Residues of Logarithmic Differential Forms in Complex Analysis and Geometry

A. G. Aleksandrov[*]

Institute for Control Sciences, Russian Academy of Sciences, 65 Profsoyunaya str., B-342, GSP-7, Moscow, 117997, Russian Federation

Abstract. In the article, we discuss basic concepts of the residue theory of logarithmic and multi-logarithmic differential forms, and describe some aspects of the theory, developed by the author in the past few years. In particular, we introduce the notion of logarithmic differential forms with the use of the classical de Rham lemma and give an explicit description of regular meromorphic differential forms in terms of residues of logarithmic or multi-logarithmic differential forms with respect to hypersurfaces, complete intersections or pure-dimensional Cohen-Macaulay spaces. Among other things, several useful applications are considered, which are related with the theory of holonomic $\mathscr{D}$-modules, the theory of Hodge structures, the theory of residual currents and others.

Key Words: Logarithmic differential forms, de Rham complex, regular meromorphic forms, holonomic $\mathscr{D}$-modules, Poincaré lemma, mixed Hodge structure, residual currents.

AMS Subject Classifications: 32S25, 14F10, 14F40, 58K45, 58K70

1 Introduction

The purpose of the present notes is to sketch broad outlines of basic concepts of the residue theory of logarithmic and multi-logarithmic differential forms, and to describe some of little-known aspects of this theory, developed by the author in the past few years. In particular, we introduce the notion of logarithmic differential forms with the use of the classical de Rham lemma and then briefly review the theory of residue originated by H. Poincaré, J. Leray, K. Saito and others in various settings. Then we construct the sheaves of multi-logarithmic differential forms with respect to arbitrary reduced pure-dimensional Cohen-Macaulay space and describe their residues. In a certain sense, the set of all such forms could be viewed as the *universal* domain of definition of the residue

---

[*]Corresponding author. *Email address:* `ag_aleksandrov@mail.ru` (A. G. Aleksandrov)

map in the framework of the general residue theory. Among other things, we give an explicit description of regular meromorphic differential forms in terms of residues of logarithmic or multi-logarithmic differential forms with respect to hypersurfaces, complete intersections or pure-dimensional Cohen-Macaulay spaces.

It seems reasonable to say that further generalizations and interpretations of the notion of logarithmic differential forms and their residues have a large number of interesting applications in different contexts and settings, among which it should be mentioned the theory of arrangements of hyperplanes and hypersurfaces, the theory of index of vector fields and differential forms, deformation theory, tropical geometry, the theory of resolutions of singularities, the theory of integral representations and residual currents, etc. Moreover, in a certain sense the theory of b-functions can be considered as a noncommutative analog or an extension of the theory of logarithmic differential forms to the category of $\mathscr{D}$-modules, etc.

The paper is organized as follows. In the first sections we discuss simple properties of logarithmic differential forms with poles along a reduced divisor. In particular, we give an unorthodox definition of this notion with the use of a version of the classical de Rham lemma adopted to the case of singular hypersurfaces. Then we consider some applications involving a logarithmic version of the classical Poincaré lemma and the classification problem of integrable holonomic $\mathscr{D}$-modules of Fuchsian and logarithmic type, etc. The basic concept of regular meromorphic differential forms is discussed in Section 4. In the next three sections we describe an explicit construction of residues for meromorphic forms with logarithmic poles along reduced divisors and for multi-logarithmic forms with respect to complete intersections. As an application, in Section 8, we show how to describe the weight filtration on the logarithmic de Rham complex directly, without the use of Hironaka's resolution of singularities. In the final section quite a general construction of multi-logarithmic differential forms with respect to reduced puredimensional Cohen-Macaulay spaces is presented and some relations with the theory of residue currents are discussed.

I would like to thank all the organizers of the Conference, especially Professor Liangwen Liao, for their complimentary invitation and hospitality in Nanjing. I am also deeply indebted to all participants for very interesting conversations and stimulating scientific discussions during the meeting.

2   The de Rham lemma

Let M be a complex manifold of dimension m, $m \geq 1$, and let $X \subset M$ be an subset in an open neighborhood U of the distinguished point $\mathfrak{o} \in U \subset M$ defined by a sequence of functions $f_1, \cdots, f_k \in \mathcal{O}_U$. We denote by Ω_X^p, $p \geq 0$, the sheaves of germs of *regular holomorphic* differential p-forms on X; they are defined as the restriction to X of the quotient module

$$\Omega_X^p = \Omega_U^p / ((f_1, \cdots, f_k)\Omega_U^p + df_1 \wedge \Omega_U^{p-1} + \cdots + df_k \wedge \Omega_U^{p-1})\big|_X.$$

Then the ordinary de Rham differentiation d equips this family of sheaves with a structure of a complex; it is called the de Rham complex on X and is denoted by $(\Omega_X^\bullet, d)$.

Let $z = (z_1, \cdots, z_m)$ be a system of local coordinates in a neighborhood of the distinguished point $\mathfrak{o} \in M$, $\omega = \sum a_i dz_i \in \Omega_{M,\mathfrak{o}}^1$ and $\mathrm{Sing}(\omega) \subset M$ the analytic subset determined by vanishing of the coefficients of ω.

The *starting point* of our approach is the classical lemma due to G. de Rham (1954). Similar versions of the following statement exist in the context of real analytic, semi-meromorphic and smooth (C^∞)-functions, polynomials, distributions or currents, etc.

Lemma 2.1 (see [12]). *Suppose that* $\dim \mathrm{Sing}(\omega) = 0$ *or, in other terms, the sequence of coefficients* $a_1, \cdots, a_m$ *of* ω *is regular in the local analytical algebra* $\mathcal{O}_{M,\mathfrak{o}}$. *If the germ of a differential form* $\eta \in \Omega_{M,\mathfrak{o}}^p$ *satisfies the condition* $\omega \wedge \eta = 0$, *then* $\eta = \omega \wedge \xi$, *where* $\xi \in \Omega_{M,\mathfrak{o}}^{p-1}$. *In particular, there exists an exact sequence of* $\mathcal{O}_{M,\mathfrak{o}}$-*modules*

$$0 \longrightarrow \Omega_{M,\mathfrak{o}}^{p-1} \wedge \omega \longrightarrow \Omega_{M,\mathfrak{o}}^p \overset{\wedge \omega}{\longrightarrow} \Omega_{M,\mathfrak{o}}^{p+1} \longrightarrow \Omega_{M,\mathfrak{o}}^{p+1}/\omega \wedge \Omega_{M,\mathfrak{o}}^p \longrightarrow 0, \quad 0 \le p < m, \qquad (2.1)$$

so that the increasing complex $(\Omega_{M,\mathfrak{o}}^\bullet, \wedge \omega)$ *is acyclic in all dimensions* $0 \le p < m$.

Remark 2.1. Assume that $s = \mathrm{codim}(\mathrm{Sing}(\omega), M) \ge 1$, that is, the codimension of $\mathrm{Sing}(\omega) \subset M$ is at least 1. Then the sequence (2.1) is *exact* for all $0 \le p < s$ and vice versa. In particular, $H^p(\Omega_{M,\mathfrak{o}}^\bullet, \wedge \omega) = 0$ whenever $0 \le p < s$.

3   Logarithmic differential forms on manifolds

Let $D \subset M$ be a Cartier divisor. Then a local equation of the hypersurface D, in a suitable neighborhood U of the distinguished point $\mathfrak{o}$, is defined by the germ of a holomorphic function $h \in \mathcal{O}_{M,\mathfrak{o}}$: if $z = (z_1, \cdots, z_m)$ is a system of local coordinates in a neighborhood of $\mathfrak{o}$, then $h(z) = 0$ is a local equation of D at $\mathfrak{o}$ and $\mathcal{O}_{D,\mathfrak{o}} \cong \mathcal{O}_{M,\mathfrak{o}}/(h)$.

Given a *reduced* divisor D, the coherent analytic sheaves of germs of *logarithmic differential forms* $\Omega_M^p(\log D)$, $p \ge 0$, are locally defined as follows.

Definition 3.1. The $\mathcal{O}_{M,\mathfrak{o}}$-module $\Omega_{M,\mathfrak{o}}^p(\log D)$ consists of germs of meromorphic p-forms ω on M such that ω and $d\omega$ have at worst simple poles along D. In other terms, $h\omega$ and $hd\omega$ (or, equivalently, $dh \wedge \omega$) are *holomorphic* at $\mathfrak{o}$, that is, $h \cdot \Omega_{M,\mathfrak{o}}^p(\log D) \subseteq \Omega_{M,\mathfrak{o}}^p$ and $h \cdot d\Omega_{M,\mathfrak{o}}^p(\log D) \subseteq \Omega_{M,\mathfrak{o}}^{p+1}$ (as well as $dh \wedge \Omega_{M,\mathfrak{o}}^p(\log D) \subseteq \Omega_{M,\mathfrak{o}}^{p+1}$).

Remark 3.1. It is well-known that this notion was introduced and studied by P. Deligne (1970) for divisors with normal crossings, and then by K. Saito (1976) for arbitrary reduced divisors in a complex analytic manifold (see [21]). There are also few versions of this notion in the real analytic, semi-meromorphic and smooth cases.

Making use of the following variant of the de Rham lemma for *singular hypersurfaces*, we are able to describe explicitly *for all* $p \ge 0$ the kernel of exterior multiplication by the total differential of a holomorphic function.

Lemma 3.1 (see [2,3]). *Let $h \in \mathcal{O}_{M,\mathfrak{o}}$ be a germ of holomorphic function without multiple factors. Then for $p = 0,1,\cdots,m-1$ there are exact sequences of $\mathcal{O}_{M,\mathfrak{o}}$-modules*

$$0 \longrightarrow \Omega^p_{M,\mathfrak{o}}(\log D) \xrightarrow{\cdot h} \Omega^p_{M,\mathfrak{o}} \xrightarrow{\wedge dh} \Omega^{p+1}_{M,\mathfrak{o}}/h \cdot \Omega^{p+1}_{M,\mathfrak{o}} \longrightarrow \Omega^{p+1}_{D,\mathfrak{o}} \longrightarrow 0, \tag{3.1}$$

where by $\wedge dh$ we denote the homomorphism of exterior multiplication by the total differential of the function h.

Proof. Taking $\vartheta \in \mathrm{Ker}(\wedge dh) \subseteq \Omega^p_{M,\mathfrak{o}}$, we see that $dh \wedge \vartheta = h \cdot \xi$, $\xi \in \Omega^{p+1}_{M,\mathfrak{o}}$. It is clear, that $\omega = \vartheta/h$ is contained in $\Omega^p_{M,\mathfrak{o}}(\log D)$, because $dh \wedge \omega$ and $hd\omega$ are holomorphic simultaneously, and vice versa. □

Remark 3.2. In other words, we obtain an *invariant* description of modules of logarithmic differential forms as the kernel of the operator of exterior multiplication

$$\Omega^\bullet_{M,\mathfrak{o}}(\log D) \cong \frac{1}{h}\mathrm{Ker}(\wedge dh \colon \Omega^p_{M,\mathfrak{o}} \longrightarrow \Omega^{p+1}_{M,\mathfrak{o}}/h \cdot \Omega^{p+1}_{M,\mathfrak{o}}).$$

That is, the set of holomorphic forms annihilated by exterior multiplication by the total differential $\omega = dh$ coincides with $h \cdot \Omega^{p+1}_{M,\mathfrak{o}}(\log D)$.

Remark 3.3. The above result can be also regarded as an analog of an observation due to G. de Rham in the context of the theory of *generalized* functions. To be more precise, he has proved that distributions T of degree zero, satisfying the condition $T \wedge \omega = 0$, are equal to multiples of the *Dirac distribution δ* (see in [12, Eq. (5)]).

Corollary 3.1. Assume that $\Omega^p_M(\log D)$, for some $p \geq 1$, is a locally free $\mathcal{O}_M$-module. Then the sequence (3.1) is a projective resolution of the $\mathcal{O}_D$-modules Ω^{p+1}_D.

Remark 3.4. The class of hypersurfaces satisfying this condition has been considered for the first time by K. Saito [21] in relation with his studies of Gauss-Manin connection in the cohomology of the relative De Rham complex associated with the versal deformation of an isolated hypersurface singularity. Somewhat later, P. Cartier [9] called such hypersurfaces free divisors or Saito free divisors.

The class of Saito free divisors contains many types of hyperplane arrangements, discriminants, Koszul free divisors, etc.; they are studied in a number of books and articles. Some basic aspects of the theory of free divisors and their singularities have been studied in [3]; the most part of considerations there can be easily adopted to algebraic, real analytic and smooth cases.

Lemma 3.2. *For all $p \geq 0$ there are exact sequences of $\mathcal{O}_{M,\mathfrak{o}}$-modules*

$$0 \longrightarrow \Omega^p_{M,\mathfrak{o}}/dh \wedge \Omega^{p-1}_{M,\mathfrak{o}}(\log D) \xrightarrow{\cdot h} \Omega^p_{M,\mathfrak{o}}/dh \wedge \Omega^{p-1}_{M,\mathfrak{o}} \longrightarrow \Omega^p_{D,\mathfrak{o}} \longrightarrow 0,$$

where $\cdot h$ is the homomorphism of ordinary multiplication.

Proof. Similarly to Lemma 3.1.   □

Remark 3.5. It is easy to see that the exact sequences from Lemma 3.1 and Lemma 3.2 yield exact sequences of the corresponding *increasing* complexes endowed with the differential induced by the de Rham differentiation d.

Proposition 3.1 (see [2,3]). Assume that the hypersurface D is reduced, that is, the function h has no multiple factors. Then there exist the following exact sequences of $\mathcal{O}_{M,\mathfrak{o}}$-modules

$$0 \longrightarrow dh \wedge \Omega_{M,\mathfrak{o}}^{p-1}(\log D) \longrightarrow \Omega_{M,\mathfrak{o}}^{p} \oplus (dh/h) \wedge \Omega_{M,\mathfrak{o}}^{p-1} \longrightarrow \Omega_{M,\mathfrak{o}}^{p}(\log D) \overset{\cdot h}{\longrightarrow} \mathrm{Tors}\,\Omega_{D,\mathfrak{o}}^{p} \longrightarrow 0,$$

where $\mathrm{Tors}\,\Omega_D^p$ is the torsion submodule of Ω_D^p, $p \geq 1$.

Corollary 3.2. For all $1 \leq p < c = \mathrm{codim}(\mathrm{Sing}\,D, D)$ there are natural isomorphisms

$$\Omega_{M,\mathfrak{o}}^{p}(\log D) \cong \Omega_{M,\mathfrak{o}}^{p} + (dh/h) \wedge \Omega_{M,\mathfrak{o}}^{p-1}.$$

Remark 3.6. By analogy with the case of isolated singularities the most important analytical invariants of *non-isolated* singularities are the Milnor numbers $\mu^{(p)}$, each of which is equal to the rank of the corresponding *torsion* $\mathcal{O}_D$-module $\mathrm{Tors}\,\Omega_D^p$. It is well-known that $\mu^{(p)} = 0$ when $1 \leq p < c$. On the other hand, for the case $c = 1$ explicit calculations show that the union of the coordinate hyperplanes in 3-dimensional space, that is, a divisor with *normal crossings* has two *non-zero* Milnor numbers $\mu^{(1)} = 2$, $\mu^{(2)} = 1$, while for the *swallow tail*, the discriminant of an A_3-polynomial, one obtains $\mu^{(1)} = 2$, $\mu^{(2)} = 2$, and so on (see further details in [1]).

4   Holonomic $\mathscr{D}$-modules and the Poincaré lemma

Given a meromorphic differential form $\omega \in \Omega_M^{\bullet}(\log D)$ having logarithmic poles along a reduced divisor D with *normal crossings* on a smooth complex quasi-projective variety M, the restriction of ω to the complement $M \setminus D$ is a *closed* form (see [11]). This statement is a direct consequence of the degeneracy of a certain spectral sequence (see (3.2.14) in [11]). An analytic proof of a similar result for compact Kähler manifolds is described in [19], where the author uses the classical harmonic integral theory and the standard reduction to the case of divisors with normal crossings, based on Hironaka's resolution of singularities.

Herein, making use of the theory of holonomic $\mathscr{D}$-modules, we show how to obtain an *explicit* representation of closed meromorphic 1-forms with logarithmic poles along *arbitrary* reduced divisor $D \subset M$. As a consequence, we shall see that such forms are *exact* (see details in [5]).

By definition, a holonomic system of differential equations is represented by the following Pfaffian system:

$$du = \nabla u, \tag{4.1}$$

where the unknown u is either a vector or a matrix. Here ∇ is a coefficient matrix whose entries are *meromorphic* differential 1-forms on M. The singular locus of the system consists of points where the entries of the matrix ∇ are not holomorphic.

Proposition 4.1. Assume that system (4.1) is regular singular, integrable, and of dimension one. Then $\nabla = [\omega]$ and the coefficient form ω has the following representation:

$$\omega = d\log((g_1^{\lambda_1} \cdots g_p^{\lambda_p})g) = \sum \lambda_i \left(\frac{dg_i}{g_i}\right) + \frac{dg}{g}, \qquad (4.2)$$

where $\lambda_1, \cdots, \lambda_p \in \mathbb{C}$ are the characteristic numbers of the system and $g_1, \cdots, g_p, g, g^{-1} \in \mathcal{O}_{M,\varnothing}$.

This statement implies the following two assertions. In a certain sense, the first one can be regarded as a *logarithmic* variant of the classical Poincaré lemma for logarithmic differential 1-forms.

Corollary 4.1. Any closed logarithmic differential 1-form ω with poles along D has representation (4.2). In particular, the differential 1-form ω is exact.

In other terms, a *closed* logarithmic form is determined by the eigenvalues of the monodromy associated with the corresponding (logarithmic) holonomic $\mathscr{D}$-module. A method of computation of eigenvalues of the monodromy, the most important invariants of a regular singular holonomic integrable system, is also described in [5].

Corollary 4.2. Any one-dimensional regular singular integrable $\mathscr{D}$-module is meromorphically equivalent to a $\mathscr{D}$-module with the following coefficient matrix:

$$\nabla = d\log(g_1^{\lambda_1} \cdots g_p^{\lambda_p}), \quad \lambda_j \in \mathbb{C}, \quad j = 1, \cdots, p.$$

The Systems 4.1 with coefficient 1×1-matrices of such type are often called linear differential equations of Fuchs' type (see $n^o 2$ in [10]). It should be noted that in the many-dimensional case any integrable Fuchsian $\mathscr{D}$-module is *meromorphically* isomorphic to an integrable $\mathscr{D}$-module of *logarithmic* type.

5   Regular meromorphic forms

Throughout this section we assume that $\dim X = n$ and there exists a dualizing complex on X, for example, X is a *Cohen-Macaulay* space. Recall that the $\mathcal{O}_X$-module

$$\omega_X^n = \mathrm{Ext}_{\mathcal{O}_M}^{m-n}(\mathcal{O}_X, \Omega_M^m)$$

is called the Grothendieck *dualizing module* of X.

Definition 5.1. The coherent sheaf of $\mathcal{O}_X$-modules ω_X^p, $p \geq 0$, is locally defined as the set of germs of meromorphic differential forms ω of degree p on X such that $\omega \wedge \eta \in \omega_X^n$ for any $\eta \in \Omega_X^{n-p}$. In other terms (see [8, 16]),

$$\omega_X^p \cong \mathrm{Hom}_{\mathcal{O}_X}(\Omega_X^{n-p}, \omega_X^n) \cong \mathrm{Ext}_{\mathcal{O}_M}^{m-n}(\Omega_X^{n-p}, \Omega_M^m), \quad 0 \leq p \leq n.$$

The elements of ω_X^p are called *regular meromorphic* differential p-forms on X; such forms are described in different settings: in terms of Noether normalization and trace (see [8, 13, 16]), in terms of residual currents (see [7]), in the context of the theory of V-varieties (see [23]), and others.

Recall some useful properties of regular meromorphic differential forms:

1. $\omega_X^p = 0$, if $p < 0$ or $p > n$;

2. ω_X^p has no torsion, i.e., $\mathrm{Tors}\,\omega_X^p = 0$, $p \geq 0$;

3. the de Rham differential d acting on Ω_X^p is extended on the family of modules ω_X^p, $0 \leq p \leq n$, and equips this family with a structure of a complex $(\omega_X^\bullet, d)$;

4. there exist inclusions $\omega_X^p \subseteq j_* j^* \Omega_X^p$, $p \geq 0$, where $j \colon X \setminus Z \longrightarrow X$ is the canonical inclusion and $Z = \mathrm{Sing}\,X$; moreover, if X is a *normal* space, then $\omega_X^p \cong j_* j^* \Omega_X^p$.

Lemma 5.1 (see [8]). *Assume X is a complete intersection given by a regular sequence of functions $f_1, \cdots, f_k \in \mathcal{O}_U$ in a neighborhood U of $\mathfrak{o} \in X$. Then $n = m - k$ and for all $p \geq 0$ there exist exact sequences of $\mathcal{O}_{M,\mathfrak{o}}$-modules*

$$0 \longrightarrow \omega_{X,\mathfrak{o}}^p \xrightarrow{c_X^M} \mathrm{Ext}_{M,\mathfrak{o}}(\mathcal{O}_{X,\mathfrak{o}}, \Omega_{M,\mathfrak{o}}^{p+k}) \xrightarrow{\mathcal{E}} \left[\mathrm{Ext}_{M,\mathfrak{o}}(\mathcal{O}_{X,\mathfrak{o}}, \Omega_{M,\mathfrak{o}}^{p+k}) \right]^k,$$

where the map c_X^M is induced by multiplication by the fundamental class of $X \subset M$ and the map $\mathcal{E}$ is given by the rule: $\mathcal{E}(e) = (e \wedge dh_1, \cdots, e \wedge dh_k)$.

In particular, $\omega_{X,\mathfrak{o}}^n = \mathrm{Ext}_{\mathcal{O}_{M,\mathfrak{o}}}^k(\mathcal{O}_{X,\mathfrak{o}}, \Omega_{M,\mathfrak{o}}^m) \cong \mathcal{O}_{X,\mathfrak{o}}(dz_1 \wedge \cdots \wedge dz_m / df_1 \wedge \cdots \wedge df_k)$, so that the Grothendieck dualizing module ω_X^n is a *locally free* $\mathcal{O}_X$-module of rank one.

Corollary 5.1. Under the assumptions of Lemma 5.1 suppose $X = Y \cup Z$ is any irredundant decomposition. Then there is a natural inclusion $\omega_Y^\bullet \oplus \omega_Z^\bullet \hookrightarrow \omega_X^\bullet$.

Proof. The inclusion is induced from the long exact sequence of the functor Ext associated with the short exact sequence $0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \oplus \mathcal{O}_Z \longrightarrow \mathcal{O}_{Y \cap Z} \longrightarrow 0$. □

6 The residue of logarithmic forms

A series of interesting results is closely related with the concept of *residue*, the core of the complex analysis and geometry. First recall a reformulation of Definition 3.1.

Lemma 6.1 (see [21]). *In the notations of Section 3 let $D \subset U$ be a reduced hypersurface determined by a function h, and ω a meromorphic p-form on U, $p \geq 1$, with simple poles along D. Then the form ω has logarithmic poles along D if and only if there exists a holomorphic function g determining the hypersurface $V \subset U$, a holomorphic $(p-1)$-form ξ and a holomorphic p-form η on U such that*

(a) $\dim_{\mathbb{C}} D \cap V \leq m - 2$,

(b) $g\omega = \dfrac{dh}{h} \wedge \xi + \eta.$

Obviously, the function g determines a non-zero divisor of $\mathcal{O}_{D,x}$ for all $x \in D$. In particular, the restriction of $g^{-1}\xi$ to D is well-defined.

Definition 6.1 (see [21]). Under the assumptions of the Lemma above the residue form of ω is defined as follows:

$$\mathrm{res}(\omega) = \frac{1}{g}\xi|_D.$$

It is clear that the residue differential form is contained in the space $\mathcal{M}_D \otimes_{\mathcal{O}_D} \Omega_D^{p-1} \cong \mathcal{M}_{\widetilde{D}} \otimes_{\mathcal{O}_{\widetilde{D}}} \Omega_{\widetilde{D}}^{p-1}$, $p \geq 1$, where $\widetilde{D}$ is the normalization of D, and $\mathcal{M}_D$ and $\mathcal{M}_{\widetilde{D}}$ are sheaves of meromorphic functions on D and $\widetilde{D}$, respectively. On the other hand, if ω is holomorphic then $\mathrm{res}(\omega) = 0$ and vice versa.

Remark 6.1. In fact, the above definition is a generalization of the notion of residue originated by H.Poincaré who studied the case of hypersurfaces *without* singularities. In such case the function germ g is *invertible* and the multiplier g^{-1} is no longer required. The corresponding presentation for *closed* forms was described by J.-P.Leray (see [18]). Moreover, J.-B. Poly has proved later that this presentation is well-defined for *any* semi-meromorphic differential forms ω as soon as ω and $d\omega$ have simple poles along a hypersurface [20].

Proposition 6.1 (see [3]). There exists an exact sequence of complexes of $\mathcal{O}_M$-modules endowed with the differential induced by the de Rham differentiation

$$0 \longrightarrow \Omega_M^{\bullet} \longrightarrow \Omega_M^{\bullet}(\log D) \xrightarrow{\mathrm{res}} \omega_D^{\bullet-1} \longrightarrow 0,$$

and natural isomorphisms

$$\mathcal{H}_{DR}^p(\Omega_M^{\bullet}(\log D)) \cong \mathcal{H}_{DR}^{p-1}(\omega_D^{\bullet}), \quad p \geq 1,$$

where $\omega_D^{\bullet}$ is the complex of *regular meromorphic* differential forms on D and $\mathcal{H}_{DR}^*$ is the functor of cohomologies.

Recall that the complex $(\omega_D^{\bullet}, d)$ is *acyclic* in positive dimensions if D is a graded rational normal hypersurface singularity (see [14, Remark (4.8)(2)]), i.e., $\omega_D^{\bullet}$ is a *resolution* of the constant sheaf $\mathbb{C}_D$. Consequently, $\Omega_M^{\bullet}(\log D)$ is *acyclic* in all dimensions $p > 1$ in the case where D is a simple rational singularity of type A_k, D_k, E_6, E_7 or E_8 of dimension $n \geq 2$, and $\dim_{\mathbb{C}} \mathcal{H}_{DR}^1(\Omega_M^{\bullet}(\log D))$ is equal to the number of irreducible components of D.

7   Multi-logarithmic differential forms

Following [4], we now describe an extension of the above results in the framework of the many-dimensional residue theory.

Let D be a reduced divisor of complex manifold M of dimension m, and let $D = D_1 \cup \cdots \cup D_k$, $k \geq 1$, be any *irredundant* (not necessarily irreducible) decomposition of D. Then the component D_j is locally determined by a function $h_j \in \mathcal{O}_{M,o}$, $j = 1, \cdots, k$. The hypersurface $D = D_1 \cup \cdots \cup D_k$ and the intersection $C = D_1 \cap \cdots \cap D_k$ is locally defined by the function $h = h_1 \cdots h_k$ and by the ideal $\mathcal{I} = (h_1, \cdots, h_k)$ of $\mathcal{O}_{M,o}$, respectively. For convenience of notations we set $\widehat{D_j} = D_1 \cup \cdots \cup D_{j-1} \cup D_{j+1} \cup \cdots \cup D_k$ and $\widehat{D_1} = \varnothing$ for $k = 1$.

We denote the sheaves of meromorphic differential forms of degree $p \geq 1$ formed by differential p-forms with *simple* poles along D by $\Omega_M^p(D)$. Similarly, the sheaves $\Omega_M^p(\star D)$ consists of meromorphic differential forms with poles along D of *any* order so that $\Omega_M^p(\star \widehat{D_j})$ is also well-defined. By definition, $\Omega_M^p(\widehat{D_1}) = \Omega_M^p(\star \widehat{D_1}) \cong \Omega_M^p$.

Definition 7.1. Assume that $h_1, \cdots, h_k$ is a *regular* sequence. Then an element $\omega \in \Omega_M^p(D)$ locally satisfying the conditions

$$dh_j \wedge \omega \in \sum_{i=1}^{k} \Omega_{M,o}^{p+1}(\widehat{D_i}), \quad j = 1, \cdots, k,$$

is called the *multi-logarithmic* differential p-form with respect to the complete intersection C; the set of all such forms and the corresponding sheaf is denoted by $\Omega_M^p(\log C)$.

Thus, $\Omega_M^p(\log C)$, $p \geq 0$, are coherent sheaves of $\mathcal{O}_M$-modules and there are natural inclusions $h_j \cdot \Omega_M^p(\log C) \subseteq \Omega_M^p(\widehat{D_j})$, $j = 1, \cdots, k$, and $h \cdot \Omega_M^p(\log C) \subseteq (\mathcal{I}) \Omega_M^p$. Further, $\Omega_M^p(\log D) \subseteq \Omega_M^p(\log C)$, $p \geq 0$, $\Omega_M^0(\log D) \cong \Omega_M^0(\log C) \cong \mathcal{O}_M$, and $\Omega_M^m(\log D) \cong \Omega_M^m(\log C) \cong \Omega_M^m / h$. Moreover, there is an analog of the presentation from Remark 3.2.

Proposition 7.1. For all $p \geq 0$ there are exact sequences of $\mathcal{O}_{M,o}$-modules

$$0 \longrightarrow h \cdot \Omega_{M,o}^p(\log C) \longrightarrow \Omega_{M,o}^p \overset{\mathcal{E}}{\longrightarrow} \left(\Omega_{M,o}^{p+1} / (h_1, \cdots, h_k) \Omega_{M,o}^{p+1} \right)^k,$$

where the mapping $\mathcal{E}$ is locally defined by the rule: $\mathcal{E}(e) = (e \wedge dh_1, \cdots, e \wedge dh_k)$; so that

$$\Omega_{M,o}^\bullet(\log C) \cong \frac{1}{h} \mathrm{Ker}(\mathcal{E} \colon \Omega_{M,o}^p \longrightarrow \Omega_{M,o}^{p+1} / (h_1, \cdots, h_k) \Omega_{M,o}^{p+1}).$$

Remark 7.1. If $k = 1$, then multi-logarithmic differential forms are *logarithmic* in the sense of Definition 3.1. Indeed, we have $C = D$, so that $\Omega_M^p(\log C) = \Omega_M^p(\log D)$, $p \geq 0$.

8   The multiple residue

Let us now discuss the concept of *multiple residue* of multi-logarithmic differential forms with respect to a complete intersection.

Proposition 8.1 (see [6, 7])**.** For any multi-logarithmic differential form $\omega \in \Omega^p(\log C)$ there is a holomorphic function g, which is not identically zero on every irreducible component of the complete intersection C, a holomorphic differential form $\xi \in \Omega_U^{p-k}$ and a meromorphic p-form $\eta \in \sum_{i=1}^k \Omega_U^p(\widehat{D_i})$ such that there exists the following representation

$$g\omega = \frac{dh_1}{h_1} \wedge \cdots \wedge \frac{dh_k}{h_k} \wedge \xi + \eta. \tag{8.1}$$

First, note that the function g in Decomposition 8.1 is not a zero divisor in the local ring $\mathcal{O}_{C,\mathfrak{o}} \cong \mathcal{O}_{X,\mathfrak{o}}/(h_1, \cdots, h_k)\mathcal{O}_{X,\mathfrak{o}}$. Therefore, one can consider the restriction of the form ξ/g to the germ of the complete intersection $C = D_1 \cap \cdots \cap D_k$.

Definition 8.1. The restriction of differential form ξ/g to the complete intersection C is called the *multiple residue form* or *multiple residue* of the differential form ω; this form is denoted by

$$\mathrm{Res}_C(\omega) = \frac{\xi}{g}\Big|_C.$$

By definition, the multiple residue differential form is contained in the space

$$\mathcal{M}_C \otimes_{\mathcal{O}_C} \Omega_C^{q-k} \cong \mathcal{M}_{\widetilde{C}} \otimes_{\mathcal{O}_{\widetilde{C}}} \Omega_{\widetilde{C}}^{q-k}, \quad q \geq k,$$

where $\widetilde{C}$ is the normalization of C, and $\mathcal{M}_C$ and $\mathcal{M}_{\widetilde{C}}$ are quasicoherent sheaves of meromorphic functions on C and $\widetilde{C}$, respectively.

Remark 8.1. An analog of the decomposition (8.1) was studied by J.-P. Leray in the case where all the divisors D_i are *smooth* (see (42.4) in [18]). In such case the germ g is *invertible* similarly to the situation described in Remark 6.1.

Theorem 8.1 (see [7])**.** *Under the assumptions of Section 7 there are exact sequences of $\mathcal{O}_M$-modules*

$$0 \longrightarrow \sum_{i=1}^k \Omega_M^p(\star\widehat{D_i}) \longrightarrow \Omega_M^p(\log C) \xrightarrow{\mathrm{Res}_C} \omega_C^{p-k} \longrightarrow 0, \quad p \geq k \geq 1, \tag{8.2}$$

where ω_C^{p-k} is the module of regular meromorphic differential $(p-k)$-forms on C.

It is clear that $\Omega_M^p(\log D) \subset \Omega_M^p(\log C)$, so that the restriction of $\mathrm{Res}_C(\omega)$ to $\Omega_M^p(\log D)$ is also well-defined. Moreover, if $\omega \in \Omega_M^p(\log D)$ then we can choose the decomposition (8.1) in such a way that the form η is also *logarithmic* along D.

Theorem 8.2 (see [4]). *Under the same assumptions there exist exact sequences of $\mathcal{O}_M$-modules*

$$0 \longrightarrow \sum_{i=1}^{k} \Omega_M^p(\log \widehat{D_i}) \longrightarrow \Omega_M^p(\log D) \xrightarrow{\operatorname{Res}_C} \omega_C^{p-k} \longrightarrow 0, \quad p \geq k \geq 1. \qquad (8.3)$$

Remark 8.2. Note that there exist also some other explicit representations for the multiple residue p-form $(\xi/g)|_C$. For example, one is described by a $\bar{\partial}$-closed meromorphic current:

$$\left\langle \left[\frac{\xi}{g} \right] \Big|_C, \varphi \right\rangle = \lim_{\varepsilon \longrightarrow 0} \int_{C \cap \{|g| > \varepsilon\}} \frac{\xi}{g} \wedge \varphi,$$

where $\varphi \in \mathscr{D}_U^{2m-k-p}$ is a differential C^∞-form on an open set U with compact support defined by the map $(h_1, \cdots, h_k) \colon U \longrightarrow \mathbb{C}^k$ (see details in in [7, Théorème 3.1]). Slightly modifying some arguments from Section 2–Section 3 in [15], we can obtain another integral representation using a variant of Grothendieck residue for complete intersections.

9 The weight filtration

The concept of the weight filtration on the logarithmic de Rham complex for divisors with *normal crossings* in a nonsingular manifold was introduced by Deligne (1971) in describing the mixed Hodge structure on the cohomology of the complement of a divisor (see [11]). Since then, this theory has been extensively developed in many directions for various types of varieties and cohomology theories. However, almost all known generalizations are based on the reduction of the situation under consideration to the case of a divisor with normal crossings, on general theorems on resolution of singularities, and on the functoriality of the notion of the mixed Hodge structure.

Following [4, 6], now we proceed to a description of the weight filtration on the logarithmic de Rham complex for divisors whose irreducible components are given locally by a *regular* sequence of holomorphic functions. In particular, this allows us to compute the mixed Hodge structure on the cohomology of the complement of divisors of certain types *without resorting* to the above-mentioned reduction.

Given an analytic manifold M, let $D \subset M$ be a *reduced* divisor and $D = D_1 \cup \cdots \cup D_k$ be its *irreducible* decomposition. For the sake of simplicity we also suppose that components D_i, $i = 1, \cdots, k$, have no *self-intersections*. For any n-tuple $I = (i_1 \cdots i_n)$, $1 \leq i_1 < \cdots < i_n \leq k$, of length $n = \#(I)$ consider the germs:

$$D_I = D_{(i_1 \cdots i_n)} = D_{i_1} \cup \cdots \cup D_{i_n}, \quad C^I = C^{(i_1 \cdots i_n)} = D_{i_1} \cap \cdots \cap D_{i_n}.$$

We use $C^{(n)}$ to denote the germ of the analytic subspace of M determined by the unions of germs $C^{(i_1 \cdots i_n)}$ for all admissible n-tuples, so that $C^{(1)} = D$, $C^{(k)} = C^{(i_1 \cdots i_k)} = C$, and so on. It is also convenient to set $D_0 = C^0 = \varnothing$.

Definition 9.1. The *weight filtration*, or the *filtration of weights* W on the logarithmic de Rham complex $\Omega_M^p(\log D)$ is locally defined as follows:

$$W_n(\Omega_{M,\mathfrak{o}}^p(\log D)) = \begin{cases} 0, & n < 0, \\ \Omega_{M,\mathfrak{o}}^p, & n = 0, \\ \sum_{\#(I)=p} \Omega_{M,\mathfrak{o}}^p(\log D_I), & n \geq p, \quad 0 < p < k_{\mathfrak{o}}, \\ \sum_{\#(I)=n} \Omega_{M,\mathfrak{o}}^p(\log D_I), & \text{otherwise}, \end{cases}$$

where the number of irreducible components of D passing through the point $\mathfrak{o} \in M$ is denoted by $k_{\mathfrak{o}}$.

For example, for $k = 3$ the first non-trivial elements of the weight filtration one can represent in the following form:

$$
\begin{array}{ccccc}
W_0 & \Omega_M^1 & \Omega_M^2 & \Omega_M^3 & \Omega_M^4 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
W_1 & \sum \Omega_M^1(\log D_i) & \sum \Omega_M^2(\log D_i) & \sum \Omega_M^3(\log D_i) & \sum \Omega_M^4(\log D_i) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
W_2 & \sum \Omega_M^1(\log D_i) & \sum \Omega_M^2(\log(D_i \cup D_j)) & \sum \Omega_M^3(\log(D_i \cup D_j)) & \sum \Omega_M^4(\log(D_i \cup D_j)) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
W_3 & \sum \Omega_M^1(\log D_i) & \sum \Omega_M^2(\log(D_i \cup D_j)) & \Omega_M^3(\log D) & \Omega_M^4(\log D) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{array}
$$

Thus, $W_n(\Omega_{M,\mathfrak{o}}^p(\log D)) = \Omega_{M,\mathfrak{o}}^p(\log D)$, if $n \geq p \geq k_{\mathfrak{o}} \geq 1$. Next, W is an increasing filtration and, in view of d- and $\wedge$-closeness of the exterior algebra $\Omega_M^\bullet(\log D)$, there exist natural inclusions

$$d(W_n(\Omega_M^\bullet(\log D))) \subset W_n(\Omega_M^\bullet(\log D)),$$
$$W_n(\Omega_M^p(\log D)) \wedge W_\ell(\Omega_M^q(\log D)) \subset W_{n+\ell}(\Omega_M^{p+q}(\log D)),$$

for all integers p, q, n and ℓ.

The following statement can be regarded as a generalization of isomorphism (3.1.5.2) described in [11] for divisors with normal crossings to the case of divisors whose components are defined by a *regular* sequence of functions.

Let $\pi \colon \widetilde{C}^{(n)} \to C^{(n)}$ be the normalization morphism, so that $\widetilde{C}^{(n)}$ is the disjoint union of the normalizations $\widetilde{C}^{(i_1 \cdots i_n)}$ for all for all n-tuples with $n \geq 1$. Let ι denote the projection of $\widetilde{C}^{(n)}$ to M, so that $\iota = i \circ \pi$, where $i \colon C^{(n)} \to M$ is a natural inclusion.

Theorem 9.1. *Suppose that D satisfies the above assumptions and the normalization morphism induces an isomorphism of complexes*

$$\pi_* \colon \omega_{\widetilde{C}^{(n)}}^\bullet \cong \omega_{C^{(n)}}^\bullet.$$

Then the multiple residue map (8.3)

$$\mathrm{Res}_n^\bullet \colon W_n(\Omega_M^\bullet(\log D)) \longrightarrow \iota_* \omega_{\widetilde{C}^{(n)}}^\bullet[-n]$$

induces an isomorphism of complexes of $\mathcal{O}_M$-*modules*

$$\mathrm{Gr}_n^W(\Omega_M^\bullet(\log D)) \cong \iota_* \omega_{\widetilde{C}^{(n)}}^\bullet[-n],$$

where the complexes of sheaves of regular meromorphic differential forms on $C^{(n)}$ *and its normalization* $\widetilde{C}^{(n)}$ *are denoted by* $\omega_{C^{(n)}}^\bullet$ *and* $\omega_{\widetilde{C}^{(n)}}^\bullet$, *respectively.*

Further analysis shows that, under one of the standard assumptions on the ambient manifold M (smoothness, Kählerness, completeness, etc.), this filtration can be used for the computation of the canonical mixed Hodge structure on the cohomology of complements $H^*(M \backslash D, \mathbb{C})$ similarly to [23, pp. 532], without the use of theorems on resolution of singularities and the standard reduction to the case of normal crossings.

10   The multiple residue with respect to Cohen-Macaulay spaces

Now, we briefly discuss an explicit construction of multi-logarithmic differential forms and their residues with respect to *non-complete* intersections.

In the notations of Section 2, let $X \subset M$ be an analytic space defined by a sheaf of ideals $\mathcal{I}_X$ so that $\mathcal{O}_X = \mathcal{O}_M / \mathcal{I}_X$. Then $\mathcal{I}_X$ is locally generated by a sequence of function germs $f_1, \cdots, f_\ell \in \mathcal{O}_{M,\mathfrak{o}}$, that is, $\mathcal{I} = \mathcal{I}_{X,\mathfrak{o}} = (f_1, \cdots, f_\ell)\mathcal{O}_{M,\mathfrak{o}}$. Assume that X is a *reduced pure-dimensional Cohen-Macaulay* space. In particular, the height of $\mathcal{I}$ is equal to $k = m - n$, where n is the dimension of X, and $\ell \geq k$.

Taking a *regular* sequence $h_1, \cdots, h_k \in \mathcal{I}$ of *maximal* length, we denote by $D \subset M$ the divisor whose components D_i are defined as the zero-sets of the functions h_i, $i = 1, \cdots, k$. Set $h = h_1 \cdots h_k$ and denote by $C = D_1 \cap \cdots \cap D_k$ the corresponding complete intersection, similarly to the notations of Section 7. It is clear that $X \subseteq C$.

Remark 10.1. Thus, any pure-dimensional Cohen-Macaulay space X can be regarded as a component (not necessarily irreducible) of an *irredundant* decomposition of the complete intersection C (not necessarily unique) which has been just considered.

In what follows we shall analyze the local situation and, in order to simplify notations, often do not indicate the distinguished point.

Definition 10.1. A meromorphic differential form $\omega \in \Omega_M^p(D)$ locally satisfying the conditions

$$g\omega \in \sum_{i=1}^k \Omega_M^p(\widehat{D_i}), \quad dg \wedge \omega \in \sum_{i=1}^k \Omega_M^{p+1}(\widehat{D_i}), \quad \text{for all } g \in \mathcal{I},$$

is called the *multi-logarithmic* differential p-form with respect to the pair (X, C). The corresponding sheaf is denoted by $\Omega_M^p(\log X | C)$.

Thus, $\Omega_M^p(\log X|C)$, $p \geq 0$, are coherent sheaves of $\mathcal{O}_M$-modules and there are natural inclusions $h_j \cdot \Omega_M^p(\log X|C) \subseteq \Omega_M^p(\widehat{D_j})$, $h \cdot \Omega_M^p(\log X|C) \subseteq \Omega_M^p$, $\Omega_M^p(\log X|C) \subseteq \Omega_M^p(\log C)$, and so on.

Remark 10.2. If $X = D$ is a reduced hypersurface or $X = C$ is a complete intersection, then $\Omega_M^p(\log X|C)$ coincides with $\Omega_M^p(\log D)$ or $\Omega_M^p(\log C)$, respectively.

By Definition 10.1, if $\omega \in \Omega_M^{\bullet}(\log X|C)[k] = \Omega_M^{\bullet+k}(\log X|C)$ then $\omega = \vartheta/h$, where the holomorphic form $\vartheta \in \Omega_M^{\bullet}[k] = \Omega_M^{\bullet+k}$ satisfies the following congruences:

$$g\vartheta \equiv 0 \quad \mod (h_1, \cdots, h_k), \quad dg \wedge \vartheta \equiv 0 \quad \mod (h_1, \cdots, h_k), \quad \text{for all } g \in \mathcal{I}. \tag{10.1}$$

In such a case one can define the *residue symbol* $\left[\begin{smallmatrix} \vartheta \\ h_1, \cdots, h_k \end{smallmatrix} \right]$ which is an element of the corresponding *Cousin complex* $C_\Omega(M)$ on M (see details in [13]).

Theorem 10.1. *There exists a functorial residue map*

$$\mathrm{res}_X \colon \Omega_M^{\bullet}[k] \longrightarrow \omega_X^{\bullet}, \quad \mathrm{res}_X(\vartheta) = \left[\begin{matrix} \vartheta \\ h_1, \cdots, h_k \end{matrix} \right],$$

compatible with the canonical extension of the ordinary de Rham differentiation d to $\omega_X^{\bullet}$, so that

$$d\left[\begin{matrix} \vartheta \\ h_1, \cdots, h_k \end{matrix} \right] = (-1)^k \left(\left[\begin{matrix} d\vartheta \\ h_1, \cdots, h_k \end{matrix} \right] - \sum_{j=1}^k \left[\begin{matrix} dh_j \wedge \vartheta \\ h_1, \cdots, h_j^2, \cdots h_k \end{matrix} \right] \right).$$

Moreover, for any $p \geq 0$ the $\mathcal{O}_X$-module ω_X^p is generated by all the elements $\mathrm{res}_X(\vartheta)$, where $\vartheta \in \Omega_M^{p+k}$. In particular, res_X is a surjective map.

Proof. It is a direct consequence of considerations from [13]. $\qquad \square$

Corollary 10.1. There is a functorial map

$$\mathrm{Res}_{X|C} \colon \Omega_M^{\bullet}(\log X|C)[k] \longrightarrow \omega_X^{\bullet}, \quad \mathrm{Res}_{X|C}(\omega) = \mathrm{res}_X(h\omega).$$

Proof. This follows from the Theorem above since $h \cdot \Omega_M^{\bullet}(\log X|C) \subseteq \Omega_M^{\bullet}$. $\qquad \square$

Proposition 10.1. For any $\omega \in \Omega_M^{\bullet}(\log X|C)$ we get $\mathrm{Res}_{X|C}(\omega) = \mathrm{Res}_C(\omega)|_X \in \omega_X^{\bullet}$.

Proof. In fact, $\omega \in \Omega_M^{\bullet}(\log C)$ since $\Omega_M^{\bullet}(\log X|C) \subseteq \Omega_M^{\bullet}(\log C)$. Hence, the multiple residue $\mathrm{Res}_C(\omega) \in \omega_C^{\bullet}$ is given by decomposition (8.1) in view of Definition 8.1. On the other hand, $\omega_X^{\bullet}$ is naturally included in $\omega_C^{\bullet}$ in view of Corollary 5.1 and one can take the restriction of the residue-form $\mathrm{Res}_C(\omega)$ to $\omega_X^{\bullet}$. $\qquad \square$

Theorem 10.2. *Under the above assumptions there exist exact sequences of $\mathcal{O}_M$-modules*

$$\sum_{i=1}^k \Omega_M^p(\star \widehat{D_i}) \cap \Omega_M^p(\log X|C) \longrightarrow \Omega_M^p(\log X|C) \overset{\mathrm{Res}_{X|C}^p}{\longrightarrow} \omega_X^{p-k} \longrightarrow 0, \quad p \geq k \geq 1,$$

induced by sequences (8.2) from Theorem 8.1.

Proof. We have already remarked above that there are natural inclusions $\Omega^p_M(\log X|C) \subseteq \Omega^p_M(\log C)$. Thus, one can restrict the sequences of Theorem 8.1 to $\Omega^p_M(\log X|C)$. Furthermore, by Corollary 5.1 one has $\omega^\bullet_X \hookrightarrow \omega^\bullet_C$. It remains to verify that $\mathrm{Im}(\mathrm{Res}^p_{X|C}) \subseteq \omega^p_X$. $\square$

Proposition 10.2. The family of the multiple residue maps $\mathrm{Res}_{X|C}$ satisfy the transformation law. To be more precise, if $C \subseteq C'$ are two ambient complex intersections given by two regular sequences $(h_1, \cdots, h_k) \supseteq (h'_1, \cdots, h'_k)$ of maximal length containing in the ideal $\mathcal{J}$, then

$$\mathrm{Res}_{X|C}(\omega) = \mathrm{Res}_{X|C'}(\Delta\omega'),$$

where Δ is the determinant of a transformation $h'_i = \sum \alpha_{ij} h_j$, and $\omega' \in \Omega^\bullet_M(\log X|C')$ and $\omega \in \Omega^\bullet_M(\log X|C)$ are the corresponding multi-logarithmic differential forms.

Proof. By definition, there is a congruence

$$dh'_1 \wedge \cdots \wedge dh'_k \equiv \Delta \cdot dh_1 \wedge \cdots \wedge dh_k \ (\mathrm{mod}\ (h_1, \cdots, h_k)).$$

On the other hand, $\Delta \cdot \delta_{ij} = \sum_k A_{ik} \alpha_{kj}$, where A_{ik} is the cofactor of the element α_{ik}, and, consequently,

$$\Delta \cdot h_i = \sum_j \Delta \delta_{ij} h_j = \sum_{j,k} A_{ik} \alpha_{kj} h_j = \sum_k A_{ik} h'_k \Longrightarrow \Delta(h_1, \cdots, h_k) \subseteq (h'_1, \cdots, h'_k).$$

Now, taking $\Delta\omega' = \Delta\vartheta/h' \in \Omega^\bullet_M(\log X|C')$, we see that the decomposition (8.1)

$$g\Delta\omega' = \frac{dh'_1}{h'_1} \wedge \cdots \wedge \frac{dh'_k}{h'_k} \wedge \xi + \eta, \quad \eta \in \sum \Omega^\bullet_M(\widehat{D}'_i),$$

is equivalent to the inclusion

$$g\Delta\vartheta \in dh'_1 \wedge \cdots \wedge dh'_k \wedge \xi + (h'_1, \cdots, h'_k)\eta,$$

which implies

$$g\vartheta \in dh_1 \wedge \cdots \wedge dh_k \wedge \xi + (h_1, \cdots, h_k)\eta'.$$

Equivalently, for $\omega = \vartheta/h \in \Omega^\bullet_M(\log X|C)$ there is a decomposition

$$g\omega = \frac{dh_1}{h_1} \wedge \cdots \wedge \frac{dh_k}{h_k} \wedge \xi + \eta', \quad \eta' \in \sum \Omega^\bullet_M(\widehat{D}_i).$$

As a result,

$$\mathrm{Res}_C(\omega) = \frac{\xi}{g}\Big|_C = \mathrm{Res}_{C'}(\Delta\omega') = \frac{\xi}{g}\Big|_{C'},$$

where $C \subseteq C'$. It is not difficult to verify that similar identities remain valid for any other presentation $h'_i = \sum \beta_{ij} h_j$ (see [17]). $\square$

Finally, we also note that for *arbitrary* regular sequences $(h_1,\cdots,h_k)$ and $(h'_1,\cdots,h'_k)$ of maximal length from the ideal $\mathcal{I}$ we can choose a regular sequence $(h''_1,\cdots,h''_k)$ of the *same* length in the intersection $(h_1,\cdots,h_k)\cap(h'_1,\cdots,h'_k)$ and then apply the above considerations to the pairs (X,C''), (X,C'), (X,C), and so on (see [17, Korollar 3.4]).

Remark 10.3. As a result, one can consider the union $\bigcup_{\{C:\ C\supset X\}}\Omega^\bullet_M(\log X|C)$ as the natural "universal" *domain of definition* of the multiple residue map. Further considerations show that this set can be naturally identified with a subspace of the local cohomology module $H^k_{\{X\}}(\Omega^p_M)$ and can be expressed in terms of the corresponding *Cousin* complex on M presented by Čech cocycles with poles of the first order along the divisors D associated with the corresponding ambient complete intersections $C\supset X$ similarly to [3, Section 4] or [7, Théorème 2.4].

We see that Theorem 10.2 implies that

$$\mathrm{Ker}(\mathrm{Res}_{X|C})\supseteq\sum_{i=1}^k\Omega^\bullet_M(\star\widehat{D_i})\cap\Omega^\bullet_M(\log X|C).$$

Now we describe the kernel explicitly.

Corollary 10.2. In the above notations

$$\mathrm{Res}_{X|C}(\omega)=0\Leftrightarrow\exists C'\supseteq C,\ \text{so that}\ \mathrm{Res}_{X|C'}(\Delta\omega')=0.$$

Proof. It follows directly from the above considerations based on decomposition (8.1). One can also use another method described in [13, Satz 2.6]. □

In slightly different terms similar ideas can be used for a description of explicit integral representations of the multiple residue for non-complete intersections in the framework of the theory of *currents*.

More precisely, let X be as above. Then there is an analytic function g on X vanishes at the singular subset $\mathrm{Sing}X$ of X such that any section $w\in H^0(X,\omega^\bullet_X)$ can be presented over the open set $\{g\neq0\}$ as the quotient v/g, where $v\in H^0(X,\Omega^\bullet_X)$. That is, all sections of $\omega^\bullet_X$ are *uniformely meromorphic*. Then there is a $\bar\partial$-*closed* meromorphic current T_w on X defined by the following rule:

$$\langle T_w,\varphi\rangle=\langle VP_g(v/g),\varphi\rangle=\lim_{\varepsilon\longrightarrow0}\int_{|g|>\varepsilon}w\wedge\varphi,$$

where $\varphi\in\mathscr{D}^{n-\bullet,n}_X$ is a differential C^∞-form on X with compact support of the type $(n{-}\bullet,n)$, and VP_g is the symbol of *principal value* in the sense of Herrera-Lieberman (see [8, Proposition 4]). The aim of the proof is to check the relation $\bar\partial T_w=0$; it follows from basic properties of currents defined on a suitable ambient complete intersection $C\supseteq X$. Similar approaches are also discussed in [22, Théorème] and [7, Théorème 3.1, Remarque (3.3)]. In contrast with our approach these constructions are based on the standard reduction to the case of a normal crossing divisor (with the use of Hironaka's resolution of singularities) when the principal value is well-defined.

References

[1] A. G. Aleksandrov, Milnor numbers of nonisolated Saito singularities, Funct. Anal. Appl., 21(1) (1987), 1–9; translation from Funkts. Anal. Prilozh., 21(1) (1987), 1–10.

[2] A. G. Aleksandrov, On the De Rham complex of nonisolated singularities, Funct. Anal. Appl., 22(2) (1988), 131–133; translation from Funkts. Anal. Prilozh., 22(2) (1988), 59–60.

[3] A. G. Aleksandrov, Nonisolated hypersurface singularities,–In: Advances in Soviet Mathematics, (V. I.Arnol'd, editor), 1 (1990), 211–246.

[4] A. G. Aleksandrov, Multidimensional residue theory and the logarithmic de Rham complex, J. Singularities, 5 (2012), 1–18.

[5] A. G. Aleksandrov and A. N. Kuznetsov, Holonomic $\mathscr{D}$-modules of Fuchsian and logarithmic types, Complex Variables and Elliptic Equations, 56(1-4) (2011), 171–191.

[6] A. G. Aleksandrov, The multiple residue and the weight filtration on the logarithmic de Rham complex, Funct. Anal. Appl., 47(4) (2013), 247–260; translation from Funkts. Anal. Prilozh., 47(4) (2013), 1–17.

[7] A. G. Aleksandrov and A. K. Tsikh, Théorie des résidus de Leray et formes de Barlet sur une intersection complète singulière, C. R. Acad. Sci. Paris, Sér. I, 333(11) (2001), 973–978.

[8] D. Barlet, Le faisceau $\omega_X^\bullet$ sur un espace analytique X de dimension pure, Lecture Notes in Mathematics, 670 (1978), Springer-Verlag, 187–204.

[9] P. Cartier, Les arrangements d'hyperplans: un chapitre de geometric combinatoire, Lecture Notes in Mathematics, 901 (1981), Springer-Verlag, 1–22.

[10] D. Cerveau, Cohomologie relative des formes fermés méromorphes, C. R. Acad. Sci. Paris, Sér. I, 303(14) (1986), 685–688.

[11] P. Deligne, Théorie de Hodge II, Publ. l'I.H.É.S. Math., 40 (1971), 5–57.

[12] G. de Rham, Sur la division de formes et de courants par une forme linéaire, Comment. Math. Helvet., 28 (1954), 346–352.

[13] M. Kersken, Cousin complex und Nennersysteme, Math. Zetschr., 182 (1983), 389–402.

[14] M. Kersken, Reguläre Differentialformen, Manuscripta Math., 46(1) (1984), 1–25.

[15] O. Klehn, Real and complex indices of vector fields on complete intersection curves with isolated singularity, Compos. Math., 141 (2005), 525–540.

[16] E. Kunz, Holomorphe Differentialformen auf algebraischen Varietäten mit Singularitäten. I, Manuscr. Math., 15 (1975), 91–108.

[17] E. Kunz, Residuen von Differentialformen auf Cohen-Macaulay-Varietäten, Math. Zetschr., 152 (1977), 165–189.

[18] J.-P. Leray, La calcul différentiel et intégral sur une variété analytique complexe (Problème de Cauchy. III), Bull. Soc. Math. de France, 87 (1959), 81–180.

[19] J. Noguchi, A short analytic proof of closedness of logarithmic forms, Kodai Math. J., 18 (1995), 295–299.

[20] J.-B. Poly, Sur un théorème de J.Leray en théorie des résidus, C. R. Acad. Sci. Paris, Sér. A-B, 274 (1972), A171–A174.

[21] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields, J. Fac. Sci. Univ. Tokyo, Ser. IA, 27(2) (1980), 265–291.

[22] J. E. Solomin, Le résidu logarithmique dans les intersections non complètes, C. R. Acad. Sci. Paris, Sér. A-B, 284(17) (1977), A1061–A1064.

[23] J. H. M. Steenbrink, Mixed Hodge structure on the vanishing cohomology–In: Real and Complex Singularities (P. Holm, ed.) Proc. Nordic Summer School, Symp. Math., Oslo–76. Oslo: Sijthoff–Noordhoff Publ., Alphen aan den Rijn, 1977.