Analysis in Theory and Applications
Anal. Theory Appl., Vol. **30**, No. 1 (2014), pp. 130-135
DOI: 10.4208/ata.2014.v30.n1.9

A Sufficient Condition for Rigidity in Extremality of Teichmüller Equivalence Classes by Schwarzian Derivative

Masahiro Yanagishita*

Departments in Fundamental Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku, Tokyo 169-8555, Japan

Received 27 November 2013; Accepted (in revised version) 5 March 2014

Available online 31 March 2014

Abstract. The Strebel point is a Teichmüller equivalence class in the Teichmüller space that has a certain rigidity in the extremality of the maximal dilatation. In this paper, we give a sufficient condition in terms of the Schwarzian derivative for a Teichmüller equivalence class of the universal Teichmüller space under which the class is a Strebel point. As an application, we construct a Teichmüller equivalence class that is a Strebel point and that is not an asymptotically conformal class.

Key Words: Strebel points, the Schwarzian derivative, asymptotically conformal maps.

AMS Subject Classifications: 30F60, 30C62

---

1 Introduction

The Teichmüller space is the deformation space of marked Riemann surfaces. Indeed, the space is defined as the quotient space of the family of marked Riemann surfaces by a certain topological equivalence relation. Each element of the Teichmüller space is called the Teichmüller equivalence class. Especially, the Teichmüller space of the hyperbolic plane is called the universal Teichmüller space, denoted by T.

It is known that each Teichmüller equivalence class has a quasiconformal mapping with smallest maximal dilatation in its class, which is called extremal. It is generally difficult to find an extremal quasiconformal mapping in each Teichmüller equivalence class. However the extremal quasiconformal mapping is uniquely determined in a certain Teichmüller equivalence class named the Strebel point, which means the class where the boundary dilatation is less than the maximal dilatation. This result is called Strebel's frame mapping theorem (see Chapter 4 in [5]). There exist several studies of the Strebel

*Corresponding author. *Email address:* `m-yanagishita@asagi.waseda.jp` (M. Yanagishita)

point. For example, Lakic [6] proved that the set of Strebel points is open and dense in the Teichmüller space and Earle and Li [4] showed that a Teichmüller equivalence class τ is a Strebel point if and only if there exists exactly one geodesic connecting the basepoint to τ in the Teichmüller distance. Hence the Strebel point is an important concept for the Teichmüller theory.

In this paper, we deal with Strebel points of the universal Teichmüller space T. In other words, we will give a sufficient condition for a Teichmüller equivalence class under which the class is a Strebel point. We take $\Delta^* = \{|z| > 1\} \cup \{\infty\}$ as the model of the hyperbolic plane. The main tool is the Schwarzian derivative, which induces a homeomorphic embedding of T into the Banach space $\mathcal{B}$ of holomorphic functions on the unit disk $\Delta = \{|z| < 1\}$ with finite hyperbolic sup-norm

$$\|\varphi\|_{\mathcal{B}} = \sup_{z\in\Delta}(1-|z|^2)^2|\varphi(z)|.$$

We have preliminaries in Section 2 and Section 3. Section 2 is devoted to introduce some properties of the Schwarzian derivative in the Teichmüller theory. Especially, we discuss the quasiconformal extensibility of a meromorphic function on Δ to the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by estimating the hyperbolic sup-norm of its Schwarzian derivative. Section 3 contains a brief summary of the universal Teichmüller space and Strebel points. We introduce an example of Strebel points named the asymptotically conformal class, whose boundary dilatation vanishes. The set T_0 of asymptotically conformal classes is a closed submanifold of T. In fact, T_0 is embedded into a closed subspace $\mathcal{B}_0$ of the Banach space $\mathcal{B}$, where each element of $\mathcal{B}$ vanishes in the semi-norm

$$\|\varphi\|_{\mathcal{B}_0} = \limsup_{|z|\to 1}(1-|z|^2)^2|\varphi(z)|.$$

We call this norm the boundary hyperbolic sup-norm. The subspace T_0 is studied in its analytic and metric structure (see [1,2]).

In Section 4 our main result is stated and proved. We will require a comparison of the hyperbolic and boundary hyperbolic sup-norm in the sufficient condition. As an application, we construct a Teichmüller equivalence class that is a Strebel point and that is not an asymptotically conformal class. Since it is generally difficult to find such a class specifically, it is considered that such an example is significant.

2 Schwarzian derivative and quasiconformal mappings

In this section, we discuss a relation between the Schwarzian derivative and quasiconformal mappings. Let f be a meromorphic function in a simply connected domain A of $\hat{\mathbb{C}}$. Then the expression

$$S_f = \Big(\frac{f''}{f'}\Big)' - \frac{1}{2}\Big(\frac{f''}{f'}\Big)^2$$

is called the *Schwarzian derivative* of f. A sense-preserving homeomorphism g is *quasiconformal* if g is ACL (absolutely continuous on lines) in A and there exists a constant $0<k<1$ such that

$$|\bar{\partial}g(z)| \le k|\partial g(z)|, \tag{2.1}$$

for a.e. $z\in A$. By the definition, the function

$$\mu(z)=\frac{\bar{\partial}g(z)}{\partial g(z)}$$

is determined for a.e. $z\in A$, which is called the *Beltrami coefficient* of g. Since g is continuous, μ is a measurable function in A, and from the inequality (2.1) we see that

$$\|\mu\|_\infty=\operatorname*{ess\,sup}_{z\in A}|\mu(z)|\le k<1.$$

Conversely, it is known that for every measurable function μ in A with L^∞-norm less than 1, there exists a quasiconformal mapping g in A with Beltrami coefficient μ.

We first state that the Schwarzian derivative can be prescribed.

Proposition 2.1. Let φ be a holomorphic function in Δ. Then there exists a meromorphic function f in Δ such that

$$S_f=\varphi.$$

The solution f is unique up to post-composition with Möbius transformation.

The Schwarzian derivative S_f of a meromorphic function f relates the univalence and quasiconformal extensibility of f to $\hat{\mathbb{C}}$ through the hyperbolic sup-norm.

Proposition 2.2. Let f be a meromorphic function in Δ. If

$$\|S_f\|_{\mathcal{B}}<2, \tag{2.2}$$

then f is univalent and can be extended to a quasiconformal mapping of $\hat{\mathbb{C}}$ with Beltrami coefficient

$$\nu(1/\bar{z})=-\frac{1}{2}(z/\bar{z})^2(1-|z|^2)^2S_f(z), \tag{2.3}$$

for $z\in\Delta$.

Conversely, if f is a quasiconformal mapping of $\hat{\mathbb{C}}$ that is conformal in Δ, then the hyperbolic sup-norm of the Schwarzian derivative $S_{f|_\Delta}$ can be estimated by the L^∞-norm of its Beltrami coefficient.

Proposition 2.3. Let f be a quasiconformal mapping of $\hat{\mathbb{C}}$ that has the Beltrami coefficent μ and that is conformal in Δ. Then

$$\|S_{f|_\Delta}\|_{\mathcal{B}}\le 6\|\mu\|_\infty. \tag{2.4}$$

These propositions can be generalized to the case of meromorphic functions on an arbitrary simply connected domain A. In this case, the constants 2, 6 in inequalities (2.2), (2.4) are replaced to the ones depending only on A, respectively (see Chapter II in [7]).

3 Universal Teichmüller space and Strebel points

In this section, we introduce the universal Teichmüller space and Strebel points.

Let Bel be the set of Beltrami coefficients on Δ^*. For $\mu\in$Bel, we extend μ to $\hat{\mathbb{C}}$ by letting $\mu\equiv 0$ on Δ. Then there exists a quasiconformal mapping of $\hat{\mathbb{C}}$ with extended Beltrami coefficient μ. Let f_μ be such a quasiconformal mapping normalized to fix 1, i, -1. Given $\mu,\nu\in$ Bel, μ and ν are *Teichmüller equivalent* if

$$f_\mu|_\Delta = f_\nu|_\Delta. \tag{3.1}$$

Then the *universal Teichmüller space* T is defined as the quotient space of Bel by this equivalence relation. For $\mu\in$ Bel, let $[\mu]$ be the Teichmüller equivalence class represented by μ. The point of T determined by $\mu\equiv 0$ on Δ^* is especially called the *base point* of T and denoted by 0.

For $p,q\in T$, define

$$d_T(p,q)=\frac{1}{2}\inf\log\frac{1+\|(\mu-\nu)/(1-\bar{\mu}\nu)\|_\infty}{1-\|(\mu-\nu)/(1-\bar{\mu}\nu)\|_\infty}, \tag{3.2}$$

where the infimum is taken over all $\mu\in p$ and $\nu\in q$. This function is called the *Teichmüller distance* on T. The metric space (T,d_T) is complete and contractible (cf. Chapter III in [7]).

For $\tau\in T$, the expression

$$k(\tau)=\inf\{\|\mu\|_\infty|\mu\in\tau\}$$

is called the *maximal dilatation* of τ. Note that $k(\tau)=\tanh d_T(0,\tau)$, which means that k measures the difference between the base point and τ with respect to the deformation of marked conformal structures on Δ^*. Similarly to the maximal dilatation, define

$$h^*(\mu)=\inf\{\|\mu|_{\Delta^*\setminus E}\|_\infty|E\subset\Delta^*:\text{compact in }\hat{\mathbb{C}}\}$$

for $\mu\in$ Bel and

$$h(\tau)=\inf\{h^*(\mu)|\mu\in\tau\}$$

for $\tau\in T$. The expression $h(\tau)$ is called the *boundary dilatation* of τ. By the definition, it follows immediately that for any $\tau\in T$,

$$h(\tau)\le k(\tau). \tag{3.3}$$

A Teichmüller equivalence class $\tau\in T$ is called a *Strebel point* if $h(\tau)<k(\tau)$.

As we mentioned previously, the Strebel point has a certain rigidity in the exremality of Teichmüller equivalence classes. For any $\tau\in T$, there exists a Beltrami coefficient on Δ such that $k(\tau)=\|\mu\|_\infty$. Such a Beltrami coefficient μ is said to be *extremal* in τ. If τ is a

Strebel point, then there exists a holomorphic function φ on Δ^* with $\iint_{\Delta^*}|\varphi(z)|dxdy=1$ such that the extremal Beltrami coefficient of τ is uniquely given by

$$k(\tau)\frac{\overline{\varphi}}{|\varphi|}. \tag{3.4}$$

A typical example of Strebel points is the asymptotically conformal class. A Teichmüller equivalence class $\tau\in T$ is *asymptotically conformal* if $h(\tau)=0$. Recall that T_0 is the set of asymptotically conformal classes in T and $\mathcal{B}_0$ be the set of holomorphic functions on Δ with $\|\varphi\|_{\mathcal{B}_0}=0$. It follows that $[\mu]\in T_0$ if and only if $S_{f_\mu|_\Delta}\in\mathcal{B}_0$ (cf. [3]).

4 Main theorem and an application

In this section, we state and show the main theorem.

Theorem 4.1. *If a Teichmüller equivalence class* $[\mu]\in T$ *satisfies*

$$3\|S_{f_\mu|_\Delta}\|_{\mathcal{B}_0}<\|S_{f_\mu|_\Delta}\|_{\mathcal{B}}<2, \tag{4.1}$$

then $[\mu]$ *is a Strebel point.*

Proof. The second inequality of condition (4.1) corresponds to condition (2.2). Then it follows from Proposition 2.2 that $f_\mu|_\Delta$ has another quasiconformal extension to $\hat{\mathbb{C}}$ with Beltrami coefficient ν of form (2.3). Since $f_\nu|_\Delta=f_\mu|_\Delta$, ν belongs to $[\mu]$. By inequality (2.4) and the first inequality of condition (4.1), we have

$$h([\mu])\le h^*(\nu)=\frac{\|S_{f_\mu|_\Delta}\|_{\mathcal{B}_0}}{2}<\frac{\|S_{f_\mu|_\Delta}\|_{\mathcal{B}}}{6}\le k([\mu]).$$

Therefore $[\mu]$ is a Strebel point. □

As an application of this theorem, we construct a Teichmüller equivalence class that is a Strebel point and that is not an asymptotically conformal class.

For $1<R<2^{5/4}-1$, set

$$\varphi_R(z)=\frac{(R-z)^2}{(1-z)^2}$$

on Δ. It clearly follows that φ is holomorphic in Δ. Fix $0\le r<1$ and let

$$\alpha_R(\theta)=|\varphi_R(re^{i\theta})|=\frac{R^2-2Rr\cos\theta+r^2}{1-2r\cos\theta+r^2}$$

for $0\le\theta<2\pi$. By differentiating α_R, we have

$$\alpha_R'(\theta)=\frac{2r(R-r^2)(1-R)\sin\theta}{(1-2r\cos\theta+r^2)^2}.$$

Then it follows that

$$\max_{0\le\theta<2\pi}\alpha_R(\theta)=\alpha_R(0)=\frac{(R-r)^2}{(1-r)^2}.$$

Let $\beta_R(r)=(1-r^2)^2\max_{0\le\theta<2\pi}\alpha_R(\theta)=(1+r)^2(R-r)^2$. By a simple calculation, we have

$$\|\varphi_R\|_{\mathcal{B}}=\sup_{0\le r<1}\beta_R(r)=\beta_R((R-1)/2)=\Big(\frac{R+1}{2}\Big)^4$$

and

$$\|\varphi_R\|_{\mathcal{B}_0}=\limsup_{r\to 1-0}\beta_R(r)=4(R-1)^2. \tag{4.2}$$

Since φ_R is holomorphic in Δ, it follows from Proposition 2.1 that there exists a meromorphic function f_R in Δ such that

$$S_{f_R}=\varphi_R.$$

This fact and $R<2^{5/4}-1$ imply that $\|S_{f_R}\|_{\mathcal{B}}<2$. By Proposition 2.2, f_R is univalent in Δ and has a quasiconformal extension to $\hat{\mathbb{C}}$ with Beltrami coefficient

$$\mu_R(1/\bar{z})=-\frac{1}{2}(z/\bar{z})^2(1-|z|^2)^2S_{f_R}(z)$$

for $z\in\Delta$. It follows from formula (4.2) that the Teichmüller equivalence class $[\mu_R]$ is not asymptotically conformal for $1<R<2^{5/4}-1$. By a simple computation, we have

$$3\|S_{f_R}\|_{\mathcal{B}_0}<\|S_{f_R}\|_{\mathcal{B}}$$

for $1<R<2^{5/4}-1$. From Theorem 4.1, $[\mu_R]$ is a Strebel point and is not an asymptotically conformal for $1<R<2^{5/4}-1$.

References

[1] C. J. Earle, F. P. Gardiner and N. Lakic, Asymptotic Teichmüller space, part I: the complex structure, Contemp. Math., 256 (2000), 17–38.

[2] C. J. Earle, F. P. Gardiner and N. Lakic, Asymptotic Teichmüller space, part II: the metric structure, Contemp. Math., 355 (2004), 187–219.

[3] C. J. Earle, V. Markovic and D. Saric, Barycentric extension and the Bers embedding for asymptotic Teichmüller space, Contemp. Math., 311 (2002), 87–105.

[4] C. J. Earle and Zhong Li, Isometrically embedded polydisks in infinite dimensional Teichmüller spaces, J. Geom. Anal., 9 (1999), 51–71.

[5] F. P. Gardiner and N. Lakic, Quasiconformal Teichmüller theory, Amer. Math. Soc., 2000.

[6] N. Lakic, Strebel points, Contemp. Math., 211 (1997), 417–431, Amer. Math. Soc., 1997.

[7] O. Lehto, Univalent Functions and Teichmüller Spaces, Graduate Texts in Mathmatics Vol. 109, Springer-Verlag, New York, 1987.