Some Inequalities Concerning the Polar Derivative of a Polynomial-II

Abdullah Mir* and Bilal Dar

Department of Mathematics, University of Kashmir, Srinagar, 190006, India

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Abstract. In this paper, we consider the class of polynomials

\[ P(z) = a_n z^n + \sum_{\mu=1}^{n} a_{n-\nu} z^{n-\nu}, \quad 1 \leq \mu \leq n, \]

having all zeros in \(|z| \leq k, k \leq 1\) and thereby present an alternative proof, independent of Laguerre’s theorem, of an inequality concerning the polar derivative of a polynomial.

Key Words: Polar derivative of a polynomial.

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1 Introduction and statement of results

Let \(P(z)\) be a polynomial of degree \(n\) and \(P'(z)\) be its derivative, then according to the well-known Bernstein’s inequality [2] on the derivative of a polynomial, we have

\[ \text{Max}_{|z|=1} |P'(z)| \leq n \text{Max}_{|z|=1} |P(z)|. \] (1.1)

The equality (1.1) is best possible and equality holds if and only if \(P(z)\) has all its zeros at the origin.

For the class of polynomials \(P(z)\) of degree \(n\) having all zeros in \(|z| \leq 1\), Turan [7] proved that

\[ \text{Max}_{|z|=1} |P'(z)| \geq \frac{n}{2} \text{Max}_{|z|=1} |P(z)|. \] (1.2)

The inequality (1.2) is best possible and become equality for polynomials having all zeros on \(|z| = 1\).

*Corresponding author. Email addresses: mabdullah.mir@yahoo.co.in (A. Mir), darbilal85@ymail.com (B. Dar)

As a refinement of (1.3), Malik [6] proved that if $P(z)$ has all its zeros in $|z| \leq k, k \leq 1$, then

$$\text{Max}_{|z|=1}|P'(z)| \geq \frac{n}{1+k} \text{Max}_{|z|=1}|P(z)|.$$  \hspace{1cm} (1.3)

As a refinement of (1.3), Govil [5] under the same hypothesis proved that

$$\text{Max}_{|z|=1}|P'(z)| \geq \frac{n}{1+k} \left\{ \text{Max}_{|z|=1}|P(z)| + \frac{1}{k^{n-1}} \text{Min}_{|z|=k}|P(z)| \right\}. \hspace{1cm} (1.4)$$

Aziz and Shah [1] generalized (1.4) in a different direction and proved that, if $P(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}, \mu \geq 1$, is a polynomial of degree $n$ having all its zeros in $|z| \leq k, k \leq 1$, then

$$\text{Max}_{|z|=1}|P'(z)| \geq \frac{n}{1+k} \left\{ \text{Max}_{|z|=1}|P(z)| + \frac{1}{k^{n-\mu}} \text{Min}_{|z|=k}|P(z)| \right\}. \hspace{1cm} (1.5)$$

For $\mu = 1$, inequality (1.5) reduces to inequality (1.4).

Let $D_\alpha P(z)$ denotes the polar derivative of the polynomial $P(z)$ of degree $n$ with respect to the point $\alpha \in \mathbb{C}$. Then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n - 1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \left[ \frac{D_\alpha P(z)}{\alpha} \right] = P'(z).$$

Dewan, Singh and Lal [4] extend the inequality (1.5) to the polar derivative of a polynomial $P(z)$ and proved that if $P(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}, 1 \leq \mu \leq n$, has all its zeros in $|z| \leq k, k \leq 1$, then for every real or complex number $\alpha$ with $|\alpha| \geq k^\mu$,

$$\text{Max}_{|z|=1}|D_\alpha P(z)| \geq n \left( \frac{|\alpha|-k^\mu}{1+k^\mu} \right) \text{Max}_{|z|=1}|P(z)| + \frac{n(|\alpha|+1)}{k^{n-\mu}(1+k^\mu)} \text{Min}_{|z|=k}|P(z)|. \hspace{1cm} (1.6)$$

As a refinement of (1.6), Dewan, Singh and Mir [3] proved the following result:

**Theorem 1.1.** Let $P(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}, 1 \leq \mu \leq n$, be a polynomial of degree $n$ having all its zeros in $|z| \leq k, k \leq 1$, then for every real or complex number $\alpha$ with $|\alpha| \geq k^\mu$, we have

$$\text{Max}_{|z|=1}|D_\alpha P(z)| \geq n \left( \frac{|\alpha|-A_\mu}{1+k^\mu} \right) \text{Max}_{|z|=1}|P(z)| + \frac{n}{k^n} \left( \frac{|\alpha| k^\mu + A_\mu}{1+k^\mu} \right) \text{Min}_{|z|=k}|P(z)|,$$

where

$$A_\mu = \frac{n(|a_n| - \frac{m}{k^n}) k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n(|a_n| - \frac{m}{k^n}) k^{\mu-1} + \mu |a_{n-\mu}|}. \hspace{1cm} (1.7)$$
In the proof of above Theorem 1.1, the authors while using Laguerre’s theorem claim to have deduced on page 814 that if $P(z) - m\lambda z^n/k^n$ has all its zeros in $|z| < k$, $k \leq 1$, then for $|\alpha| \geq k^\mu$, $1 \leq \mu \leq n$, the polynomial $D_\alpha[P(z) - m\lambda z^n/k^n]$ also has all its zeros in $|z| < k$, $k \leq 1$, which is true when $|\alpha| \geq k^\mu$ and not for $|\alpha| \geq k^\mu$, $1 \leq \mu \leq n$, in general. It is worth to mention here that the result still follows without using Laguerre’s theorem.

The main aim of this paper is to present an alternative proof of Theorem 1.1 which is independent of Laguerre’s theorem.

For the proof of this theorem, we need the following lemmas. The following lemmas from 1.1-1.5 are all due to Dewan, Singh and Mir [3].

Lemma 1.1. Let $P(z) = a_n z^n + \sum_{\nu=1}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, be a polynomial of degree $n$ having all zeros in $|z| \leq k$, $k \leq 1$, then for every real or complex number $\alpha$ with $|\alpha| \geq S_\mu$, we have

$$\max_{|z|=1}|D_\alpha P(z)| \geq n\left(\frac{|\alpha| - S_\mu}{1 + k^\mu}\right) \max_{|z|=1}|P(z)|,$$

(1.8)

where

$$S_\mu = \frac{n|a_n| k^{2\mu} + \mu|a_{n-\mu}| k^{\mu-1}}{n|a_n| k^{n-1} + \mu|a_{n-\mu}|}.$$  

(1.9)

Lemma 1.2. Let $P(z) = a_n z^n + \sum_{\nu=1}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, be a polynomial of degree $n$ having all zeros in $|z| \leq k$, $k \leq 1$, then on $|z| = 1$, we have

$$|Q'(z)| \leq k^\mu |P'(z)| - \frac{n}{k^\mu - \mu} \min_{|z|=k} |P(z)|,$$

(1.10)

where $Q(z) = z^n P(\frac{1}{z^{-1}})$.

Lemma 1.3. The function

$$S_\mu(x) = \frac{n x k^{2\mu} + \mu|a_{n-\mu}| k^{\mu-1}}{n x k^{n-1} + \mu|a_{n-\mu}|},$$

(1.11)

where $k \leq 1$ and $\mu \geq 1$, is a non-increasing function of $x$.

Lemma 1.4. If $P(z) = \sum_{\nu=0}^{n} a_{n} z^\nu$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, $k > 0$, then $|Q(z)| \geq m/k^n$, for $|z| \leq 1/k$ and in particular

$$|a_n| > \frac{m}{k^n},$$

(1.12)

where $m = \min_{|z|=k} |P(z)|$ and $Q(z) = z^n P(\frac{1}{z^{-1}})$.

Lemma 1.5. If $P(z) = a_n z^n + \sum_{\nu=1}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu \leq n$, is a polynomial of degree $n$ having all zeros in $|z| \leq k$, $k \leq 1$, then

$$A_\mu \leq k^\mu,$$

(1.13)

where $A_\mu$ is defined in (1.7).
Proof of Theorem 1.1. By hypothesis, the polynomial $P(z) = a_n z^n + \sum_{\nu=1}^{n} a_{n-\nu} z^{n-\nu} \geq 1 \leq k$, has all its zeros in $|z| \leq k$, $k \leq 1$. If $P(z)$ has a zero on $|z| = k$, then $m = 0$ and the result follows from Lemma 1.1 in this case. Henceforth, we assume that all the zeros of $P(z)$ lie in $|z| < k$, $k \leq 1$, so that $m > 0$ and $m \leq |P(z)|$ for $|z| = k$. Therefore, if $\lambda$ is any complex number such that $|\lambda| < 1$, then

$$\left| \frac{m \lambda z^n}{k^n} \right| < |P(z)| \quad \text{for } |z| = k. \quad (1.14)$$

Since all the zeros of $P(z)$ lie in $|z| < k$, it follows by Rouche’s theorem that all the zeros of $P(z) - m \lambda z^n / k^n$ also lie in $|z| < k$, $k \leq 1$. Hence, we can apply Lemma 1.1 to $P(z) - m \lambda z^n / k^n$ and obtain for $|\alpha| \geq k^\mu \geq S'_\mu$

$$\left| D_{\alpha} \left\{ P(z) - \frac{m \lambda z^n}{k^n} \right\} \right| \geq n \left( |\alpha| - S'_\mu \right) \left| P(z) - \frac{m \lambda z^n}{k^n} \right| \quad \text{for } |z| = 1, \quad (1.15)$$

where

$$S'_\mu = \frac{n |a_n - \frac{m \lambda}{k^n}| k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n |a_n - \frac{m \lambda}{k^n}| k^{2\mu} + \mu |a_{n-\mu}|}. \quad (1.16)$$

Since for every $\lambda$ with $|\lambda| < 1$, we have

$$\left| a_n - \frac{m \lambda}{k^n} \right| \geq |a_n| - \frac{m |\lambda|}{k^n} \geq |a_n| - \frac{m}{k^n}, \quad (1.17)$$

and $|a_n| > m / k^n$ by Lemma 1.4. Now combining (1.16), (1.17) and Lemma 1.3 for every $\lambda$ with $|\lambda| < 1$, we get

$$S'_\mu \leq \frac{n |a_n - \frac{m \lambda}{k^n}| k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n |a_n - \frac{m \lambda}{k^n}| k^{2\mu} + \mu |a_{n-\mu}|}$$

$$= \frac{n (|a_n| - \frac{m}{k^n}) k^{2\mu} + \mu |a_{n-\mu}| k^{\mu-1}}{n (|a_n| - \frac{m}{k^n}) k^{2\mu} + \mu |a_{n-\mu}|} = A_\mu. \quad (1.18)$$

Since by Lemma 1.5, we have $A_\mu \leq k^\mu$, it follows from (1.15) and (1.18) that for every $\alpha$ with $|\alpha| \geq k^\mu$,

$$\left| D_{\alpha} P(z) - \frac{\lambda m n \alpha z^{n-1}}{k^n} \right| \geq n \left( \frac{|\alpha|-A_\mu}{1+k^\mu} \right) \left| P(z) - \frac{m \lambda z^n}{k^n} \right| \quad \text{for } |z| = 1. \quad (1.19)$$

Let $z_0$ be on $|z| = 1$ such that $|P(z_0)| = \max_{|z|=1} |P(z)|$, then from (1.19), we get

$$\left\{ D_{\alpha} P(z) \right\}_{z=z_0} - \frac{\lambda m n z_0^{n-1}}{k^n} \geq n \left( \frac{|\alpha|-A_\mu}{1+k^\mu} \right) \left\{ |P(z_0)| - \frac{m |\lambda|}{k^n} \right\}. \quad (1.20)$$
Again since the polynomial \( P(z) - m\lambda z^n / k^n \) has all its zeros in \(|z| < k, k \leq 1\) with \(|\lambda| < 1\), therefore, by Guass-Lucas theorem, the polynomial \( P'(z) - mn\lambda z^{n-1} / k^n \) also has all its zeros in \(|z| < k, k \leq 1\) and hence
\[
|P'(z)| \geq \frac{mn|z|^{n-1}}{k^n} \quad \text{for} \quad |z| \geq k. \tag{1.21}
\]
Because, if (1.21) is not true, then there is a point \( z = z_0 \) with \(|z_0| \geq k\) such that
\[
|P'(z_0)| < \frac{mn|z_0|^{n-1}}{k^n}.
\]
If we take \( \lambda = k^n P'(z_0) / mnz_0^{n-1} \), so that \(|\lambda| < 1\), then with this choice of \( \lambda \), we have
\[
P'(z_0) - \frac{mn\lambda z_0^{n-1}}{k^n} = 0,
\]
where \(|z_0| \geq k\), which contradicts the fact that all the zeros of \( P'(z) - mn\lambda z^{n-1} / k^n \) lie in \(|z| < k, k \leq 1\).

Also for \(|z| = 1\), we have
\[
|D_\alpha P(z)| = |nP(z) + (\alpha-z)P'(z)| \geq |\alpha||P'(z)| - |nP(z) - zP'(z)| = |\alpha||P'(z)| - |Q'(z)|. \tag{1.22}
\]
Combining this with inequality (1.10) of Lemma 1.2, we get for \(|z| = 1\) and \(|\alpha| \geq k^n\),
\[
|D_\alpha P(z)| \geq (|\alpha| - k^n) |P'(z)| + \frac{mn}{k^n-\mu} \tag{1.23}
\]
Inequality (1.23) in conjunction with (1.21) gives for \(|z| = 1\) and \(|\alpha| \geq k^n\),
\[
|D_\alpha P(z)| \geq \frac{|\alpha|mn}{k^n}. \tag{1.24}
\]

If in (1.20), we choose the argument of \( \lambda \) such that
\[
\left| \left\{ D_\alpha P(z) \right\} \right|_{z=z_0} = \left| \frac{\lambda mn\alpha z_0^{n-1}}{k^n} \right| = \left| \left\{ D_\alpha P(z) \right\} \right|_{z=z_0} \frac{mn|\alpha||\lambda||z_0|^{n-1}}{k^n},
\]
which easily follows from (1.24), we obtain
\[
\left| \left\{ D_\alpha P(z) \right\} \right|_{z=z_0} \frac{mn|\alpha||\lambda||z_0|^{n-1}}{k^n} \geq n \left( \frac{|\alpha| - A_{\mu}}{1+k^n} \right) |P(z_0)| - n \left( \frac{|\alpha| - A_{\mu}}{1+k^n} \right) m|\lambda|. \tag{1.25}
\]
Since \( z_0 \) lies on \(|z| = 1\) and \(|P(z_0)| = \text{Max}_{|z|=1} |P(z)|\), inequality (1.25) is equivalent to
\[
\left| \left\{ D_\alpha P(z) \right\} \right|_{z=z_0} \geq n \left( \frac{|\alpha| - A_{\mu}}{1+k^n} \right) \text{Max}_{|z|=1} |P(z)| - n \left( \frac{|\alpha| - A_{\mu}}{1+k^n} \right) m|\lambda| + \frac{mn|\alpha||\lambda|}{k^n}. \tag{1.26}
\]
Now, if in (1.26) we make \(|\lambda| \rightarrow 1\), we get
\[
\text{Max}_{|z|=1} |D_\alpha P(z)| \geq n \left( \frac{|\alpha| - A_{\mu}}{1+k^n} \right) \text{Max}_{|z|=1} |P(z)| + \frac{mn|\alpha|}{k^n} \left( \frac{|\alpha|}{1+k^n} + A_{\mu} \right),
\]
which is equivalent to (1.7) and which proves Theorem 1.1 completely.
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References