Global Meromorphic Solutions of Partial Differential Equations

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Abstract. In this paper, we survey some recent progress on analytic theory of partial differential equations.

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1 Introduction

We [19] studied meromorphic solutions of homogeneous linear partial differential equations of the second order in two independent complex variables

$$a_0 \frac{\partial^2 u}{\partial t^2} + 2a_1 \frac{\partial^2 u}{\partial t \partial z} + a_2 \frac{\partial^2 u}{\partial z^2} + a_3 \frac{\partial u}{\partial t} + a_4 \frac{\partial u}{\partial z} + a_6 u = 0, \tag{1.1}$$

where $a_k = a_k(t,z)$ are holomorphic functions for $(t,z) \in \Sigma$, where Σ is a region on $\mathbb{C}^2$.

When t and z are real variables, Hilbert's 19-th problem conjectures that if all $a_k = a_k(t,z)$ are analytic on t and z, then any solution $u = u(t,z)$ of an elliptic equation of the form (1.1) also is analytic on its existing region, which was confirmed by S. N. Bernšteĭn [3] provided one knows that $u \in C^3$.

H. Lewy [23], using the solvability of the initial value problem for hyperbolic equations, gave a simple proof by extending t and z to a domain of $\mathbb{C}^2$. Further, I. G. Petrovskiĭ [29] extended this result to general non-linear elliptic systems. It also is known that all regular solutions of linear elliptic equations of the second order have bounded derivatives up to order k, provided all coefficients have bounded derivatives up to order k.

We follow Lewy's idea to study the Eq. (1.1) on a region $\Sigma \subseteq \mathbb{C}^2$.

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2 Linear partial differential equations of second order related to Bessel functions

To explain our idea clearly, here we examine the following special differential equation:

$$t^2\frac{\partial^2 u}{\partial t^2} - z^2\frac{\partial^2 u}{\partial z^2} + t\frac{\partial u}{\partial t} - z\frac{\partial u}{\partial z} + t^2 u = 0. \tag{2.1}$$

Theorem 2.1. *The differential equation (2.1) has an entire solution $f(t,z)$ on $\mathbb{C}^2$ if and only if f is an entire function expressed by the series*

$$f(t,z) = \sum_{n=0}^{\infty} n! c_n J_n(t) z^n, \tag{2.2}$$

such that

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 0, \tag{2.3}$$

where $J_n(t)$ is the first kind of Bessel's function of order n. Moreover, the order $\mathrm{ord}(f)$ of the entire function f satisfies

$$\rho \le \mathrm{ord}(f) \le \max\{1,\rho\},$$

where

$$\rho = \limsup_{n\to\infty} \frac{2\log n}{\log \frac{1}{\sqrt[n]{|c_n|}}}. \tag{2.4}$$

By definition, the order of f is defined by

$$\mathrm{ord}(f) = \limsup_{r\to\infty} \frac{\log^+ \log^+ M(r,f)}{\log r},$$

where

$$\log^+ x = \begin{cases} \log x, & \text{if } x \ge 1, \\ 0, & \text{if } x < 1, \end{cases}$$

and

$$M(r,f) = \max_{|t|\le r, |z|\le r} |f(t,z)|.$$

G. Valiron [34] showed that each transcendental entire solution of a homogeneous linear ordinary differential equation with polynomial coefficients is of finite positive order. However, Theorem 2.1 shows that Valiron's theorem is not true for general partial differential equations. Here we exhibit another example that the following equation

$$t^2\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial z^2} + t\frac{\partial u}{\partial t} = 0$$

has an entire solution $\exp(te^z)$ of infinite order.

If $0 < \lambda = \mathrm{ord}(f) < \infty$, we define the type of f by

$$\mathrm{typ}(f) = \limsup_{r \to \infty} \frac{\log^+ M(r,f)}{r^\lambda}.$$

For the type of entire solutions of the Eq. (2.1), we have an analogue of Lindelöf-Pringsheim theorem, the proof of which is essentially the same as that of determining of the type for Taylor series of entire functions of one complex variable.

Theorem 2.2. *If $f(t,z)$ is an entire solution of (2.1) defined by (2.2) and (2.3) such that $1 < \lambda = \mathrm{ord}(f) < \infty$, then the type $\sigma = \mathrm{typ}(f)$ satisfies*

$$e\lambda\sigma = 2^{-\lambda/2} \limsup_{n \to \infty} 2n \sqrt[2n]{|c_n|^\lambda}.$$

Next we give one uniqueness theorem related to (2.1).

Theorem 2.3. *Let $f(t,z)$ be a nonconstant meromorphic solution of (2.1) such that $\mathrm{ord}(f) < \infty$ and let g be a nonconstant meromorphic function of finite order on $\mathbb{C}^2$. If f and g share 0, 1, ∞ counting multiplicity, then we have either $g = f$ or $gf = 1$.*

For more detail, see Hu and Yang [19].

3 A linear homogeneous partial differential equation with entire solutions represented by Bessel polynomials

As a further illustration of this observation, here we study the following special case of the Eq. (1.1) with $a_6 = 0$

$$t^2 \frac{\partial^2 u}{\partial t^2} - z^2 \frac{\partial^2 u}{\partial z^2} + (2t+2) \frac{\partial u}{\partial t} - 2z \frac{\partial u}{\partial z} = 0. \tag{3.1}$$

Let us consider the Bessel polynomials

$$y_n(t) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)!k!} \left(\frac{t}{2}\right)^k,$$

which satisfy the differential equation

$$t^2 \frac{d^2 w}{dt^2} + (2t+2) \frac{dw}{dt} - n(n+1)w = 0. \tag{3.2}$$

The question of formal expansions

$$f(t) \sim \sum_{n=0}^{\infty} \tilde{c}_n y_n(t)$$

has been asked and answered already by Krall and Frink [21].

P. Rusev [31] indicates sufficient conditions for the convergence of series in Bessel polynomials, similar to those corresponding to series in Jacobi polynomials inside an ellipse with foci at 1 and -1.

However, the determination of necessary and sufficient conditions on $f(t)$, for the convergence, or the summability to $f(t)$ of the series appears to be an open problem (see [8]).

This problem has an answer if we study functions $f(t,z)$ of two variables replacing $f(t)$, and is closely related to the partial differential equation (3.1).

Theorem 3.1. *The partial differential equation* (3.1) *has an entire solution* $u = f(t,z)$ *on* $\mathbb{C}^2$ *if and only if* $u = f(t,z)$ *has a series expansion*

$$f(t,z) = \sum_{n=0}^{\infty} \frac{c_n}{n!} y_n(t) z^n, \qquad (3.3)$$

such that

$$\limsup_{n \to \infty} \sqrt[n]{|c_n|} = 0. \qquad (3.4)$$

For an entire function of several variables, little are known for determining relations between its growth and the coefficients of Taylor expansion, except for the Cauchy's inequality. However, for an entire solution of (3.1), or equivalently, the entire function (3.3) satisfying (3.4), we can prove an analogue of Lindelöf-Pringsheim theorem determining order of the function by using its coefficients c_n:

Theorem 3.2. *If* $f(t,z)$ *is an entire solution of* (3.1) *defined by* (3.3) *and* (3.4), *then*

$$\text{ord}(f) = \limsup_{n \to \infty} \frac{2 \log n}{\log \frac{1}{\sqrt[n]{|c_n|}}}.$$

By using Theorem 3.1 and Theorem 3.2, we can construct entire solutions of (3.1) with arbitrary order. We also have an analogue of Lindelöf-Pringsheim theorem determining the type of an entire function:

Theorem 3.3. *If* $f(t,z)$ *is an entire solution of* (3.1) *defined by* (3.3) *and* (3.4), *such that* $0 < \lambda = \text{ord}(f) < \infty$, *then the type* $\sigma = \text{typ}(f)$ *satisfies*

$$e\lambda\sigma = 2^{\lambda/2} \limsup_{n \to \infty} 2n \sqrt[2n]{|c_n|^{\lambda}}.$$

Here we note that in the original Lindelöf-Pringsheim theorem on entire functions of one variable, those formulae do not contain the factors 2. If $f(t,z)$ is an entire solution of (2.1) of finite order >1, there are results similar to Theorem 3.2 and Theorem 2.2, but there are no such results for the case of order ≤ 1.

Theorem 3.4. *Let $f(t,z)$ be a non-constant meromorphic solution of (3.1) such that $\mathrm{ord}(f) < \infty$ and let $g(t,z)$ be a non-constant meromorphic function of finite order on $\mathbb{C}^2$. Assume that f and g share 0, 1, ∞ counting multiplicities. Then g must be a Möbius transformation of f. Moreover, one of the following four cases is occurred:*

(a) $g = f$;
(b) $gf = 1$;
(c) $gf = f + g$;
(d) there exist a constant $b \neq 1$ and a polynomial β such that

$$f = \frac{1}{b-1}\left(e^\beta - 1\right), \quad g = \frac{b}{b-1}\left(1 - e^{-\beta}\right).$$

W. H. J. Fuchs [7] asked whether formulations of the Wiman-Valiron method, applicable to entire functions of a single variable, might be generalized to entire functions of two variables. Schumitzky's thesis (see [32,33]), which develops Rosenbloom's probabilistic methods [30], was in part the prompt for Fuchs's question. One of the main results is an inequality for the maximum modulus of an entire functions of two complex variables (in fact, Schumitzky deals with the general n-variable case) in terms of the maximum term of the Taylor series of the function. Fenton [6] gave an alternative account of Wiman-Valiron theory by way of comparison series following from Hayman [9] that allows, among other things, a sharpening of the Schumitzky's inequality. However, we can do this job better for the entire solutions of (3.1). Following Wiman-Valiron theory, for $r \geq 0$ we define the maximum term

$$\mu(r) = \max_{n \geq 0} \frac{|c_n|}{n!} y_n(r) r^n$$

and central index

$$\nu(r) = \max\left\{ m \,\middle|\, \mu(r) = \frac{|c_m|}{m!} y_m(r) r^m \right\}$$

of the entire function (3.3). Similar to the cases of entire functions of one complex variable, the functions $\mu(r)$ and $\nu(r)$ have good behavior, and determine completely the growth of f.

Theorem 3.5. *If $f(t,z)$ is an entire solution of (3.1) defined by (3.3) and (3.4), we have*

$$\mathrm{ord}(f) = \limsup_{r \to \infty} \frac{\log^+ \log^+ \mu(r)}{\log r} = \limsup_{r \to \infty} \frac{\log^+ \nu(r)}{\log r}.$$

Theorem 3.6. *If $f(t,z)$ is a transcendental entire solution of (3.1) defined by (3.3) and (3.4) such that $\mathrm{ord}(f) < \infty$, we have*

$$\lim_{r \to \infty} \frac{\log M(r,f)}{\log \mu(r)} = 1.$$

For more detail, see [20].

4 Linear partial differential equations of second order related to orthogonal polynomials

4.1 Main results

We consider a special form of the partial differential equation (1.1) as follows:

$$\alpha(t)\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}\alpha''(t)z^2\frac{\partial^2 u}{\partial z^2} + \{\alpha'(t)+\beta(t)\}\frac{\partial u}{\partial t} - \{\alpha''(t)+\beta'(t)\}z\frac{\partial u}{\partial z} = 0, \tag{4.1}$$

where $\alpha(\not\equiv 0)$, β are polynomials of t satisfying the conditions:

(A) Take two polynomials

$$\alpha(t)=\alpha_0+\alpha_1 t+\alpha_2 t^2, \quad \beta(t)=\beta_0+\beta_1 t,$$

such that the ordinary differential equation of second order

$$\alpha(t)\frac{d^2 w}{dt^2} + \{\alpha'(t)+\beta(t)\}\frac{dw}{dt} - n\{(n+1)\alpha_2+\beta_1\}w = 0 \tag{4.2}$$

has a polynomial solution $Q_n(t)$ of degree n for each $n \geq 0$.

In fact, up to a complex linear change of variable, the only polynomial set $\{Q_n(t)\}_{n=0}^{\infty}$ in the Assumption (**A**) are

(1) Jacobi polynomials $\{P_n^{(\lambda,\mu)}(t)\}_{n=0}^{\infty}$ for $\lambda,\mu,\lambda+\mu+1\notin\{-1,-2,\cdots\}$;

(2) Laguerre polynomials $\{L_n^{(\lambda)}(t)\}_{n=0}^{\infty}$ for $\lambda\notin\{-1,-2,\cdots\}$;

(3) Hermite polynomials $\{H_n(t)\}_{n=0}^{\infty}$;

(4) Bessel polynomials $\{y_n^{(\lambda,\mu)}(t)\}_{n=0}^{\infty}$ for $\lambda\notin\{0,-1,-2,\cdots\}$ and $\mu\neq 0$; or

(5) $\{t^n\}_{n=0}^{\infty}$.

For example, $Q_n(t)=t^n$ satisfies the differential equation

$$t^2\frac{d^2 w}{dt^2} + t\frac{dw}{dt} - n^2 w = 0,$$

where $\alpha(t)=t^2$, $\beta(t)=-t$, and (4.1) becomes the following form

$$t^2\frac{\partial^2 u}{\partial t^2} - z^2\frac{\partial^2 u}{\partial z^2} + t\frac{\partial u}{\partial t} - z\frac{\partial u}{\partial z} = 0, \tag{4.3}$$

which has meromorphic solutions $f(zt)$, where f is any meromorphic function on $\mathbb{C}$.

Generally, we can characterize all entire solutions of the partial differential equation (4.1) as follows:

Theorem 4.1. *Under the Assumption* **(A)***, the partial differential equation* (4.1) *has an entire solution* $f(t,z)$ *on* $\mathbb{C}^2$ *if and only if* f *is an entire function expressed by the series*

$$f(t,z) = \sum_{n=0}^{\infty} c_n Q_n(t) z^n, \tag{4.4}$$

such that

$$\limsup_{n\to\infty} |c_n Q_n(t)|^{1/n} = 0, \quad t \in \mathbb{C}. \tag{4.5}$$

By using asymptotic formula of $Q_n(t)$, we easily change the condition (4.5) into a restricting condition on the coefficients c_n only. For example, when $Q_n(t) = P_n^{(\lambda,\mu)}(t)$ are Jacobi polynomials in which the partial differential equation (4.1) is of the following form

$$(1-t^2)\frac{\partial^2 u}{\partial t^2} + z^2\frac{\partial^2 u}{\partial z^2} - \{\lambda - \mu + (\lambda + \mu + 2)t\}\frac{\partial u}{\partial t} + (\lambda + \mu + 2)z\frac{\partial u}{\partial z} = 0, \tag{4.6}$$

such that $\alpha(t) = 1 - t^2$ and $\beta(t) = -(\lambda - \mu) - (\lambda + \mu)t$, we have

$$\lim_{n\to\infty} |Q_n(t)|^{1/n} = \left| t + \sqrt{t^2 - 1} \right|,$$

and hence the condition (4.5) may be replaced by

$$\limsup_{n\to\infty} |c_n|^{1/n} = 0. \tag{4.7}$$

Moreover, the order $\mathrm{ord}(f)$ of the entire function f satisfies

$$\mathrm{ord}(f) = \limsup_{n\to\infty} \frac{2\log n}{\log |c_n|^{-1/n}}. \tag{4.8}$$

If we weaken the condition (4.5) by choosing c_n such that the series (4.4) converges in a domain $\Omega \subset \mathbb{C}^2$, the function $f(t,z)$, called a generating function of polynomials $Q_n(t)$, still is a solution of (4.1). Thus (4.1) serves as a partial differential equation of generating functions for the above special functions.

Theorem 4.2. *Take two distinct complex numbers* b_1 *and* b_2*. Assume that the Assumption* **(A)** *holds. Let* f *and* g *be nonconstant meromorphic functions of finite order on* $\mathbb{C}^2$*. If* f *and* g *share* b_1*,* b_2*,* ∞ *counting multiplicity such that* f *is a solution of* (4.1)*, then one of the following four cases is occurred:*

(a) $g = f$*;*

(b) $fg = b_1(f+g) + b_2^2 - 2b_1 b_2$*, and there exists a nonconstant polynomial* β *satisfying*

$$L\beta + D\beta = 0, \tag{4.9}$$

where

$$Lu = \alpha(t)\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}\alpha''(t)z^2\frac{\partial^2 u}{\partial z^2} + \{\alpha'(t) + \beta(t)\}\frac{\partial u}{\partial t} - \{\alpha''(t) + \beta'(t)\}z\frac{\partial u}{\partial z},$$

$$Du = \alpha(t)\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\alpha''(t)z^2\left(\frac{\partial u}{\partial z}\right)^2;$$

(c) $fg = b_2(f+g) + b_1^2 - 2b_1 b_2$, and there exists a nonconstant polynomial β satisfying (4.9);

(d) there exist a nonconstant polynomial β satisfying (4.9) *such that for some constant $b \notin \{0,1\}$,*

$$f = b_1 + \frac{b_2 - b_1}{b-1}(e^\beta - 1), \quad g = b_2 + \frac{b_2 - b_1}{b-1}(1 - be^{-\beta}).$$

For more detail, see [15].

4.2 An example

Here we consider a special case. In the Assumption **(A)**, the following case

$$\alpha(t) = 1 - t^2, \quad \beta(t) = -(\lambda - \mu) - (\lambda + \mu)t,$$

where $\lambda, \mu > -1$, is associated to the Jacobi polynomial $Q_n(t) = P_n^{(\lambda,\mu)}(t)$, which satisfies

$$(1-t^2)\frac{d^2 w}{dt^2} - \{(\lambda - \mu) + (\lambda + \mu + 2)t\}\frac{dw}{dt} + n\{\lambda + \mu + n + 1\}w = 0, \qquad (4.10)$$

such that the Eq. (4.1) has the form (4.6). The following special case of Theorem 4.2 was proved in [13]:

Theorem 4.3. *Suppose that f is a nonconstant meromorphic solution of finite order to the Eq.* (4.6) *and shares b_1, b_2 and ∞ with a meromorphic function g in $\mathbb{C}^2$, where b_1, b_2 are two distinct complex numbers. Then f must be a Möbius transformation of g and, furthermore, one of the following occurs:*

(a) $f = g$;

(b) $fg = b_1(f+g) + b_2^2 - 2b_1 b_2$;

(c) $fg = b_2(f+g) + b_1^2 - 2b_1 b_2$;

(d) $fg = \frac{b_2 b - b_1}{b-1}f + \frac{b_1 b - b_2}{b-1}g - b_1 b_2$, where $b \neq 0,1$ is a constant.

Recently, we [14] removed all the cases (b), (c), and (d). That is, we have

Theorem 4.4. *Suppose that f is a nonconstant meromorphic solution of finite order to the Eq.* (4.6) *and shares b_1, b_2 and ∞ with a meromorphic function g in $\mathbb{C}^2$, where b_1, b_2 are two distinct complex numbers. Then $f = g$.*

For further examples, see [35].

5   A general uniqueness theorem

We study meromorphic solutions of linear partial differential equations of the second order in m (≥ 1) independent complex variables

$$a_{-1}+a_0 u+\sum_{k=1}^{m} a_k \frac{\partial u}{\partial z_k}+\sum_{j=1}^{m}\sum_{k=1}^{m} a_{j,k}\frac{\partial^2 u}{\partial z_j \partial z_k}=0, \tag{5.1}$$

where $a_k=a_k(z)$, $a_{j,k}=a_{j,k}(z)$ are holomorphic functions for $z=(z_1,\cdots,z_m)\in\Sigma$ with $a_{j,k}=a_{k,j}$, where Σ is a region on $\mathbb{C}^m$.

To state the uniqueness theorem, we abbreviate

$$u_{z_k}=\frac{\partial u}{\partial z_k}, \quad u_{z_j z_k}=\frac{\partial^2 u}{\partial z_j \partial z_k},$$

and set

$$Du=\sum_{j=1}^{m}\sum_{k=1}^{m} a_{j,k} u_{z_j} u_{z_k}, \quad Lu=\sum_{k=1}^{m} a_k u_{z_k}+\sum_{j=1}^{m}\sum_{k=1}^{m} a_{j,k} u_{z_j z_k},$$

which satisfy the following simple properties

$$D(\lambda u)=\lambda^2 Du, \quad L(\lambda u)=\lambda Lu,$$

for a constant λ. We make the following assumption:

(D) All coefficients a_k, $a_{j,k}$ in (5.1) are polynomials and when $a_0=0$ there are no nonconstant polynomials u satisfying the system

$$\begin{cases} Du=0, \\ Lu=0. \end{cases}$$

Theorem 5.1 (see [11]). *Take two distinct complex numbers b_1 and b_2. Assume that the Assumption **(D)** holds. Let f and g be nonconstant meromorphic functions of finite order on $\mathbb{C}^m$. If f and g share b_1, b_2, ∞, counting multiplicity such that f is a solution of (5.1), then one of the following six cases is occurred:*

(a) $g=f$;

(b) $b_1 a_0+a_{-1}=0$, $fg=b_1(f+g)+b_2^2-2b_1 b_2$, and there exists a nonconstant polynomial β satisfying

$$L\beta+D\beta+a_0=0; \tag{5.2}$$

(c) $b_2 a_0+a_{-1}=0$, $fg=b_2(f+g)+b_1^2-2b_1 b_2$, and there exists a nonconstant polynomial β satisfying (5.2);

(d) $b_2a_0+a_{-1}=b(b_1a_0+a_{-1})$ for some constant $b\notin\{0,1\}$, and there exist a nonconstant polynomial β satisfying (5.2) *such that*

$$f=b_1+\frac{b_2-b_1}{b-1}\left(e^\beta-1\right), \quad g=b_2+\frac{b_2-b_1}{b-1}\left(1-be^{-\beta}\right);$$

(e) $a_0\neq0$, $b_1a_0+a_{-1}=0$, $f^2g^2=3b_2^2fg-b_2^3(f+g)$, and there exists a nonconstant polynomial γ satisfying

$$\begin{cases} D\gamma=\dfrac{1}{2}a_0, \\ L\gamma=\dfrac{3}{2}a_0; \end{cases} \tag{5.3}$$

(f) $a_0\neq0$, $b_2a_0+a_{-1}=0$, $f^2g^2=3b_1^2fg-b_1^3(f+g)$, and there exists a nonconstant polynomial γ satisfying

$$\begin{cases} D\gamma=\dfrac{1}{2}a_0, \\ L\gamma=-\dfrac{3}{2}a_0. \end{cases} \tag{5.4}$$

A part of Theorem 5.1 was given by Hu-Yang [19]. For the cases (a), (b), (c) and (d), g is a Möbius transformation of f, but it is not true for the cases (e) and (f). Further, we note that γ is a solution of the system (5.3) if and only if $-\gamma$ is a solution of the system (5.4). In this sense, the systems (5.3) and (5.4) are equivalent.

The condition **(D)** is necessary. For example, we consider the differential equation

$$\frac{\partial^2u}{\partial t^2}+\frac{\partial^2u}{\partial z^2}=0, \tag{5.5}$$

which has an entire solution of order 1

$$f(t,z)=e^{2(t+iz)}+e^{t+iz}+1,$$

where $i=\sqrt{-1}$ is the imaginary unit. Let's compare f with the following entire function of order 1

$$g(t,z)=e^{-2(t+iz)}+e^{-t-iz}+1.$$

Obviously, f and g share 0, 1, ∞ counting multiplicity, but g is not a Möbius transformation of f. Now the differential equation (5.5) has a nonconstant polynomial solution

$$u(t,z)=t+iz$$

satisfying

$$u_t^2+u_z^2=0,$$

that is, the condition **(D)** associated to the Eq. (5.5) is not satisfied.

The condition **(D)** is meaningful. We can prove that the condition **(D)** associated to the Eq. (4.6) holds, and hence Theorem 5.1 yields immediately Theorem 4.3.

The cases (b), (c) in Theorem 5.1 may happen. For example, we consider the differential equation (cf. [19])

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial u}{\partial t} - u = 0, \tag{5.6}$$

which has an entire solution of order 1

$$f(t,z) = e^{t+z}.$$

Let's compare f with the following entire function of order 1

$$g(t,z) = e^{-t-z}.$$

Obviously, f and g share 0, 1, -1, ∞ counting multiplicity, and satisfy $gf = 1$. Now the differential equation (5.2) has the form

$$u_{tt} + u_{zz} - u_t + u_t^2 + u_z^2 - 1 = 0,$$

which has a nonconstant polynomial solution

$$u(t,z) = t + z,$$

that is, the case (b) in Theorem 5.1 can indeed occur for the differential equation (5.6), where $b_1 = 0$ and $b_2 = 1$.

The case (d) in Theorem 5.1 may happen. For example, we consider the differential equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial u}{\partial t} - \frac{\partial u}{\partial z} = 0, \tag{5.7}$$

which has an entire solution of order 1

$$f(t,z) = \frac{1}{b-1}\left\{e^{t+z} - 1\right\},$$

where $b = e^c \neq 1$ for some complex number $c \in \mathbb{C}$. Let's compare f with the following entire function of order 1

$$g(t,z) = \frac{b}{b-1}\left\{1 - e^{-t-z}\right\}.$$

Obviously, f and g share 0, 1, ∞ counting multiplicity. The differential equation (5.2) has the form

$$u_{tt} + u_{zz} - u_t - u_z + u_t^2 + u_z^2 = 0,$$

which has a nonconstant polynomial solution

$$u(t,z) = t + z.$$

Now it is not difficult to show that the system of differential equations

$$\begin{cases} u_t^2 + u_z^2 = 0, \\ u_{tt} + u_{zz} - u_t - u_z = 0, \end{cases}$$

has no nonconstant polynomial solutions, that is, the condition **(A)** holds for (5.7). Hence the case (d) in Theorem 5.1 happens really for the differential equation (5.7), where $b_1 = 0$ and $b_2 = 1$.

The cases (e), (f) in Theorem 5.1 may happen. For example, we consider the differential equation

$$18\frac{\partial^2 u}{\partial t^2} - 9\frac{\partial u}{\partial z} + u = 0, \tag{5.8}$$

which has an entire solution of order 1

$$f(t,z) = -e^{(t+z)/3} - e^{(t+z)/6}.$$

Let's compare f with the following entire function of order 1

$$g(t,z) = -e^{-(t+z)/3} - e^{-(t+z)/6}.$$

Obviously, f and g share 0, 1, ∞ counting multiplicity, and satisfy the equation

$$f^2 g^2 = 3fg - f - g.$$

Now the system (5.3) of differential equations has the following form

$$\begin{cases} 18u_t^2 = \dfrac{1}{2}, \\ 18u_{tt} - 9u_z = \dfrac{3}{2}, \end{cases} \tag{5.9}$$

which has a nonconstant polynomial solution

$$u(t,z) = -\frac{1}{6}(t+z),$$

that is, the case (e) in Theorem 5.1 can indeed occur for the differential equation (5.8), where $b_1 = 0$ and $b_2 = 1$.

Remark 5.1. We gave examples to show that when $a_0 \neq 0$, the cases (e), (f) in Theorem 5.1 may be ruled out.

Remark 5.2. We can obtain similar result for certain non-linear partial differential equations. For example, see [12].

Acknowledgements

References

[1] J. Anastassiadis, Recherches Algébriques sur le Théorème de Picard-Montel, Exprosés sur la Théorie des Fonctions XVIII, Hermann, Paris, 1959.

[2] C. A. Berenstein and B. Q. Li, On certain first-order partial differential equations in $\mathbb{C}^n$, Harmonic Analysis, Signal Processing, and Complexity, 29–36, Progr. Math., 238, Birkhäuser Boston, 2005.

[3] S. N. Bernšteĭn, Sur la nature analytique des solutions de certaines équations aux deriveés parttielles du second order, C. R. Acad. Sci. Paris, 137 (1903), 778–781.

[4] S. Bochner, Über Sturm-Liouvillesche Polynomsysteme, Math. Z., 29 (1929), 730–736.

[5] G. Brosch, Eindeutigkeitssätze für Meromorphe Funktionen, Thesis, Technical University of Aachen, 1989.

[6] P. C. Fenton, Wiman-Valiron theory in two variables, Trans. Amer. math. Soc., 347(11) (1995), 4403–4412.

[7] W. H. J. Fuchs, A look at Wiman-Valiron Theory, Complex Analysis, Kentucky 1976, Lecture Notes in Math., vol. 599, Springer, 1977.

[8] E. Grosswald, Bessel Polynomials, Lecture Notes in Math. 698, Springer-Verlag, 1978.

[9] W. K. Hayman, The local growth of power series: a survey of the Wiman-Valiron method, Canad. Math. Bull., 17 (1974), 317–358.

[10] Y. Z. He and X. Z. Xiao, Algebroid Functions and Ordinary Differential Equations (Chinese), Science Press, Beijing, 1988.

[11] P. C. Hu and B. Q. Li, Unicity of meromorphic solutions of partial differential equations, Journal of Mathematical Sciences 173 (2011), 201–206; Translated from Sovremennaya Matematika i Ee Prilozheniya (Contemporary Mathematics and Its Applications), Vol. 67, Partial Differential Equations, 2010.

[12] P. C. Hu and B. Q. Li, On meromorphic solutions of nonlinear partial differential equations of first order, J. Math. Anal. Appl., 377(2) (2011), 881–888.

[13] P. C. Hu and B. Q. Li, On meromorphic solutions of linear partial differential equations of second order, J. Math. Anal. Appl., 393(1) (2012), 200–211.

[14] P. C. Hu and B. Q. Li, A note on meromorphic solutions of linear partial differential equations of second order, Complex Analysis and Operator Theory, DOI 10.1007/s 11785-013-0314-6.

[15] P. C. Hu and B. Q. Li, Linear partial differential equations of second order related to orthogonal polynomials, preprint.

[16] P. C. Hu and B. Q. Li, Deformations of meromorphic functions for partial differential equations, preprint.

[17] P. C. Hu and S. Mori, Elimination of defects of non-Archimedean holomorphic curves, Proceedings of the Second ISAAC Congress (edited by H. G. W. Begehr, R. P. Gilbert and J. Kajiwara), Vol. 1, 433–449, Kluwer Academic Publishers, 2000.

[18] P. C. Hu and C. C. Yang, Malmquist type theorem and factorization of meromorphic solutions of partial differential equations, Complex Variables, 27 (1995), 269–285.

[19] P. C. Hu and C. C. Yang, Global solutions of homogeneous linear partial differential equations of the second order, Michigan Math. J., 58 (2009), 807–831.

[20] P. C. Hu and C. C. Yang, A linear homogeneous partial differential equation with entire solutions represented by Bessel polynomials, J. Math. Anal. Appl., 368 (2010), 263–280.

[21] H. L. Krall and O. Frink, A new class of orthogonal polynomials: the Bessel polynomials, Trans. Amer. Math. Soc., 65 (1949), 100–115.

[22] M. A. Lavrentieff and B. V. Shabat, Methods of Functions of a Complex Variable (Chinese Translation), Higher Education Press, Beijing, 2006.

[23] H. Lewy, Neuer Beweis des analytischen Charakters der Losungen elliptische Differential-gleichungen, Math. Annl., 101 (1929), 609–619.

[24] B. Q. Li, Entire solutions of certain partial differential equations and factorization of partial derivatives, Trans. Amer. Math. Soc., 357(8) (2005), 3169–3177 (electronic).

[25] B. Q. Li and E. G. Saleeby, Entire solutions of first-order partial differential equations, Complex Variables, 48(8) (2003), 657–661.

[26] Y. Mo, Legendre Functions (Chinese), Shandong University Press, 1988.

[27] S. Mori, Elimination of defects of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{P}^n(\mathbb{C})$, Ann. Acad. Sci. Fenn., 24 (1999), 89–104.

[28] S. Mori, Elimination of defects of meromorphic mappings by small deformation, Proc. of US-Japan Seminar of the Delaware Conference, Kluwer Acad. Publ., (1999), 247–258.

[29] I. G. Petrovskiĭ, Dokl. Akad. Nauk SSSR 17(1937), 343-346; Mat. Sb. (N.S.), 5(47) (1939), 3–70.

[30] P. C. Rosenbloom, Probability and entire functions, Studies in Mathematical Analysis (G. Szegö et al., eds.), Stanford Univ., 1963.

[31] P. Rusev, Convergence of series of Jacobi and Bessel polynomials on the boundaries of their regions of convergence, B"lgar. Akad. Nauk. Otdel. Mat. Fiz. Nauk. Izv. Mat. Inst., 10 (1969), 17–26.

[32] A. Schumitzky, Wiman-Valiron Theory for Entire Functions of Several Complex Variables, Ph.D. Thesis, Cornell Univ., 1965.

[33] A. Schumitzky, A probabilistic approach to the Wiman-Valiron theory for entire functions of several complex variables, Complex Variables, 13 (1989), 85–98.

[34] G. Valiron, Lectures on the General Theory of Integral Functions, Toulouse: Édouard privat, 1923.

[35] Xin-Li Wang, Feng-Li Zhang and Pei Chu Hu, A linear homogeneous partial differential equation with entire solutions represented by Laguerre polynomials, Abstract and Applied Analysis, Volume 2012, Article ID609862, Doi:10.1155/2012/609862(USD 1000;IF 1.318).

[36] C. C. Yang and H. X. Yi, Uniqueness Theory of Meromorphic Functions, Mathematics and Its Applications, 557, Science Press/Kluwer Academic Publishers, 2003.