Weighted Integral Means of Mixed Areas and Lengths Under Holomorphic Mappings

Jie Xiao¹,∗ and Wen Xu²

1 Department of Mathematics and Statistics, Memorial University, NL A1C 5S7, Canada
2 Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111, FI-80101 Joensuu, Finland

Received 31 October 2013; Accepted (in revised version) 1 March 2014
Available online 31 March 2014

Abstract. This note addresses monotonic growths and logarithmic convexities of the weighted \((1-t^2)^\alpha dt, -\infty < \alpha < \infty, 0 < t < 1\) integral means \(A_{\alpha,\beta}(f,\cdot)\) and \(L_{\alpha,\beta}(f,\cdot)\) of the mixed area \((\pi r^2)^{\beta}A(f,r)\) and the mixed length \((2\pi r)^{\beta}L(f,r)\) \(0 \leq \beta \leq 1\) under a holomorphic map \(f\) from the unit disk \(D\) into the finite complex plane \(C\).

Key Words: Monotonic growth, logarithmic convexity, mean mixed area, mean mixed length, isoperimetric inequality, holomorphic map, univalent function.

AMS Subject Classifications: 32A10, 32A36, 51M25

1 Introduction

From now on, \(D\) represents the unit disk in the finite complex plane \(C\), \(H(D)\) denotes the space of holomorphic mappings \(f:D \rightarrow C\), and \(U(D)\) stands for all univalent functions in \(H(D)\). For any real number \(\alpha\), positive number \(r \in (0,1)\) and the standard area measure \(dA\), let

\[dA_{\alpha}(z) = (1-|z|^2)^\alpha dA(z), \quad rD = \{z \in D: |z| < r\}, \quad rT = \{z \in D: |z| = r\}.

In their recent paper [11], Xiao and Zhu have discussed the following area 0 < \(p < \infty\)-integral mean of \(f \in H(D)\):

\[M_{p,\alpha}(f,r) = \left[ \frac{1}{A_{\alpha}(rD)} \right]^{\frac{1}{p}} \left[ \int_D |f|^p dA_{\alpha} \right]^{\frac{1}{p}},\]

∗Corresponding author. Email addresses: jxiao@mun.ca (J. Xiao), wen.xu@uef.fi (W. Xu)
proving that \( r \mapsto M_{p,a}(f,r) \) is strictly increasing unless \( f \) is a constant, and \( \log r \mapsto \log M_{p,a}(f,r) \) is not always convex. This last result suggests such a conjecture that \( \log r \mapsto \log M_{p,a}(f,r) \) is convex or concave when \( p \leq 0 \) or \( p > 0 \). But, motivated by [11, Example 10, (ii)] we can choose \( p = 2, a = 1, f(z) = z + c \) and \( c > 0 \) to verify that the conjecture is not true. At the same time, this negative result was also obtained in Wang-Zhu’s manuscript [10]. So far it is unknown whether the conjecture is generally true for \( p \neq 2 \) – see [9] for a recent development.

The foregoing observation has actually inspired the following investigation. Our concentration is the fundamental case \( p = 1 \). To understand this new approach, let us take a look at \( M_{1,a}(\cdot,\cdot) \) from a differential geometric viewpoint. Note that

\[
M_{1,a}(f',r) = \frac{\int_{\partial D}|f'||dA_a|}{A_a(rD)} = \frac{\int_0^\alpha [(2\pi t)^{-1} \int_\Gamma |f'(z)||dz|] (1-t^2)^a dt^2}{\int_0^\alpha (1-t^2)^a dt^2}.
\]

So, if \( f \in U(D) \), then

\[
(2\pi t)^{-1} \int_\Gamma |f'(z)||dz|
\]

is a kind of mean of the length of \( \partial f(tD) \), and hence the square of this mean dominates a sort of mean of the area of \( f(tD) \) in the isoperimetric sense:

\[
\Phi_A(f,t) = (\pi t^2)^{-1} \int_{tD} |f'(z)|^2 dA(z) \leq \left( (2\pi t)^{-1} \int_{t\partial D} |f'(z)||dz| \right)^2 = [\Phi_L(f,t)]^2.
\]

In accordance with the well-known Pólya-Szegő monotone principle [8, Problem 309] (or [2, Proposition 6.1]) and the area Schwarz’s lemma in Burckel, Marshall, Minda, Poggi-Corradini and Ransford [2, Theorem 1.9], \( \Phi_L(f,\cdot) \) and \( \Phi_A(f,\cdot) \) are strictly increasing on \((0,1)\) unless \( f(z) = a_1z \) with \( a_1 \neq 0 \). Furthermore, \( \log \Phi_L(f,r) \) and \( \log \Phi_A(f,r) \), equivalently, \( \log L(f,r) \) and \( \log A(f,r) \), are convex functions of \( \log r \) for \( r \in (0,1) \), due to the classical Hardy’s convexity and [2, Section 5]. Perhaps, it is worthwhile to mention that if \( c > 0 \) is small enough then the universal cover of \( D \) onto the annulus \( \{ e^{-c\pi/2} < |z| < e^{c\pi/2} \} \):

\[
f(z) = \exp \left[ ic \log \left( \frac{1+z}{1-z} \right) \right]
\]

enjoys the property that \( \log r \mapsto \log A(f,r) \) is not convex; see [2, Example 5.1].

In the above and below, we have used the following convention:

\[
\Phi_A(f,r) = \frac{A(f,r)}{\pi r^2} \quad \text{and} \quad \Phi_L(f,r) = \frac{L(f,r)}{2\pi r},
\]

where under \( r \in (0,1) \) and \( f \in H(D) \), \( A(f,r) \) and \( L(f,r) \) stand respectively for the area of \( f(rD) \) (the projection of the Riemannian image of \( rD \) by \( f \)) and the length of \( \partial f(rD) \) (the boundary of the projection of the Riemannian image of \( rD \) by \( f \)) with respect to the standard Euclidean metric on \( C \). For our purpose, we choose a shortcut notation

\[
d\mu_a(t) = (1-t^2)^a dt^2 \quad \text{and} \quad \nu_a(t) = \mu_a([0,t]), \quad \forall t \in (0,1),
\]

for our purpose, we choose a shortcut notation.
and for $0 \leq \beta \leq 1$ define
\[
\Phi_{A,\beta}(f, t) = \frac{A(f, t)}{(\pi t^2)^\beta} \quad \text{and} \quad \Phi_{L,\beta}(f, t) = \frac{L(f, t)}{(2\pi t)^\beta},
\]
and then introduce two natural analytic-geometric quantities
\[
A_{a,\beta}(f, r) = \frac{\int_0^r \Phi_{A,\beta}(f, t)d\mu_a(t)}{\int_0^r d\mu_a(t)} \quad \text{and} \quad L_{a,\beta}(f, r) = \frac{\int_0^r \Phi_{L,\beta}(f, t)d\mu_a(t)}{\int_0^r d\mu_a(t)},
\]
which are respectively called the weighted integral means of the mixed area and the mixed length for $f(r\mathbb{D})$ and $\partial f(r\mathbb{D})$.

In this note, we consider two fundamental properties: monotonic growths and logarithmic convexities of both $A_{a,\beta}(f, r)$ and $L_{a,\beta}(f, r)$, thereby giving two applications: (i) if $r \mapsto \Phi_{L}(f, r)$ is monotone increasing on $(0, 1)$, then so is the isoperimetry-induced function:
\[
r \mapsto \int_0^r \frac{[\Phi_{L,1}(f, t)]^2d\mu_a(t)}{\int_0^r d\mu_a(t)} \geq A_{a,1}(f, r);
\]
(ii) the log-convexity for $L_{a,1}(f, r)$ essentially settles the above-mentioned conjecture. The non-trivial details (results and their proofs) are arranged in the forthcoming two sections.

## 2 Monotonic growth

In this section, we deal with the monotonic growths of $A_{a,\beta}(f, r)$ and $L_{a,\beta}(f, r)$, along with their associated Schwarz type lemmas. In what follows, $\mathbb{N}$ is used as the set of all natural numbers.

### 2.1 Two lemmas

The following two preliminary results are needed.

**Lemma 2.1** (see [5]). Let $f \in H(\mathbb{D})$ be of the form $f(z) = a_0 + \sum_{k=1}^n a_k z^k$ with $n \in \mathbb{N}$. Then:

(i) $\pi^n [\frac{|f^{(n)}(0)|}{n!}]^2 \leq A(f, r), \quad \forall r \in (0, 1)$.

(ii) $2\pi^n [\frac{|f^{(n)}(0)|}{n!}] \leq L(f, r), \quad \forall r \in (0, 1)$.

Moreover, equality in (i) or (ii) holds if and only if $f(z) = a_0 + a_n z^n$.

**Proof.** This may be viewed as the higher order Schwarz type lemma for area and length. See also the proofs of Theorems 1 and 2 in [5], and their immediate remarks on equalities. Here it is worth noticing three matters: (a) $f^{(n)}(0)/n!$ is just $a_n$; (b) [4, Corollary 3] presents a different argument for the area case; (c) $L(f, r)$ is greater than or equal to the length $l(r, f)$ of the outer boundary of $f(r\mathbb{D})$ (defined in [5]) which is not less than the length $l^0(r, f)$ of the exact outer boundary of $f(r\mathbb{D})$ (introduced in [12]). \(\square\)
Lemma 2.2. Let \( 0 \leq \beta \leq 1 \).

(i) If \( f \in H(D) \), then \( r \mapsto \Phi_{A,\beta}(f,r) \) is strictly increasing on \((0,1)\) unless

\[
    f = \begin{cases} 
        \text{constant}, & \text{when } \beta < 1, \\
        \text{linear map}, & \text{when } \beta = 1.
    \end{cases}
\]

(ii) If \( f \in U(D) \) or \( f(z) = a_0 + a_n z^n \) with \( n \in \mathbb{N} \), then \( r \mapsto \Phi_{L,\beta}(f,r) \) is strictly increasing on \((0,1)\) unless

\[
    f = \begin{cases} 
        \text{constant}, & \text{when } \beta < 1, \\
        \text{linear map}, & \text{when } \beta = 1.
    \end{cases}
\]

Proof. It is enough to handle \( \beta < 1 \) since the case \( \beta = 1 \) has been treated in [2, Theorem 1.9 and Proposition 6.1]. The monotonic growths in (i) and (ii) follow from

\[
    \Phi_{A,\beta}(f,r) = (\pi r^2)^{1-\beta} \Phi_{A,1}(f,r) \quad \text{and} \quad L(f,r) = (2\pi r)^{1-\beta} \Phi_{L,1}(f,r).
\]

To see the strictness, we consider two cases.

(i) Suppose that \( \Phi_{A,\beta}(f,\cdot) \) is not strictly increasing. Then there are \( r_1, r_2 \in (0,1) \) such that \( r_1 < r_2 \), and \( \Phi_{A,\beta}(f,\cdot) \) is a constant on \([r_1, r_2]\). Hence

\[
    \frac{d}{dr} \Phi_{A,\beta}(f,r) = 0, \quad \forall r \in [r_1, r_2].
\]

Equivalently,

\[
    2\beta A(f,r) = r \frac{d}{dr} A(f,r), \quad \forall r \in [r_1, r_2].
\]

But, according to [2, (4.2)],

\[
    2A(f,r) \leq r \frac{d}{dr} A(f,r), \quad \forall r \in (0,1).
\]

Since \( \beta < 1 \), we get \( A(f,r) = 0 \) for all \( r \in [r_1, r_2] \), whence finding that \( f \) is constant.

(ii) Now assume that \( \Phi_{L,\beta}(f,\cdot) \) is not strictly increasing. There are \( r_3, r_4 \in (0,1) \) such that \( r_3 < r_4 \) and

\[
    0 = \frac{d}{dr} \Phi_{L,\beta}(f,r) = (2\pi r)^{-\beta} \left[ \frac{d}{dr} L(f,r) - \frac{\beta}{r} L(f,r) \right] = 0, \quad \forall r \in [r_3, r_4].
\]

If \( f \in U(D) \), then

\[
    L(f,r) = \int_{|z|=r} |f'(z)||dz|
\]

and hence one has the following “first variation formula”

\[
    \frac{d}{dr} L(f,r) = \int_0^{2\pi} |f'(re^{i\theta})|d\theta + r \frac{d}{dr} \int_0^{2\pi} |f'(re^{i\theta})|d\theta, \quad \forall r \in [r_3, r_4].
\]
The previous three equations yield
\[ 0 = (1 - \beta) \int_0^{2\pi} |f'(re^{i\theta})|d\theta + r \frac{d}{dr} \int_0^{2\pi} |f'(re^{i\theta})|d\theta, \quad \forall r \in [r_3, r_4], \]
and so
\[ \int_0^{2\pi} |f'(re^{i\theta})|d\theta = 0, \quad \forall r \in [r_3, r_4]. \]
This ensures that \( f \) is a constant, contradicting \( f \in U(\mathbb{D}) \). Therefore, \( f(z) \) is of the form \( a_0 + a_n z^n \). But, since \( L(z^n, r) = 2\pi r^n \) is strictly increasing, \( f \) must be constant.

### 2.2 Monotonic growth of \( A_{\alpha, \beta}(f, \cdot) \)

This aspect is essentially motivated by the following Schwarz type lemma.

**Proposition 2.1.** Let \( -\infty < \alpha < \infty, 0 \leq \beta \leq 1 \), and \( f \in H(\mathbb{D}) \) be of the form \( f(z) = a_0 + \sum_{k=n}^{\infty} a_k z^k \) with \( n \in \mathbb{N} \). Then
\[
\pi^{1-\beta} \left[ \frac{|f^{(n)}(0)|}{n!} \right]^2 \leq A_{\alpha, \beta}(f, r) \left[ \frac{\nu_{\alpha}(r)}{\int_0^{2(\alpha-\beta)} d\mu_{\alpha}(t)} \right], \quad \forall r \in (0, 1),
\]
with equality if and only if \( f(z) = a_0 + a_n z^n \).

**Proof.** The inequality follows from Lemma 2.1(i) right away. When \( f(z) = a_0 + a_n z^n \), the last inequality becomes an equality due to the equality case of Lemma 2.1(i). Conversely, suppose that the last inequality is an equality. If \( f \) does not have the form \( a_0 + a_n z^n \), then the equality in Lemma 2.1(i) is not true, then there are \( r_1, r_2 \in (0, 1) \) such that \( r_1 < r_2 \) and
\[
A(f, t) > \pi^{2n} \left[ \frac{|f^{(n)}(0)|}{n!} \right]^2, \quad \forall t \in [r_1, r_2].
\]

This strict inequality forces that for \( r \in [r_1, r_2] \),
\[
\pi^{1-\beta} \left[ \frac{|f^{(n)}(0)|}{n!} \right]^2 \int_0^r t^{2(\alpha-\beta)} d\mu_{\alpha}(t)
\]
\[= \int_0^r (\pi t^2)^{-\beta} A(f, t) d\mu_{\alpha}(t) = \left( \int_{r_1}^{r_2} + \int_{r_2}^r \right) (\pi t^2)^{-\beta} A(f, t) d\mu_{\alpha}(t)
\]
\[> \pi^{1-\beta} \left[ \frac{|f^{(n)}(0)|}{n!} \right]^2 \int_0^r t^{2(\alpha-\beta)} d\mu_{\alpha}(t),
\]
a contradiction. Thus \( f(z) = a_0 + a_n z^n \). \qed

Based on Proposition 2.1, we find the monotonic growth for \( A_{\alpha, \beta}(\cdot, \cdot) \) as follows.
**Theorem 2.1.** Let $-\infty < \alpha < \infty$, $0 \leq \beta \leq 1$, and $f \in H(\mathbb{D})$. Then $r \mapsto A_{\alpha,\beta}(f,r)$ is strictly increasing on $(0,1)$ unless

$$f = \begin{cases} \text{constant,} & \text{when } \beta < 1, \\ \text{linear map,} & \text{when } \beta = 1. \end{cases}$$

Consequently,

(i) 

$$\lim_{r \to 0} A_{\alpha,\beta}(f,r) = \begin{cases} 0, & \text{when } \beta < 1, \\ |f'(0)|^2, & \text{when } \beta = 1. \end{cases}$$

(ii) If

$$\Phi_{A,\beta}(f,0) := \lim_{r \to 0} \Phi_{A,\beta}(f,r) \quad \text{and} \quad \Phi_{A,\beta}(f,1) := \lim_{r \to 1} \Phi_{A,\beta}(f,r) < \infty,$$

then

$$0 < r < s < 1 \Rightarrow 0 \leq \frac{A_{\alpha,\beta}(f,s) - A_{\alpha,\beta}(f,r)}{\log\nu_a(s) - \log\nu_a(r)} \leq \Phi_{A,\beta}(f,s) - \Phi_{A,\beta}(f,0)$$

with equality if and only if

$$f = \begin{cases} \text{constant,} & \text{when } \beta < 1, \\ \text{linear map,} & \text{when } \beta = 1. \end{cases}$$

In particular, $t \mapsto A_{\alpha,\beta}(f,t)$ is Lipschitz with respect to $\log\nu_a(t)$ for $t \in (0,1)$.

**Proof.** Note that $\nu_a(r) = \int_0^r d\mu_a(t)$. So $d\nu_a(r)$, the differential of $\nu_a(r)$ with respect to $r \in (0,1)$, equals $d\mu_a(r)$. By integration by parts we have

$$\Phi_{A,\beta}(f,r)d\nu_a(r) - \int_0^r \Phi_{A,\beta}(f,t)d\mu_a(t) = \int_0^r \left[ \frac{d}{dt} \Phi_{A,\beta}(f,t) \right] d\nu_a(t)dt.$$ 

Differentiating the function $A_{\alpha,\beta}(f,r)$ with respect to $r$ and using Lemma 2.2(i), we get

$$\frac{d}{dr} A_{\alpha,\beta}(f,r) = \frac{\Phi_{A,\beta}(f,r)2r(1-r^2)^{\alpha}d\nu_a(r) - \int_0^r \Phi_{A,\beta}(f,t)d\mu_a(t)}{\nu_a(r)^2} \left[ \int_0^r \Phi_{A,\beta}(f,t)d\mu_a(t) \right] 2r(1-r^2)^{\alpha}$$

$$= \frac{2r(1-r^2)^{\alpha} \left[ \Phi_{A,\beta}(f,t)d\nu_a(r) - \int_0^r \Phi_{A,\beta}(f,t)d\mu_a(t) \right]}{\nu_a(r)^2}$$

$$= \frac{2r(1-r^2)^{\alpha} \left[ \int_0^r \left[ \frac{d}{dt} \Phi_{A,\beta}(f,t) \right] d\nu_a(t)dt \right]}{\nu_a(r)^2} \geq 0.$$
Lemma 2.2(i) derives the equality case. This gives the desired inequality right away. Furthermore, the above argument plus
\[
\int_0^s \left( \frac{d}{dt} \Phi_{A,\beta}(f, t) \right) v_a(t) dt = 0, \quad \forall r \in [r_1, r_2],
\]
and so
\[
\int_0^r \left( \frac{d}{dt} \Phi_{A,\beta}(f, t) \right) v_a(t) dt = 0, \quad \forall r \in [r_1, r_2].
\]
Then we must have
\[
\frac{d}{dt} \Phi_{A,\beta}(f, t) = 0, \quad \forall t \in (0, r), \quad \text{with } r \in [r_1, r_2],
\]
whence getting that if \( \beta < 1 \) then \( f \) must be constant or if \( \beta = 1 \) then \( f \) must be linear, thanks to the argument for the strictness in Lemma 2.2(i).

It remains to check the rest of Theorem 2.1.

(i) The monotonic growth of \( A_{\alpha,\beta}(f, r) \) ensures the existence of the limit. An application of L’Hôpital’s rule gives
\[
\lim_{r \to 0} A_{\alpha,\beta}(f, r) = \lim_{r \to 0} \Phi_{A,\beta}(f, r) = \begin{cases} 0, & \text{when } \beta < 1, \\ |f'(0)|^2, & \text{when } \beta = 1. \end{cases}
\]

(ii) Again, the above monotonicity formula of \( A_{\alpha,\beta}(f, r) \) plus the given condition yields that for \( s \in (0, 1) \),
\[
\sup_{r \in (0, s)} A_{\alpha,\beta}(f, r) = A_{\alpha,\beta}(f, s) < \infty.
\]
Integrating by parts twice and using the monotonicity of \( \Phi_{A,\beta}(f, r) \), we obtain that under \( 0 < r < s < 1 \),
\[
0 \leq A_{\alpha,\beta}(f, s) - A_{\alpha,\beta}(f, r) = \int_r^s \frac{d}{dt} A_{\alpha,\beta}(f, t) dt
\]
\[
= \int_r^s \left( \int_0^t \left[ \frac{d}{d\tau} \Phi_{A,\beta}(f, \tau) \right] v_a(\tau) d\tau \right) \frac{d v_a(t)}{v_a(t)^2}
\]
\[
= \int_r^s \left( v_a(t) \Phi_{A,\beta}(f, t) - \int_0^t \Phi_{A,\beta}(f, \tau) d v_a(\tau) \right) \frac{d v_a(t)}{v_a(t)^2}
\]
\[
\leq \left[ \Phi_{A,\beta}(f, s) - \Phi_{A,\beta}(f, 0) \right] \int_r^s \frac{d v_a(t)}{v_a(t)}. \]
This gives the desired inequality right away. Furthermore, the above argument plus Lemma 2.2(i) derives the equality case. \( \square \)
As an immediate consequence of Theorem 2.1, we get a sort of "norm" estimate associated with $\Phi_{A,\beta}(f,\cdot)$.

**Corollary 2.1.** Let $-\infty < \alpha < \infty$, $0 \leq \beta \leq 1$ and $f \in H(D)$.

(i) If $-\infty < \alpha \leq -1$, then

$$\int_0^1 \Phi_{A,\beta}(f,t)d\mu_{\alpha}(t) = \sup_{r \in (0,1)} \int_0^r \Phi_{A,\beta}(f,t)d\mu_{\alpha}(t) < \infty,$$

if and only if $f$ is constant. Moreover, $\sup_{r \in (0,1)} A_{\alpha,\beta}(f,r) = \Phi_{A,\beta}(f,1)$.

(ii) If $-1 < \alpha < \infty$, then

$$A_{\alpha,\beta}(f,r) \leq A_{\alpha,\beta}(f,1) : = \sup_{s \in (0,1)} A_{\alpha,\beta}(f,s), \ \forall r \in (0,1),$$

where the inequality becomes an equality for all $r \in (0,1)$ if and only if

$$f = \begin{cases} \text{constant,} & \text{when } \beta < 1, \\ \text{linear map,} & \text{when } \beta = 1. \end{cases}$$

(iii) The following function $\alpha \mapsto A_{\alpha,\beta}(f,1)$ is strictly decreasing on $(-1,\infty)$ unless

$$f = \begin{cases} \text{constant,} & \text{when } \beta < 1, \\ \text{linear map,} & \text{when } \beta = 1. \end{cases}$$

**Proof.** (i) By Theorem 2.1, we have

$$A_{\alpha,\beta}(f,r) \leq \frac{\int_0^r \Phi_{A,\beta}(f,t)d\mu_{\alpha}(t)}{\nu_{\alpha}(s)}, \ \forall r \in (0,1).$$

Note that

$$\lim_{s \to 1} \nu_{\alpha}(s) = \infty \ \text{and} \ \lim_{s \to 1} \int_0^s \Phi_{A,\beta}(f,t)d\mu_{\alpha}(t) = \int_0^1 \Phi_{A,\beta}(f,t)d\mu_{\alpha}(t).$$

So, the last integral is finite if and only if

$$\Phi_{A,\beta}(f,r) = 0, \ \forall r \in (0,1),$$

equivalently, $A(f,r) = 0$ holds for all $r \in (0,1)$, i.e., $f$ is constant.

For the remaining part of (i), we may assume that $f$ is not a constant map. Due to $\lim_{r \to 1} \nu_{\alpha}(r) = \infty$, we obtain

$$\lim_{r \to 1} \int_0^r \Phi_{A,\beta}(f,t)d\mu_{\alpha}(t) = \int_0^1 \Phi_{A,\beta}(f,t)d\mu_{\alpha}(t) = \infty.$$
So, an application of L’Hôpital’s rule yields
\[
\sup_{0 < r < 1} A_{\alpha, \beta}(f, r) = \lim_{r \to 1} \frac{\int_{0}^{r} \Phi_{A, \beta}(f, t) d\mu_{\alpha}(t)}{v_{\alpha}(r)} = \lim_{r \to 1} \frac{\Phi_{A, \beta}(f, r) r(1-r^2)^{\alpha}}{r(1-r^2)^{\alpha}} = \Phi_{A, \beta}(f, 1).
\]

(ii) Under \(-1 < \alpha < \infty\), we have
\[
\lim_{r \to 1} v_{\alpha}(r) = v_{\alpha}(1) \quad \text{and} \quad \lim_{r \to 1} \int_{0}^{r} \Phi_{A, \beta}(f, t) d\mu_{\alpha}(t) = \int_{0}^{1} \Phi_{A, \beta}(f, t) d\mu_{\alpha}(t).
\]
Thus, by Theorem 2.1 it follows that for \(r \in (0,1)\),
\[
A_{\alpha, \beta}(f, r) \leq \lim_{s \to 1} A_{\alpha, \beta}(f, s) = [v_{\alpha}(1)]^{-1} \int_{0}^{1} \Phi_{A, \beta}(f, t) d\mu_{\alpha}(t) = \sup_{s \in (0,1)} A_{\alpha, \beta}(f, s).
\]
The equality case just follows from a straightforward computation and Theorem 2.1.

(iii) Suppose \(-1 < \alpha_{1} < \alpha_{2} < \infty\) and \(A_{\alpha_{1}, \beta}(f, 1) < \infty\), then integrating by parts twice, we obtain
\[
A_{\alpha_{2}, \beta}(f, 1) = [v_{\alpha_{2}}(1)]^{-1} \int_{0}^{1} \Phi_{A, \beta}(f, r) d\mu_{\alpha_{2}}(r)
\]
\[
= [v_{\alpha_{2}}(1)]^{-1} \int_{0}^{1} (1-r^2)^{\alpha_{2} - \alpha_{1}} \frac{d}{dr} \left[ \int_{0}^{r} \Phi_{A, \beta}(f, t) d\mu_{\alpha_{1}}(t) \right] dr
\]
\[
= [v_{\alpha_{2}}(1)]^{-1} \left[ - \int_{0}^{1} \left( \int_{0}^{r} \Phi_{A, \beta}(f, t) d\mu_{\alpha_{1}}(t) \right) d(1-r^2)^{\alpha_{2} - \alpha_{1}} \right]
\]
\[
\leq [v_{\alpha_{2}}(1)]^{-1} A_{\alpha_{1}, \beta}(f, 1) \int_{0}^{1} v_{\alpha_{1}}(r) d[-(1-r^2)^{\alpha_{2} - \alpha_{1}}]
\]
\[
= A_{\alpha_{1}, \beta}(f, 1) [v_{\alpha_{2}}(1)]^{-1} \left[ \int_{0}^{1} (1-r^2)^{\alpha_{2} - \alpha_{1}} d\mu_{\alpha_{1}}(r) \right]
\]
\[
= A_{\alpha_{1}, \beta}(f, 1),
\]
thereby establishing \(A_{\alpha_{2}, \beta}(f, 1) \leq A_{\alpha_{1}, \beta}(f, 1)\). If this last inequality becomes an equality, then the above argument forces
\[
\int_{0}^{r} \Phi_{A, \beta}(f, t) d\mu_{\alpha_{1}}(t) = A_{\alpha_{1}, \beta}(f, 1) v_{\alpha_{1}}(r), \quad \forall r \in (0,1),
\]
whence yielding (via the just-verified (ii))
\[
f = \begin{cases} 
\text{constant}, & \text{when } \beta < 1, \\
\text{linear map}, & \text{when } \beta = 1.
\end{cases}
\]
Thus, we complete the proof. \(\square\)
2.3 Monotonic growth of \( L_{\alpha,\beta}(f, \cdot) \)

Correspondingly, we first have the following Schwarz type lemma.

**Proposition 2.2.** Let \(-\infty < \alpha < \infty, 0 \leq \beta \leq 1, \) and \( f \in H(D) \) be of the form \( f(z) = a_0 + \sum_{k=n}^{\infty} a_k z^k \) with \( n \in \mathbb{N} \). Then

\[
(2\pi)^{1-\beta} \left[ \frac{|f^{(n)}(0)|}{n!} \right] \leq L_{\alpha,\beta}(f, r) \left[ \frac{\nu_{\alpha}(r)}{\int_0^r t^{\alpha-\beta} d\mu_{\alpha}(t)} \right], \quad \forall r \in (0,1),
\]

with equality when and only when \( f = a_0 + a_n z^n \).

**Proof.** This follows from Lemma 2.1(ii) and its equality case. \( \square \)

The coming-up-next monotonicity contains a hypothesis stronger than that for Theorem 2.1.

**Theorem 2.2.** Let \(-\infty < \alpha < \infty, 0 \leq \beta \leq 1, \) and \( f \in U(D) \) or \( f(z) = a_0 + a_n z^n \) with \( n \in \mathbb{N} \). Then \( r \mapsto L_{\alpha,\beta}(f, r) \) is strictly increasing on \((0,1)\) unless

\[
f = \begin{cases} \text{constant,} & \text{when } \beta < 1, \\ \text{linear map,} & \text{when } \beta = 1. \end{cases}
\]

Consequently,

(i) \[
\lim_{r \to 0} L_{\alpha,\beta}(f, r) = \begin{cases} 0, & \text{when } \beta < 1, \\ |f'(0)|, & \text{when } \beta = 1. \end{cases}
\]

(ii) If \( \Phi_{L,\beta}(f,0) := \lim_{r \to 0} \Phi_{L,\beta}(f, r) \) and \( \Phi_{L,\beta}(f,1) := \lim_{r \to 1} \Phi_{L,\beta}(f, r) < \infty, \) then

\[
0 < r < s < 1 \Rightarrow \frac{L_{\alpha,\beta}(f, s) - L_{\alpha,\beta}(f, r)}{\log \nu_{\alpha}(s) - \log \nu_{\alpha}(r)} \leq \Phi_{L,\beta}(f, s) - \Phi_{L,\beta}(f, 0)
\]

with equality if and only if

\[
f = \begin{cases} \text{constant,} & \text{when } \beta < 1, \\ \text{linear map,} & \text{when } \beta = 1. \end{cases}
\]

In particular, \( t \mapsto L_{\alpha,\beta}(f, t) \) is Lipschitz with respect to \( \log \nu_{\alpha}(t) \) for \( t \in (0,1). \)

**Proof.** Similar to that for Theorem 2.1, but this time by Lemma 2.2(ii). \( \square \)

Naturally, we can establish the so-called “norm” estimate associated to \( \Phi_{L,\beta}(f, \cdot). \)
Corollary 2.2. Let \(0 \leq \beta \leq 1\) and \(f \in U(D)\) or \(f(z) = a_0 + a_n z^n\) with \(n \in \mathbb{N}\),
(i) If \(-\infty < \alpha \leq -1\), then
\[
\int_0^1 \Phi_{L,\beta}(f, t) d\mu_{\alpha}(t) = \sup_{r \in (0,1)} \int_0^r \Phi_{L,\beta}(f, t) d\mu_{\alpha}(t) < \infty
\]
if and only if \(f\) is constant. Moreover, \(\sup_{r \in (0,1)} L_{a,\beta}(f, r) = \Phi_{L,\beta}(f, 1)\).
(ii) If \(-1 < \alpha < \infty\), then
\[
L_{a,\beta}(f, r) \leq L_{a,\beta}(f, 1) := \sup_{s \in (0,1)} L_{a,\beta}(f, s), \quad \forall r \in (0,1),
\]
where the inequality becomes an equality for all \(r \in (0,1)\) if and only if
\[
f = \begin{cases} \text{constant}, & \text{when } \beta < 1, \\ \text{linear map}, & \text{when } \beta = 1. \end{cases}
\]
(iii) \(\alpha \mapsto L_{a,\beta}(f, 1)\) is strictly decreasing on \((-1, \infty)\) unless
\[
f = \begin{cases} \text{constant}, & \text{when } \beta < 1, \\ \text{linear map}, & \text{when } \beta = 1. \end{cases}
\]

Proof. The argument is similar to that for Corollary 2.1, but via Lemma 2.2(ii).

3 Logarithmic convexity

In this section, we treat the convexities of the following two functions: \(\log r \mapsto \log A_{a,\beta}(f, r)\)
and \(\log r \mapsto \log L_{a,\beta}(f, r)\) for \(r \in (0,1)\).

3.1 Two more lemmas

The following are two technical preliminaries.

Lemma 3.1 (see [10]). Suppose that \(f(x)\) and \(\{h_k(x)\}_{k=0}^{\infty}\) are positive and twice differentiable
for \(x \in (0,1)\) such that the function \(H(x) = \sum_{k=0}^{\infty} h_k(x)\) is also twice differentiable for \(x \in (0,1)\).
Then:
(i) \(\log x \mapsto \log f(x)\) is convex if and only if \(\log x \mapsto \log f(x^2)\) is convex.
(ii) The function \(\log x \mapsto \log f(x)\) is convex if and only if the D-notation of \(f\)
\[
D(f(x)) := \frac{f'(x)}{f(x)} + x \left( \frac{f''(x)}{f(x)} \right)' \geq 0, \quad \forall x \in (0,1).
\]
(iii) If for each \(k\) the function \(\log x \mapsto \log h_k(x)\) is convex, then \(\log x \mapsto \log H(x)\) is also convex.
Lemma 3.2. Let \( f \in H(\mathbb{D}) \). Then \( f \) belongs to \( U(\mathbb{D}) \) provided that one of the following two conditions is valid:

(i) see [7] or [1, Lemma 2.1]

\[
f(0) = f'(0) - 1 = 0 \quad \text{and} \quad \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1, \quad \forall z \in \mathbb{D}.
\]

(ii) see [6, Theorem 1] or [3, Theorem 8.12]

\[
\left| \left[ \frac{f''(z)}{f'(z)} \right]' - \frac{1}{2} \frac{f'''(z)}{f'(z)^2} \right| \leq 2(1 - |z|^2)^{-2}, \quad \forall z \in \mathbb{D}.
\]

3.2 Log-convexity for \( A_{a,\beta}(f,r) \)

Such a property is given below.

Theorem 3.1. Let \( 0 \leq \beta \leq 1 \) and \( 0 < r < 1 \).

(i) If \( \alpha \in (-\infty, -3) \), then there exist two maps \( f, g \in H(\mathbb{D}) \) such that \( \log r \mapsto \log A_{a,\beta}(f,r) \) is not convex and \( \log r \mapsto \log A_{a,\beta}(g,r) \) is not concave.

(ii) If \( \alpha \in [-3,0] \), then \( \log r \mapsto \log A_{a,1}(a_n z^n, r) \) is convex for \( a_n \neq 0 \) with \( n \in \mathbb{N} \). Consequently,

\[
\log r \mapsto \log A_{a,1}(f, r)
\]

is convex for all \( f \in U(\mathbb{D}) \).

(iii) If \( \alpha \in (0,\infty) \), then \( \log r \mapsto \log A_{a,\beta}(a_1 z^n, r) \) is not convex for \( a_n \neq 0 \) and \( n \in \mathbb{N} \).

Proof. The key issue is to check whether or not \( \log r \mapsto \log A_{a,\beta}(z^n, r) \) is convex for \( r \in (0,1) \).

To see this, let us borrow some symbols from [10]. For \( \lambda \geq 0 \) and \( 0 < x < 1 \), we define

\[
f_{\lambda}(x) = \int_0^x t^\lambda (1-t)^{\alpha} dt
\]

and

\[
\Delta(\lambda, x) = \frac{f'_{\lambda}(x)}{f_{\lambda}(x)} + x \left( \frac{f''_{\lambda}(x)}{f'_{\lambda}(x)} \right)' - \left[ \frac{f'_0(x)}{f_0(x)} + x \left( \frac{f''_0(x)}{f'_0(x)} \right)' \right].
\]

Given \( n \in \mathbb{N} \). A simple calculation shows \( \Phi_{A,\beta}(z^n, t) = \pi^{1-\beta} t^2 (1-t)^{\alpha-\beta} \), and then a change of variable derives

\[
A_{a,\beta}(z^n, r) = \int_0^r \Phi_{A,\beta}(z^n, t) d\mu_a(t) = \pi^{1-\beta} \int_0^r t^{n-\beta} (1-t)^\alpha dt \frac{t^{\alpha-\beta}}{f_0(t)} = \pi^{1-\beta} \left[ \frac{f_{n-\beta}(r^2)}{f_0(r^2)} \right].
\]

In accordance with Lemma 3.1(i)-(ii), it is easy to work out that \( \log r \mapsto \log A_{a,\beta}(z^n, r) \) is convex for \( r \in (0,1) \) if and only if \( \Delta(n-\beta, x) \geq 0 \) for any \( x \in (0,1) \).
(i) Under $\alpha \in (-\infty, -3)$, we follow the argument for [10, Proposition 6] to get
\[
\lim_{x \to 1} \Delta(\lambda, x) = \frac{\lambda(\alpha+1)(\lambda+2+\alpha)}{(\alpha+2)^2(\alpha+3)}.
\]
Choosing
\[
f(z) = z^n = \begin{cases} 
z, & \text{when } \beta < 1, \\
z^2, & \text{when } \beta = 1,
\end{cases}
\]
and $\lambda = n - \beta$, we find $\lim_{x \to 1} \Delta(\lambda, x) < 0$, whence deriving that $\log r \mapsto \log A_{\alpha}(f, r)$ is not convex.

In the meantime, picking $n \in \mathbb{N}$ such that $n > \beta - (2 + \alpha)$ and putting $g(z) = z^n$, we obtain
\[
\lim_{x \to 1} \Delta(n - \beta, x) = \frac{(n - \beta)(\alpha+1)(n-\beta+2+\alpha)}{(\alpha+2)^2(\alpha+3)} > 0,
\]
whence deriving that $\log r \mapsto \log A_{\alpha, \beta}(g, r)$ is not concave.

(ii) Under $\alpha \in [-3, 0]$, we handle the two situations.

Situation 1: $f \in U(D)$. Upon writing $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we compute
\[
\Phi_{\alpha,1}(f(z), t) = (\pi t^2)^{-1} A(f, t) = \sum_{n=0}^{\infty} n |a_n|^2 t^{2(n-1)},
\]
and consequently,
\[
A_{\alpha,1}(f, r) = \frac{\sum_{n=0}^{\infty} n |a_n|^2 \int_{0}^{1} (\pi t^2)^{-1} A(z^n, t) d\mu_{\alpha}(t)}{\nu_{\alpha}(r)} = \sum_{n=0}^{\infty} n |a_n|^2 A_{\alpha,1}(z^n, r).
\]
So, by Lemma 3.1(iii), we see that the convexity of
\[
\log r \mapsto \log A_{\alpha,1}(f, r) \quad \text{under } f \in U(D),
\]
follows from the convexity of
\[
\log r \mapsto \log A_{\alpha,1}(z^n, r) \quad \text{under } n \in \mathbb{N}.
\]
So, it remains to verify this last convexity via the coming-up-next consideration.

Situation 2: $f(z) = a_n z^n$ with $a_n \neq 0$. Three cases are required to control.

Case 1: $\alpha = 0$. An easy computation shows
\[
A_{0,1}(z^n, r) = n^{-1} r^{2(n-1)}
\]
and so $\log r \mapsto \log A_{0,1}(z^n, r)$ is convex.

Case 2: $-2 \leq \alpha < 0$. Under this condition, we see from the arguments for [10, Propositions 4-5] that
\[
\Delta(n-1, x) \geq 0, \quad \forall n \geq 0, \quad 0 < x < 1,
\]
and so that \( \log r \mapsto \log A_{\alpha,1}(z^n,r) \) is convex.

Case 3: \(-3 \leq \alpha < -2\). With the assumption, we also get from the arguments for [10, Propositions 4-5] that

\[
\Delta(n-1,x) \geq \Delta(-2-a,x) > 0, \quad \forall x \in (0,1), \quad n-1 \in [-2-a,\infty),
\]
and so that \( \log r \mapsto \log A_{\alpha,1}(z^n,r) \) is convex when \( n \geq 2 \). Here it is worth noting that the convexity of \( \log r \mapsto \log A_{\alpha,1}(z^n,r) = 0 \) is trivial.

(iii) Under \( 0 < \alpha < \infty \), from the argument for [10, Proposition 6] we know that \( \Delta(n-\beta,x) < 0 \) as \( x \) is sufficiently close to 1. Thus \( \log r \mapsto \log A_{\alpha,\beta}(a_nz^n,r) \) is not convex under \( a_n \neq 0 \).

The following illustrates that the function \( \log r \mapsto \log A_{\alpha,\beta}(f,r) \) is not always concave for \( \alpha > 0, \beta \leq 1, \) and \( f \in U(D) \).

Example 3.1. Let \( \alpha = 1, \beta \in \{0,1\} \) and \( f(z) = z + z^2/2 \). Then the function \( \log r \mapsto \log A_{\alpha,\beta}(f,r) \) is neither convex nor concave for \( r \in (0,1) \).

Proof. A direct computation shows

\[
\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| = \left| \frac{z^2(1+z)}{(z+2)^2} - 1 \right| = \frac{|z|^2}{|z+2|^2} < 1,
\]

since

\[
|z| < 1 < 2 - |z| \leq |z+2|, \quad \forall z \in D.
\]

So, \( f \in U(D) \) owing to Lemma 3.2(i). By \( f'(z) = z + 1 \) we have

\[
A(f,t) = \int_D |z+1|^2 dA(z) = \pi \left( t^2 + \frac{t^4}{2} \right),
\]

plus

\[
\int_0^r \Phi_{A,\beta}(f,t) d\mu_1(t) = \begin{cases} 
\frac{\pi}{2} \left( r^4 - \frac{r^6}{3} - \frac{r^8}{4} \right), & \text{when } \beta = 0, \\
\frac{r^2}{2} - \frac{r^4}{4} - \frac{r^6}{6}, & \text{when } \beta = 1.
\end{cases}
\]

Meanwhile,

\[
v_1(r) = \int_0^r (1-t^2) dt^2 = r^2 - \frac{r^4}{2}.
\]

So, we get

\[
A_{1,\beta}(f,r) = \begin{cases} 
\frac{\pi(12r^2 - 4r^4 - 3r^6)}{12(2-r^2)}, & \text{when } \beta = 0, \\
\frac{12-3r^2 - 2r^4}{6(2-r^2)}, & \text{when } \beta = 1.
\end{cases}
\]
and in turn consider the logarithmic convexities of the following function

\[ h_\beta(x) = \begin{cases} 
\frac{12x - 4x^2 - 3x^3}{2 - x}, & \text{when } \beta = 0, \\
\frac{12 - 3x - 2x^2}{2 - x}, & \text{when } \beta = 1,
\end{cases} \]

for \( x \in (0, 1) \).

Using the so-called D-notation in Lemma 3.1, we have

\[ D(h_\beta(x)) = \begin{cases} 
D(12x - 4x^2 - 3x^3) - D(2 - x), & \text{when } \beta = 0, \\
D(12 - 3x - 2x^2) - D(2 - x), & \text{when } \beta = 1,
\end{cases} \]

for \( x \in (0, 1) \). By an elementary calculation, we get

\[ \begin{align*}
D(12x - 4x^2 - 3x^3) &= -48 - 144x + 12x^2, \\
D(2 - x) &= -\frac{2}{(2 - x)^2}, \\
D(12 - 3x - 2x^2) &= -\frac{36 - 96x + 6x^2}{(12 - 3x - 2x^2)^2}.
\end{align*} \]

Consequently,

\[ D(h_\beta(x)) = \begin{cases} 
\frac{2g_\beta(x)}{(12 - 4x - 3x^2)^2(2 - x)^2}, & \text{when } \beta = 0, \\
\frac{2g_\beta(x)}{(12 - 3x - 2x^2)^2(2 - x)^2}, & \text{when } \beta = 1,
\end{cases} \]

where

\[ g_\beta(x) = \begin{cases} 
48 - 288x + 232x^2 - 72x^3 + 15x^4, & \text{when } \beta = 0, \\
72 - 192x + 147x^2 - 48x^3 + 7x^4, & \text{when } \beta = 1.
\end{cases} \]

Now, under \( x \in (0, 1) \) we find

\[ g_0'(x) = -288 + 464x - 216x^2 + 60x^3 \quad \text{and} \quad g_0''(x) = 464 - 432x + 180x^2. \]

Clearly, \( g_0''(x) \) is an open-upward parabola with the axis of symmetry \( x = 6/5 > 1 \). By \( g_0''(1) = 212 > 0 \) and the monotonicity of \( g_0'' \) on \((0, 1)\), we have \( g_0''(x) > 0 \) for all \( x \in (0, 1) \). Thus \( g_0' \) is increasing on \((0, 1)\). The following condition

\[ g_0'(0) = -288 < 0 \quad \text{and} \quad g_0'(1) = 20 > 0 \]

yields an \( x_1 \in (0, 1) \) such that \( g_0'(x) < 0 \) for \( x \in (0, x_1) \) and \( g_0'(x) > 0 \) for \( x \in (x_1, 1) \). Since \( g_0(0) = 48 \) and \( g_0(1) = -65 \), there exists an \( x_0 \in (0, 1) \) such that \( g_0(x) > 0 \) for \( x \in (0, x_0) \) and \( g_0(x) < 0 \) for \( x \in (x_0, 1) \). Thus the function \( \log x \mapsto \log h_0(x) \) is neither convex nor concave.
Similarly, under $x \in (0,1)$ we have
\[
g'_1(x) = -192 + 294x - 144x^2 + 28x^3 \quad \text{and} \quad g''_1(x) = 294 - 288x + 84x^2.
\]
Obviously, $g''_1(x)$ is an open-upward parabola with the axis of symmetry $x = 12/7 > 1$. By $g''_1(1) = 90 > 0$ and the monotonicity of $g''_1$ on $(0,1)$, we have $g''_1(x) > 0$ for all $x \in (0,1)$. Thus $g'_1$ is increasing on $(0,1)$. The following condition
\[
g'_1(0) = -192 < 0 \quad \text{and} \quad g'_1(1) = -14 < 0
\]
yields $g'_1(x) < 0$ for $x \in (0,1)$. Since $g'_1(0) = 72$ and $g'_1(1) = -14$, there exists an $x_0 \in (0,1)$ such that $g'_1(x) > 0$ for $x \in (0,x_0)$ and $g'_1(x) < 0$ for $x \in (x_0,1)$. Thus the function $\log x \mapsto \log h_1(x)$ is neither convex nor concave.

3.3 Log-convexity for $L_{\alpha,\beta}(f, \cdot)$

Analogously, we can establish the expected convexity for the mixed lengths.

**Theorem 3.2.** Let $0 \leq \beta \leq 1$ and $0 < r < 1$.

(i) If $\alpha \in (-\infty, -3)$, then there exist two maps $f, g \in H(D)$ such that $\log r \mapsto \log L_{\alpha,\beta}(f, r)$ is not convex and $\log r \mapsto \log L_{\alpha,\beta}(g, r)$ is not concave.

(ii) If $\alpha \in [-3, 0]$, then $\log r \mapsto \log L_{\alpha,1}(a_n z^n, r)$ is convex for $a_n \neq 0$ with $n \in \mathbb{N}$. Consequently, $\log r \mapsto \log L_{\alpha,1}(f, r)$ is convex for $f \in U(D)$.

(iii) If $\alpha \in (0, \infty)$, then $\log r \mapsto \log L_{\alpha,\beta}(a_n z^n, r)$ is not convex for $a_n \neq 0$ and $n \in \mathbb{N}$.

**Proof.** The argument is similar to that for Theorem 3.1 except using the following statement for $\alpha \in [-3, 0]$—If $f \in U(D)$, then there exists $g(z) = \sum_{n=0}^{\infty} b_n z^n$ such that $g$ is the square root of the zero-free derivative $f'$ on $D$ and $f'(0) = g^2(0)$, and hence
\[
\Phi_{\alpha,1}(f, t) = (2\pi t)^{-1} \int_{|z|=1} |f'(z)||dz| = (2\pi t)^{-1} \int_{|z|=1} |g(z)|^2|dz| = \sum_{n=0}^{\infty} |b_n|^2 t^{2n}
\]
Thus, we complete the proof.

Our concluding example shows that under $0 < \alpha < \infty$ and $0 \leq \beta \leq 1$ one cannot get that $\log L_{\alpha,\beta}(f, r)$ is convex or concave in $\log r$ for all functions $f \in U(D)$.

**Example 3.2.** Let $\alpha = 1$, $\beta \in \{0, 1\}$ and $f(z) = (z + 2)^3$. Then the function $\log r \mapsto \log L_{\alpha,\beta}(f, r)$ is neither convex nor concave for $r \in (0,1)$.

**Proof.** Clearly, we have
\[
f'(z) = 3(z + 2)^2 \quad \text{and} \quad f''(z) = 6(z + 2)
\]
as well as the Schwarzian derivative
\[
\left[ \frac{f''(z)}{f'(z)} \right]' - \frac{1}{2} \left[ \frac{f''(z)}{f'(z)} \right]^2 = \frac{-4}{(z+2)^2}.
\]

It is easy to see that
\[
\sqrt{2}(1 - |z|^2) \leq 2 - |z|, \quad \forall z \in \mathbb{D}.
\]
So,
\[
\left| \frac{f''(z)}{f'(z)} \right|' - \frac{1}{2} \left[ \frac{f''(z)}{f'(z)} \right]^2 \leq \frac{4}{|z+2|^2} \leq \frac{2}{(1 - |z|^2)^2}.
\]
By Lemma 3.2(ii), \( f \) belongs to \( U(\mathbb{D}) \). Consequently,
\[
L(f, t) = \int_0^{2\pi} |f'(te^{i\theta})| t d\theta = 6\pi t(t^2 + 4)
\]
and
\[
\int_0^r \Phi_{L, \beta}(f, t) d\mu_1(t) = \begin{cases} 
12\pi \left( \frac{4}{3} r^3 - \frac{3}{5} r^5 - \frac{1}{7} r^7 \right), & \text{when } \beta = 0, \\
12r^2 - \frac{9}{2} r^4 - r^6, & \text{when } \beta = 1.
\end{cases}
\]
Note that \( v_1(r) = r^2 - r^4/2 \). So,
\[
L_{1, \beta}(f, r) = \begin{cases} 
\frac{24\pi(140r - 63r^3 - 15r^5)}{105(2 - r^2)}, & \text{when } \beta = 0, \\
\frac{24 - 9r^2 - 2r^4}{2 - r^2}, & \text{when } \beta = 1.
\end{cases}
\]
To gain our conclusion, we only need to consider the logarithmic convexity of the function
\[
h_\beta(x) = \begin{cases} 
\frac{140x - 63x^3 - 15x^5}{2 - x^2}, & \text{when } \beta = 0, \\
\frac{24 - 9x - 2x^2}{2 - x}, & \text{when } \beta = 1.
\end{cases}
\]
Case 1: \( \beta = 0 \). Applying the definition of \( D \)-notation, we obtain
\[
D(140x - 63x^3 - 15x^5) = \frac{-35280x - 33600x^3 + 3780x^5}{(140 - 63x^2 - 15x^4)^2}
\]
and
\[
D(2 - x^2) = \frac{-8x}{(2 - x)^2},
\]
whence reaching
\[
D(h_0(x)) = D(140x - 63x^3 - 15x^5) - D(2 - x^2) = \frac{4xh_0(x)}{(140 - 63x^2 - 15x^4)^2(2 - x^2)^2}.
\]
where
\[ g_0(x) = 3920 - 33600x^2 + 28098x^4 - 8400x^6 + 1395x^8. \]

Obviously,
\[ g_0(0) = 3920 > 0 \quad \text{and} \quad g_0(1) = -8587 < 0. \]

Now letting \( s = x^2 \), we get
\[ g_0(x) = G_0(s) = 3920 - 33600s + 28098s^2 - 8400s^3 + 1395s^4, \]
and
\[ G'_0(s) = -33600 + 56196s - 25200s^2 + 5580s^3 \quad \text{and} \quad G''_0(s) = 56196 - 50400s + 16740s^2. \]

Since the axis of symmetry of \( G'_0(s) \) is \( s = 140/93 > 1 \), \( G''_0(s) \) is decreasing on \((0,1)\). Due to \( G''_0(1) = 22536 > 0 \), we have \( G''_0(s) > 0 \) for all \( s \in (0,1) \), i.e., \( G'_0(s) \) is increasing on \((0,1)\). By
\[ G'_0(0) = -33600 < 0 \quad \text{and} \quad G'_0(1) = 2976 > 0, \]
we conclude that there exists an \( s_0 \in (0,1) \) such that \( G'_0(s) < 0 \) for \( s \in (0,s_0) \) and \( G'_0(s) > 0 \) for \( s \in (s_0,1) \). Then there exists an \( x_0 \in (0,1) \) such that \( g_0(x) \) is decreasing for \( x \in (0,x_0) \) and \( g_0(x) \) is increasing for \( x \in (x_0,1) \). Thus there exists an \( x_1 \in (0,1) \) such that \( g_0(x) > 0 \) for \( x \in (0,x_1) \) and \( g_0(x) < 0 \) for \( x \in (x_1,1) \). As a result, we find that \( \log r \mapsto \log L_{a,0}(f,r) \) is neither concave nor convex.

Case 2: \( \beta = 1 \). Again using the \( D \)-notation, we obtain
\[ D(24 - 9x - 2x^2) = \frac{-216 - 192x + 18x^2}{(24 - 9x - 2x^2)^2} \]
and
\[ D(2 - x) = \frac{-2}{(2 - x)^2}, \]
whence deriving
\[ D(h_1(x)) = D(24 - 9x - 2x^2) - D(2 - x) = \frac{2g_1(x)}{(24 - 9x - 2x^2)^2(2 - x)^2}, \]
where
\[ g_1(x) = 144 - 384x + 297x^2 - 96x^3 + 13x^4. \]

Now we have
\[ g'_1(x) = -384 + 594x - 288x^2 + 52x^3 \quad \text{and} \quad g''_1(x) = 594 - 576x + 165x^2. \]

Since the axis of symmetry of \( g''_1(x) \) is \( x = 24/13 > 1 \), \( g''_1(x) \) is decreasing on \((0,1)\). Due to \( g''_1(1) = 174 > 0 \), we have \( g''_1(x) > 0 \) for all \( x \in (0,1) \), i.e., \( g'_1(x) \) is increasing on \((0,1)\). By
\[ g'_1(0) = -384 < 0 \quad \text{and} \quad g'_1(1) = -26 < 0, \]
we conclude that \( g'_1(x) < 0 \) for \( x \in (0,1) \). Obviously,
\[
g_1(0) = 144 > 0 \quad \text{and} \quad g_1(1) = -26 < 0.
\]
Hence there exists an \( x_0 \in (0,1) \) such that \( g_1(x) > 0 \) for \( x \in (0,x_0) \) and \( g_1(x) < 0 \) for \( x \in (x_0,1) \). Consequently, we find that \( \log r \mapsto \log L_{\alpha,\beta} = 1(f,r) \) is neither concave nor convex. \( \square \)

Acknowledgements

J. Xiao and W. Xu were in part supported by NSERC of Canada and the Finnish Cultural Foundation, respectively.

References