1}{[n][n+3]} x^2 \\
&\quad + \frac{2[n]^2+2nq([n-1]+[n+2])+2n}{[n][n+3]([n]+\beta)} x \\
&\quad - \frac{2n[n]^2+nq([n-1]+[n+2])+\alpha^2}{[n][n+3]([n]+\beta)^2},
\end{align*}
\]
(4.3a)
\[
\begin{align*}
\mu_{n,1}^\alpha(x) &= U_{n,q}^{\alpha,\beta}(t-x,x) = 0, \\
\mu_{n,2}^\alpha(x) &= U_{n,q}^{\alpha,\beta}((t-x)^2,x) = -
\frac{q[n-1]+q[n+2]+1}{[n][n+3]} x^2 \\
&\quad + \frac{2[n]^2+2nq([n-1]+[n+2])+2n}{[n][n+3]([n]+\beta)} x \\
&\quad - \frac{2n[n]^2+nq([n-1]+[n+2])+\alpha^2}{[n][n+3]([n]+\beta)^2}. \\
\end{align*}
\]
(4.3b)

Theorem 4.1. Let \( f \in C(I_n) \), \( x \in I_n \) and \( 0 < q < 1 \). Then for \( n \in \mathbb{N} \), there exists a constant \( L > 0 \) such that
\[
\left| \left| U_{n,q}^{\alpha,\beta}(f,x) - f(x) \right| \right| \leq L \omega_2\left( f, \sqrt{\mu_{n,2}^\alpha(x)} \right).
\]

Proof. Let \( g \in C(I_n) \) and \( t, x \in I_n \). By Taylor's expansion, we have
\[
g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.
\]
(4.4)

Applying \( U_{n,q}^{\alpha,\beta} \) to (4.4), we get
\[
U_{n,q}^{\alpha,\beta}(g,x) - g(x) = g'(x)U_{n,q}^{\alpha,\beta}(t-x,x) + U_{n,q}^{\alpha,\beta} \left( \int_x^t (t-u)g''(u)du,x \right).
\]
Obviously, we see
\[
\left| \left| U_{n,q}^{\alpha,\beta}(g,x) - g(x) \right| \right| \leq \omega_2^\alpha \omega_2^{\alpha,\beta,\beta} |g''|.
\]
Since
\[
\left| \left| U_{n,q}^{\alpha,\beta}(f,x) - f(x) \right| \right| \leq |f|,
\]
we obtain
\[
\left| \left| U_{n,q}^{\alpha,\beta}(f,x) - f(x) \right| \right| \leq \left| \left| U_{n,q}^{\alpha,\beta}(f-g,x) - (f-g)(x) \right| \right| + \left| \left| U_{n,q}^{\alpha,\beta}(g,x) - g(x) \right| \right| \\
\leq 2 |f-g| + \omega_2^\alpha \omega_2^{\alpha,\beta,\beta} |g''|.
\]
Taking infimum overall \( g \in C(I_n) \), we have
\[
\left| \left| U_{n,q}^{\alpha,\beta}(f,x) - f(x) \right| \right| \leq K_2(f, \mu_{n,2}^\alpha).
\]
In view of (3.2), we have
\[
\left| \left| U_{n,q}^{\alpha,\beta}(f,x) - f(x) \right| \right| \leq L \omega_2\left( f, \sqrt{\mu_{n,2}^\alpha} \right).
\]
Thus, the theorem is proved. \( \square \)
5 Statistical convergence

A sequence \((x_n)_n\) is said to be statistically convergent to a number \(l\), denoted by \(st\-\lim_n x_n = l\) if for every \(\varepsilon > 0\),

\[
\delta \{ n \in \mathbb{N} : |x_n - l| \geq \varepsilon \} = 0,
\]

where

\[
\delta(K) = \lim_n \frac{1}{n} \sum_{j=1}^{n} \chi_K(j)
\]

is the natural density of set \(K \subseteq \mathbb{N}\) and \(\chi_K\) is the characteristic function of \(K\). For instance \(x_n = \left\{ \begin{array}{ll} \log n, & n \in \{10^k, k \in \mathbb{N}\} \\ 1, & \text{otherwise} \end{array} \right.\), series \((x_n)_{n \in \mathbb{N}}\) converge statistically, but \(\lim_n x_n\) does not exist. We note that every convergent sequence is a statistical convergent, but converse need not to be true (details can be found in [4]).

As an application of the Bohman-Korovkin type theorem, we have the following theorem for our operators:

**Theorem 5.1.** Let \((q_n)_n\) be a sequence satisfying

\[
st\-\lim_n a_n = 1 \quad \text{and} \quad st\-\lim_n q^n_n = a, \quad a < 1, (5.1)
\]

then for any function \(f \in C[0,1]\), the operator \(U^{(\alpha, \beta)}_{n, q_n} f\) statistically converges to \(f\), that is

\[
st\-\lim_n \| U^{(\alpha, \beta)}_{n, q_n} f - f \| = 0.
\]

**Proof.** It is clear that

\[
st\-\lim_n \| U^{(\alpha, \beta)}_{n, q_n} (1, x) - 1 \| = 0. (5.2)
\]

Based on Lemma 2.1, we have

\[
\max_{x \in [0,1]} | U^{(\alpha, \beta)}_{n, q_n} (t, x) - x | = \max_{x \in [0,1]} \left| \left( \frac{n^2}{[n+2]_{q_n} ([n+\beta]_{q_n} + \beta)} - 1 \right) x + \frac{\alpha}{[n]_{q_n} + \beta} \right|
\]

\[
\leq \left( 1 - \frac{[n]_{q_n}}{[n+2]_{q_n} ([n]_{q_n} + \beta)} \right) + \frac{\alpha}{[n]_{q_n} + \beta}.
\]

Using the conditions (5.1), we get

\[
st\-\lim_n \left( 1 - \frac{[n]_{q_n}^2}{[n+2]_{q_n} ([n]_{q_n} + \beta)} \right) = 0 \quad \text{and} \quad st\-\lim_n \frac{\alpha}{[n]_{q_n} + \beta} = 0.
\]
For each $\varepsilon > 0$, we define the following sets:

\[
D := \left\{ n \in \mathbb{N} : \left| U_{n,q_n}^{(\alpha, \beta)}(t,x) - x \right| \geq \varepsilon \right\},
\]

\[
D_1 := \left\{ n \in \mathbb{N} : \left( 1 - \frac{[n]^2}{[n+2][n+3]([n] + \beta + 1)} \right) \geq \varepsilon \right\},
\]

\[
D_2 := \left\{ n \in \mathbb{N} : \frac{\alpha}{[n] + \beta + 1)} \geq \varepsilon \right\}.
\]

Thus, we obtain $D \subseteq D_1 \cup D_2$, i.e., $\delta(D) \leq \delta(D_1) + \delta(D_2) = 0$. Therefore,

\[
st - \lim_n \left\| U_{n,q_n}^{(\alpha, \beta)}(t,x) - x \right\| = 0. \quad (5.3)
\]

A similar calculation reveals that

\[
\max_{x \in [0,1]} \left| U_{n,q_n}^{(\alpha, \beta)}(t^2,x) - x^2 \right| = \max_{x \in [0,1]} \left| \left( \frac{q^2 [n]^3 [n-1]}{[n+2][n+3]([n] + \beta + 1)} - 1 \right) x^2 + \frac{2 [n]^3 + 2\alpha [n]^2 [n+1]}{[n+2][n+3]([n] + \beta + 1)} x + \left( \frac{\alpha}{[n] + \beta + 1)} \right)^2 \right| 
\]

\[
\leq 1 - \frac{q^2 [n]^3 [n-1]}{[n+2][n+3]([n] + \beta + 1)} + \frac{2 [n]^3 + 2\alpha [n]^2 [n+1]}{[n+2][n+3]([n] + \beta + 1)} + \left( \frac{\alpha}{[n] + \beta + 1)} \right)^2.
\]

Using the conditions (5.1), we get

\[
st - \lim_n \left( 1 - \frac{q^2 [n]^3 [n-1]}{[n+2][n+3]([n] + \beta + 1)} \right) = 0,
\]

\[
st - \lim_n \frac{2 [n]^3 + 2\alpha [n]^2 [n+1]}{[n+2][n+3]([n] + \beta + 1)} = 0,
\]

and

\[
st - \lim_n \left( \frac{\alpha}{[n] + \beta + 1)} \right)^2 = 0.
\]

For each $\varepsilon > 0$, we define the following sets:

\[
B := \left\{ n \in \mathbb{N} : \left| U_{n,q_n}^{(\alpha, \beta)}(t^2,x) - x^2 \right| \geq \varepsilon \right\},
\]

\[
B_1 := \left\{ n \in \mathbb{N} : 1 - \frac{q^2 [n]^3 [n-1]}{[n+2][n+3]([n] + \beta + 1)} \geq \varepsilon \right\},
\]

\[
B_2 := \left\{ n \in \mathbb{N} : \frac{2 [n]^3 + 2\alpha [n]^2 [n+1]}{[n+2][n+3]([n] + \beta + 1)} \geq \varepsilon \right\},
\]

\[
B_3 := \left\{ n \in \mathbb{N} : \left( \frac{\alpha}{[n] + \beta + 1)} \right)^2 \geq \varepsilon \right\}.
\]
Thus, we obtain \( B \subseteq B_1 \cup B_2 \cup B_3 \), i.e., \( \delta(B) \leq \delta(B_1) + \delta(B_2) + \delta(B_3) = 0 \). Therefore,

\[
\text{st} \lim_{n} \left\| U_{n,\delta}^{(a,\delta)} (t^2, x) - x^2 \right\| = 0. \tag{5.4}
\]

Thus, by using Eqs. (5.2), (5.3), (5.4) and the Bohman-Korovkin type theorem, we get the result.

References