Analysis in Theory and Applications
Anal. Theory Appl., Vol. **30**, No. 1 (2014), pp. 51-81

DOI: 10.4208/ata.2014.v30.n1.4

Complex and p-Adic Meromorphic Functions $f'P'(f)$, $g'P'(g)$ Sharing a Small Function

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Received 10 September 2013; Accepted (in revised version) 1 March 2014

Available online 31 March 2014

Abstract. Let $\mathbb{K}$ be a complete algebraically closed p-adic field of characteristic zero. We apply results in algebraic geometry and a new Nevanlinna theorem for p-adic meromorphic functions in order to prove results of uniqueness in value sharing problems, both on $\mathbb{K}$ and on $\mathbb{C}$. Let P be a polynomial of uniqueness for meromorphic functions in $\mathbb{K}$ or $\mathbb{C}$ or in an open disk. Let f, g be two transcendental meromorphic functions in the whole field $\mathbb{K}$ or in $\mathbb{C}$ or meromorphic functions in an open disk of $\mathbb{K}$ that are not quotients of bounded analytic functions. We show that if $f'P'(f)$ and $g'P'(g)$ share a small function α counting multiplicity, then $f=g$, provided that the multiplicity order of zeros of P' satisfy certain inequalities. A breakthrough in this paper consists of replacing inequalities $n\geq k+2$ or $n\geq k+3$ used in previous papers by Hypothesis (G). In the p-adic context, another consists of giving a lower bound for a sum of q counting functions of zeros with $(q-1)$ times the characteristic function of the considered meromorphic function.

Key Words: Meromorphic, nevanlinna, sharing value, unicity, distribution of values.

AMS Subject Classifications: 12J25, 30D35, 30G06

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1 Introduction

Notation and Definition 1.1. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, complete with respect to an ultrametric absolute value $|\cdot|$. We will denote by $\mathbb{E}$ a field that is either $\mathbb{K}$ or $\mathbb{C}$. Throughout the paper we denote by a a point in $\mathbb{K}$. Given $R\in[0,+\infty]$ we define disks $d(a,R)=\{x\in\mathbb{K}||x-a|\leq R\}$ and disks $d(a,R^-)=\{x\in\mathbb{K}||x-a|<R\}$.

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A polynomial $Q(X)\in\mathbb{E}[X]$ is called *a polynomial of uniqueness for a family of functions $\mathcal{F}$ defined in a subset of* $\mathbb{E}$ if $Q(f)=Q(g)$ implies $f=g$. The definition of polynomials of uniqueness was introduced in [19] by P. Li and C. C. Yang and was studied in many papers [11,13,20] for complex functions and in [1,2,9,10,17,18], for p-adic functions.

Throughout the paper we will denote by $P(X)$ a polynomial in $\mathbb{E}[X]$ such that $P'(X)$ is of the form $b\prod_{i=1}^{l}(X-a_i)^{k_i}$ with $l\geq 2$ and $k_1\geq 2$. The polynomial P will be said *to satisfy Hypothesis (G)* if $P(a_i)+P(a_j)\neq 0$, $\forall i\neq j$.

We will improve the main theorems obtained in [5] and [6] with the help of a new hypothesis denoted by Hypothesis (G) and by thorougly examining the situation with p-adic and complex analytic and meromorphic functions in order to avoid a lot of exclusions. Moreover, we will prove a new theorem completing the 2nd Main Theorem for p-adic meromorphic functions. Thanks to this new theorem we will give more precisions in results on value-sharing problems.

Notation 1.1. Let L be an algebraically closed field, let $P\in L[x]\setminus L$ and let $\Xi(P)$ be the set of zeros c of P' such that $P(c)\neq P(d)$ for every zero d of P' other than c. We denote by $\Phi(P)$ its cardinal.

We denote by $\mathcal{A}(\mathbb{E})$ the $\mathbb{E}$-algebra of entire functions in $\mathbb{E}$, by $\mathcal{M}(\mathbb{E})$ the field of meromorphic functions in $\mathbb{E}$, i.e., the field of fractions of $\mathcal{A}(\mathbb{E})$ and by $\mathbb{E}(x)$ the field of rational functions. Throughout the paper, we denote by $\mathcal{A}(d(a,R^-))$ the $\mathbb{K}$-algebra of analytic functions in $d(a,R^-)$ i.e., the $\mathbb{K}$-algebra of power series $\sum_{n=0}^{\infty}a_n(x-a)^n$ converging in $d(a,R^-)$ and we denote by $\mathcal{M}(d(a,R^-))$ the field of meromorphic functions inside $d(a,R^-)$, i.e., the field of fractions of $\mathcal{A}(d(a,R^-))$. Moreover, we denote by $\mathcal{A}_b(d(a,R^-))$ the $\mathbb{K}$-subalgebra of $\mathcal{A}(d(a,R^-))$ consisting of the bounded analytic functions in $d(a,R^-)$, i.e., which satisfy $\sup_{n\in\mathbb{N}}|a_n|R^n<+\infty$. We denote by $\mathcal{M}_b(d(a,R^-))$ the field of fractions of $\mathcal{A}_b(d(a,R^-))$ and finally, we denote by $\mathcal{A}_u(d(a,R^-))$ the set of unbounded analytic functions in $d(a,R^-)$, i.e., $\mathcal{A}(d(a,R^-))\setminus\mathcal{A}_b(d(a,R^-))$. Similarly, we set $\mathcal{M}_u(d(a,R^-))=\mathcal{M}(d(a,R^-))\setminus\mathcal{M}_b(d(a,R^-))$.

Theorem 1.1 (see [9]). *Let $P(X)\in\mathbb{K}[X]$. If $\Phi(P)\geq 2$ then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$. If $\Phi(P)\geq 3$ then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ and for $\mathcal{A}_u(d(a,R^-))$. If $\Phi(P)\geq 4$ then P is a polynomial of uniqueness for $\mathcal{M}_u(d(a,R^-))$.*

Let $P(X)\in\mathbb{C}[X]$. If $\Phi(P)\geq 3$ then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{C})$. If $\Phi(P)\geq 4$ then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$.

Concerning polynomials such that P' has exactly two distinct zeros, we know other results:

Theorem 1.2 (see [1,2,18]). *Let $P\in\mathbb{K}[x]$ be such that P' has exactly two distinct zeros γ_1 of order c_1 and γ_2 of order c_2 with $\min\{c_1,c_2\}\geq 2$. Then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$.*

Theorem 1.3 (see [9,17]). *Let $P\in\mathbb{K}[x]$ be of degree $n\geq 6$ be such that P' only has two distinct zeros, one of them being of order 2. Then P is a polynomial of uniqueness for $\mathcal{M}_u(d(0,R^-))$.*

Theorem 1.4 (see [18]). *Let $P\in\mathbb{C}[x]$ be such that P' has exactly two distinct zeros γ_1 of order c_1 and γ_2 of order c_2 with* $\min\{c_1,c_2\}\geq 2$ *and* $\max(c_1,c_2)\geq 3$. *Then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$.*

In order to state theorems and recall the definition of a small function, we must recall the definition of the classical Nevanlinna functions both on a p-adic field and on the field $\mathbb{C}$ together with a few specific properties of ultrametric analytic or meromorphic functions [7,11,13].

Notation 1.2. Let log be a real logarithm function of base $b>1$ and let $\log^+(x)=\max(0,\log(x))$. Let $f\in\mathcal{M}(\mathbb{E})$ (resp. $f\in\mathcal{M}(d(0,R^-))$) having no zero and no pole at 0. Let $r\in[0,+\infty]$ (resp. $r\in[0,R]$) and let $\gamma\in d(0,r)$. If f has a zero of order n at γ, we put $\omega_\gamma(h)=n$. If f has a pole of order n at γ, we put $\omega_\gamma(f)=-n$ and finally, if $f(\gamma)\neq 0,\infty$, we set $\omega_\gamma(f)=0$. These definitions of Nevanlinna's functions are equivalent to these defined in [7].

We denote by $Z(r,f)$ the *counting function of zeros of f* in $d(0,r)$, counting multiplicities, i.e.,

$$Z(r,f)=\max(\omega_0,0)\log r+\sum_{\omega_\gamma(f)>0,\ 0<|\gamma|\leq r}\omega_\gamma(f)(\log r-\log|\gamma|).$$

Similarly, we denote by $\overline{Z}(r,f)$ the counting function of zeros of f in $d(0,r)$, ignoring multiplicities, and set

$$\overline{Z}(r,f)=u\log r+\sum_{\omega_\gamma(f)>0,\ 0<|\gamma|\leq r}(\log r-\log|\gamma|)$$

with $u=1$ when $\omega_0(f)>0$ and $u=0$ else.

In the same way, we set $N(r,f)=Z(r,1/f)$ (resp. $\overline{N}(r,f)=\overline{Z}(r,1/f)$) to denote the *counting function of poles of f* in $d(0,r)$, counting multiplicities (resp. ignoring multiplicities).

For $f\in\mathcal{M}(\mathbb{K})$ or $f\in\mathcal{M}(d(0,R^-))$, we call *Nevanlinna function of f* the function $T(r,f)=\max\{Z(r,f),N(r,f)\}$.

Consider now a function $f\in\mathcal{M}(\mathbb{C})$. We can define a function

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{i\theta})|d\theta$$

and we call *Nevanlinna function of f* the function $T(r,f)=m(r,f)+N(r,f)$.

Now, we must recall the definition of a *small function* with respect to a meromorphic function and some pertinent properties.

Definition 1.1. Let $f\in\mathcal{M}(\mathbb{E})$ (resp. let $f\in\mathcal{M}(d(0,R^-))$) such that $f(0)\neq 0,\infty$. A function $\alpha\in\mathcal{M}(\mathbb{E})$ (resp. $\alpha\in\mathcal{M}(d(0,R^-))$) is called *a small function with respect to f*, if it satisfies

$$\lim_{r\to+\infty}\frac{T(r,\alpha)}{T(r,f)}=0,\quad\text{resp.}\quad\lim_{r\to R^-}\frac{T(r,\alpha)}{T(r,f)}=0.$$

We denote by $\mathcal{M}_f(\mathbb{E})$ (resp. $\mathcal{M}_f(d(0,R^-))$) the set of small meromorphic functions with respect to f in $\mathbb{E}$ (resp. in $d(0,R^-)$).

Remark 1.1. Thanks to classical properties of the Nevanlinna function $T(r,f)$ with respect to the operations in a field of meromorphic functions, such as $T(r,f+g)\le T(r,f)+T(r,g)+\mathcal{O}(1)$ and $T(r,fg)\le T(r,f)+T(r,g)+\mathcal{O}(1)$, for $f,g\in\mathcal{M}(\mathbb{K})$ and $r>0$, it is easily proven that $\mathcal{M}_f(\mathbb{E})$ (resp. $\mathcal{M}_f(d(0,R^-))$) is a subfield of $\mathcal{M}(\mathbb{E})$ (resp. $\mathcal{M}(d(0,R^-))$) and that $\mathcal{M}(\mathbb{E})$ (resp. $\mathcal{M}(d(0,R))$) is a transcendental extension of $\mathcal{M}_f(\mathbb{E})$ $\big($resp. of $\mathcal{M}_f(d(0,R^-))\big)$ [10].

Let us remember the following definition.

Definition 1.2. Let $f,g,\alpha\in\mathcal{M}(\mathbb{E})$ (resp. let $f,g,\alpha\in\mathcal{M}(d(0,R^-))$). We say that f and g *share the function α C.M.*, if $f-\alpha$ and $g-\alpha$ have the same zeros with the same multiplicities in $\mathbb{E}$ (resp. in $d(0,R^-)$).

In [5] and [6], we have obtained this general Theorem (where results of [5] and [6] here are gathered):

Theorem 1.5. *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, (resp. for $\mathcal{M}(\mathbb{C})$, resp. for $\mathcal{M}(d(0,R^-))$) with $l\ge 2$, $k_i\ge k_{i+1}$, $2\le i\le l-1$ and let $k=\sum_{i=2}^{l}k_i$. Suppose P satisfies the following conditions:*

(1) $k_1\ge 10+\sum_{i=3}^{l}\max(0,4-k_i)+\max(0,5-k_2)$,

(2) $k_1\ge k+2$ (resp. $k_1\ge k+3$, resp. $k_1\ge k+3$),

(3) if $l=2$, then $k_1\ne k+1,2k,2k+1,3k+1$,

(4) if $l=3$, then $k_1\ne k+1,2k+1,3k_i-k$, $\forall i=2,3$,

(5) If $l\ge 4$, then $k_1\ne k+1$.

Let $f,g\in\mathcal{M}(\mathbb{E})$ (resp. $f,g\in\mathcal{M}_u((d(a,R^-))$ be transcendental and let $\alpha\in\mathcal{M}_f(\mathbb{E})\cap\mathcal{M}_g(\mathbb{K})$ (resp. $\alpha\in\mathcal{M}_f(d(a,R^-))\cap\mathcal{M}_g(d(a,R^-))$) be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

In the field $\mathbb{K}$, several particular applications were given when the small function is a constant or a Moebius function. On $\mathbb{C}$, we can't get similar refinements because the complex Nevanlinna Theory is less accurate than the p-adic Nevanlinna Theory.

In the present paper, thanks to the new Hypothesis (G) introduced below, we mean to avoid the hypothesis $k_1\ge k+2$ for $\mathcal{M}(\mathbb{K})$ and $k_1\ge k+3$ for $\mathcal{M}(\mathbb{C})$ and for $\mathcal{M}(d(a,R^-))$.

But first, we have a new theorem for p-adic analytic functions: First we can improve results of [5] concerning p-adic analytic functions.

Theorem 1.6. *Let $P(X)\in\mathbb{K}[X]$ be a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$ (resp. for $\mathcal{A}_u(d(a,R^-))$) and let $P'(X)=b\prod_{i=1}^{l}(X-a_i)^{k_i}$. Let $f,g\in\mathcal{A}(\mathbb{K})$ be transcendental (resp. let $f,g\in\mathcal{A}_u(d(a,R^-))$), be such that $f'P'(f)$ and $g'P'(g)$ share CM a small function $\alpha\in\mathcal{A}_f(\mathbb{K})\cap\mathcal{A}_g(\mathbb{K})$ (resp. $\alpha\in\mathcal{A}_f(d(,R^-))\cap\mathcal{A}_g(d(a,R^-))$). If $\sum_{i=1}^{l}k_i\ge 2l+2$ then $f=g$. Moreover, if f,g belong to $\mathcal{A}(\mathbb{K})$, if α is a constant and if $\sum_{i=1}^{l}k_i\ge 2l+1$ then $f=g$.*

Corollary 1.1. Let $P(X)\in\mathbb{K}[X]$ be such that $\Phi(P)\geq 2$ and let $P'(X)=b\prod_{i=1}^{l}(X-a_i)^{k_i}$. Let $f,g\in\mathcal{A}(\mathbb{K})$ be transcendental such that $f'P'(f)$ and $g'P'(g)$ share CM a small function $\alpha\in\mathcal{A}_f(\mathbb{K})\cap\mathcal{A}_g(\mathbb{K})$. If $\sum_{i=1}^{l}k_i\geq 2l+2$ then $f=g$. Moreover, if α is a constant and if $\sum_{i=1}^{l}k_i\geq 2l+1$ then $f=g$.

Example 1.1. Let $c\in\mathbb{K}$ be a solution of the algebraic equation:

$$X^{11}\Big(\frac{1}{11}-\frac{1}{10}\Big)-X^9\Big(\frac{1}{9}-\frac{1}{8}\Big)+X\Big(\frac{1}{10}-\frac{1}{8}\Big)-\frac{1}{11}+\frac{1}{9}=0.$$

Let

$$P(X)=\frac{X^{11}}{11}-\frac{cX^{10}}{10}-\frac{X^9}{9}+\frac{cX^8}{8}.$$

Then we can check that $P'(X)=X^7(X-1)(X+1)(X-c)$, $P(1)=P(c)\neq 0$ and that $P(1)\neq 0$, $P(-1)\neq 0$, $P(1)+P(-1)=c(1/4-1/5)$ and $P(-1)-P(1)=2(1/11-1/9)$, hence $P(-1)\neq P(c)$.

Consequently, we can apply Corollary 1.1 and show that if $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha\in\mathcal{A}_f(\mathbb{K})\cap\mathcal{A}_g(\mathbb{K})$, then $f=g$.

Remark 1.2. Recall Hypothesis (F) due to H. Fujimoto [12]. A polynomial Q is said to satisfy Hypothesis (F) if the restriction of Q to the set of zeros of Q' is injective. In the last example, we may notice that Hypothesis (F) is not satisfied by P.

Corollary 1.2. Let $P(X)\in\mathbb{K}[X]$ be such that $\Phi(P)\geq 3$ and let $P'(X)=b\prod_{i=1}^{l}(X-a_i)^{k_i}$. Let $f,g\in\mathcal{A}_u(d(a,R^-))$ be such that $f'P'(f)$ and $g'P'(g)$ share CM a small function $\alpha\in\mathcal{A}_f(d(a,R^-))\cap\mathcal{A}_g(d(a,R^-))$. If $\sum_{i=1}^{l}k_i\geq 2l+2$ then $f=g$.

Corollary 1.3. Let $P(X)\in\mathbb{K}[X]$ be such that $\Phi(P)\geq 2$ (resp. $\Phi(P)\geq 3$) and let $P'(X)=bX^n\prod_{i=2}^{l}(X-a_i)$ with $l\geq 3$ and let $f,g\in\mathcal{A}(\mathbb{K})$ (resp. $f,g\in\mathcal{A}_u(d(a,R^-))$) be such that $f'P'(f)$ and $g'P'(g)$ share CM a small function $\alpha\in\mathcal{A}_f(\mathbb{K})\cap\mathcal{A}_g(\mathbb{K})$ (resp. $\alpha\in\mathcal{A}_f(d(a,R^-))\cap\mathcal{A}_g(d(a,R^-))$). If $n\geq l+3$ then $f=g$. Moreover, if f,g belong to $\mathcal{A}(\mathbb{K})$, if α is a constant and if $n\geq l+2$ then $f=g$.

In order to improve results of [5] on p-adic meromorphic functions and of [6] on complex meromorphic functions, we have to state Propositions 1.1 and 1.2 derived from results of [3] and [4].

Notation and Definition 1.2. Henceforth we assume that $P(a_1)=0$ and that $P'(X)$ is of the form $b\prod_{i=1}^{l}(X-a_i)^{k_i}$ with $l\geq 2$. The polynomial P will be said *to satisfy Hypothesis (G)* if $P(a_i)+P(a_j)\neq 0$, $\forall i\neq j$.

Proposition 1.1. Let $P\in\mathbb{K}[X]$ satisfy Hypothesis (G) and $n\geq 2$ (resp. $n\geq 3$). If meromorphic functions $f,g\in\mathcal{M}(\mathbb{K})$ (resp. $f,g\in\mathcal{M}(d(a,R^-))$) satisfy $P(f(x))=P(g(x))+C$ $(C\in\mathbb{K}^*)$, $\forall x\in\mathbb{K}$ (resp. $\forall x\in d(a,R^-)$), then both f and g are constant (resp. f and g belong to $\mathcal{M}_b(d(a,R^-))$).

Proposition 1.2. Let $P\in\mathbb{C}[X]$ satisfy Hypothesis (G) and $n\geq 3$. If meromorphic functions $f,g\in\mathcal{M}(\mathbb{C})$ satisfy $P(f(x))=P(g(x))+C(C\in\mathbb{C}^*)$, $\forall x\in\mathbb{C}$, then both f and g are constant.

From [5] and thanks to Propositions 1.1, we can now derive the following Theorems 1.7-1.10:

Theorem 1.7. *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, (resp for $\mathcal{M}(d(0,R^-))$) with $l\geq 2$, let $P'(X)=b\prod_{i=1}^{l}(X-a_i)^{k_i}$ with $b\in\mathbb{K}^*$, $k_i\geq k_{i+1}$, $2\leq i\leq l-1$, let $k=\sum_{i=2}^{l}k_i$, let u_5 be the biggest of the i such that $k_i>4$ and let $s_5=\max(0,u_5-3)$ and for each $m\in\mathbb{N}$, let u_m be the biggest of the i such that $k_i>m$ and let $s_m=\max(0,u_m-2)$. Suppose P satisfies the following conditions:*

(1) $k_1\geq 10+\max(0,5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\min(2l,\sum_{m=5}^{\infty}s_m)$,

(2) either $k_1\geq k+2$ (resp. $k_1\geq k+3$, resp. $k_1\geq k+3$) or P satisfies Hypothesis (G),

(3) if $l=2$, then $k_1\neq k+1,2k,2k+1,3k+1$,

(4) if $l=3$, then $k_1\neq \frac{k}{2},k_1\neq k+1,2k+1,3k_i-k$, $\forall i=2,3$,

(5) $l\geq 4$, then $k_1\neq k+1$.

Let $f,g\in\mathcal{M}(\mathbb{K})$ (resp. $f,g\in\mathcal{M}((d(a,R^-))$ be transcendental and let $\alpha\in\mathcal{M}_f(\mathbb{K})\cap\mathcal{M}_g(\mathbb{K})$ (resp. $\alpha\in\mathcal{M}_f(d(a,R^-))\cap\mathcal{M}_g(d(a,R^-)))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

Remark 1.3. The sum $\sum_{m=5}^{\infty}s_m$ is obviously finite.

Corollary 1.4. Let $P\in\mathbb{K}[x]$ satisfy $\Phi(P)\geq 3$ and hypothesis (G), let $P'=b\prod_{i=1}^{l}(X-a_i)^{k_i}$ with $b\in\mathbb{K}^*$, $l\geq 3$, $k_i\geq k_{i+1}$, $2\leq i\leq l-1$, let $k=\sum_{i=2}^{l}k_i$, and for each $m\in\mathbb{N}$, let u_m be the biggest of the i such that $k_i>4$, $s_5=\max(0,u_5-3)$ and for every $m\geq 6$, let $s_m=\max(0,u_m-2)$. Suppose P satisfies the following conditions:

(1) $k_1\geq 10+\max(0,5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\min(2l-1,\sum_{m=5}^{\infty}s_m)$,

(2) if $l=3$, then $k_1\neq k/2,k+1,2k+1,3k_i-k$, $\forall i=2,3$,

(3) if $l\geq 4$, then $k_1\neq k+1$.

Let $f,g\in\mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha\in\mathcal{M}_f(\mathbb{K})\cap\mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

Example 1.2. Let

$$P(X)=\frac{X^{20}}{20}-\frac{X^{19}}{19}-\frac{4X^{18}}{18}+\frac{4X^{17}}{17}+\frac{6X^{16}}{16}-\frac{6X^{15}}{15}-\frac{4X^{14}}{14}+\frac{4X^{13}}{13}+\frac{X^{12}}{12}-\frac{X^{11}}{11}.$$

We can check that $P'(X)=X^{10}(X-1)^5(X+1)^4$ and

$$P(0)=0,\quad P(1)=\sum_{j=0}^{4}C_4^j(-1)^j\Big(\frac{1}{10+2j}-\frac{1}{9+2j}\Big),\quad P(-1)=-\sum_{j=0}^{4}C_4^j\Big(\frac{1}{10+2j}+\frac{1}{9+2j}\Big).$$

Consequently, we have $\Phi(P)=3$ and we check that Hypothesis (G) is satisfied. Now, let $f,g\in\mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha\in\mathcal{M}_f(\mathbb{K})\cap\mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

Remark 1.4. In that example, we have $k_1=10$, $k=9$. Applying our previous work, a conclusion would have required $k_1\geq k+2=11$.

Theorem 1.8. *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$, with $l\geq 2$, let $P'(X)=b\prod_{i=1}^{l}(X-a_i)^{k_i}$ with $b\in\mathbb{K}^*$, $k_i\geq k_{i+1}$, $2\leq i\leq l-1$, let $k=\sum_{i=2}^{l}k_i$, let u_5 be the biggest of the i such that $k_i>4$ and let $s_5=u_5-3$ and for each $m\in\mathbb{N}$, let u_m be the biggest of the i such that $k_i>m$ and let $s_m=\max(0,u_m-2)$. Suppose P satisfies the following conditions:*

(1) $k_1\geq 10+\max(0,5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\min(2l,\sum_{m=5}^{\infty}s_m)$,

(2) either $k_1\geq k+3$ or P satisfies Hypothesis (G),

(3) if $l=2$, then $k_1\neq k+1,2k,2k+1,3k+1$,

(4) if $l=3$, then $k_1\neq k/2, k_1\neq k+1,2k+1,3k_i-k$, $\forall i=2,3$,

(5) If $l\geq 4$, then $k_1\neq k+1$.

Let $f,g\in\mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha\in\mathcal{M}_f(\mathbb{C})\cap\mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

Corollary 1.5. Let $P\in\mathbb{C}[X]$ satisfy $\Phi(P)\geq 4$ and Hypothesis (G), let $P'=b\prod_{i=1}^{l}(X-a_i)^{k_i}$, $k_i\geq k_{i+1}$, $2\leq i\leq l-1$, let $k=\sum_{i=2}^{l}k_i$, and for each $m\in\mathbb{N}$, let u_m be the biggest of the i such that $k_i>4$, let $s_5=\max(0,u_5-3)$ and for every $m\geq 6$, let $s_m=\max(0,u_m-2)$. Suppose P satisfies the following conditions:

(1) $k_1\geq 10+\max(0,5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\min(2l,\sum_{m=5}^{\infty}s_m)$,

(2) $k_1\neq k+1$.

Let $f,g\in\mathcal{M}(\mathbb{C})$ and let $\alpha\in\mathcal{M}_f(\mathbb{C}))\cap\mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

As noticed in [5], if f,g belong to $\mathcal{M}(\mathbb{K})$ and if α is a constant or a Moebius function, we can get a more acurate statement:

Theorem 1.9. *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P'=b\prod_{i=1}^{l}(x-a_i)^{k_i}$ with $b\in\mathbb{K}^*$, $l\geq 2$, $k_i\geq k_{i+1}$, $2\leq i\leq l-1$, let $k=\sum_{i=2}^{l}k_i$, and for each $m\in\mathbb{N}$, let u_m be the biggest of the i such that $k_i>4$, let $s_5=\max(0,u_5-3)$ and for every $m\geq 6$, let $s_m=\max(0,u_m-2)$. Suppose P satisfies the following conditions:*

(1) $k_1\geq 9+\max(0,5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\min(2l-1,\sum_{m=5}^{\infty}s_m)$,

(2) either $k_1\geq k+2$ or P satisfies (G),

(3) if $l=2$, then $k_1\neq k+1,2k,2k+1,3k+1$,

(4) if $l=3$, then $k_1\neq k/2,k+1,2k+1,3k_i-k$, $\forall i=2,3$.

Let $f,g\in\mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

By Theorem 1.4, we can derive Corollary 1.6.

Corollary 1.6. Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$, let $P' = b\prod_{i=1}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$, let $k = \sum_{i=2}^{l} k_i$, and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 3)$ and for every $m \geq 6$, let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:

(1) $k_1 \geq 9 + \max(0, 5-k_2) + \sum_{i=3}^{l} \max(0, 4-k_i) - \min(2l-1, \sum_{m=5}^{\infty} s_m)$,

(2) either $k_1 \geq k+2$ or P satisfies (G),

(3) if $l = 3$, then $k_1 \neq k/2, k+1, 2k+1, 3k_i - k$, $\forall i = 2,3$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem 1.7, we have Corollary 1.7.

Corollary 1.7. Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x-a_1)^n(x-a_2)^k$ with $k \leq n$, $\min(k,n) \geq 2$ and with $b \in \mathbb{K}^*$. Suppose P satisfies the following conditions:

(1) $n \geq 9 + \max(0, 5-k)$,

(2) either $n \geq k+2$ or P satisfies (G),

(3) $n \neq k+1, 2k, 2k+1, 3k+1$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem 1.10. *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b\prod_{i=1}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$, let $k = \sum_{i=2}^{l} k_i$, and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 4)$ and for every $m \geq 6$, let $s_m = \max(0, u_m - 3)$. Suppose P satisfies the following conditions:*

(1) either $k_1 \geq k+2$ or P satisfies (G),

(2) $k_1 \geq 9 + \max(0, 5-k_2) + \sum_{i=3}^{l} \max(0, 4-k_i) - \min(2l-1, \sum_{m=5}^{\infty} s_m)$,

(3) $k_1 \neq k+1$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem 1.4, we can derive Corollary 1.8.

Corollary 1.8. Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$, let $P' = b\prod_{i=1}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$, let $k = \sum_{i=2}^{l} k_i$, and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 3)$ and for every $m \geq 6$, let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:

(1) $k_1 \geq k+2$ or P satisfies Hypothesis (G),

(2) $k_1 \geq 9 + \max(0, 5-k_2) + \sum_{i=3}^{l} \max(0, 4-k_i) - \min(2l-1, \sum_{m=5}^{\infty} s_m)$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem 1.2, we have Corollary 1.9.

Corollary 1.9. Let $P\in\mathbb{K}[x]$ be such that P' is of the form $b(x-a_1)^n(x-a_2)^k$ with $\min(k,n)\geq 2$ and with $b\in\mathbb{K}^*$. Suppose P satisfies the following conditions:

(1) $n\geq 9+\max(0,5-k)$,

(2) either $n\geq k+2$ or P satisfies (G),

(3) $n\neq k+1$.

Let $f,g\in\mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

Example 1.3. Let

$$\begin{aligned}P(X)=&\frac{X^{24}}{24}-\frac{10X^{23}}{23}+\frac{36X^{22}}{22}-\frac{40X^{21}}{21}-\frac{74X^{20}}{20}+\frac{226X^{19}}{19}-\frac{84X^{18}}{18}\\&-\frac{312X^{17}}{17}+\frac{321X^{16}}{16}+\frac{88X^{15}}{15}-\frac{280X^{14}}{14}+\frac{48X^{13}}{13}+\frac{80X^{12}}{12}-\frac{32X^{11}}{11}.\end{aligned}$$

We can check that $P'(X)=X^{10}(X-2)^5(X+1)^4(X-1)^4$. Next, we have $P(2)<-134378$, $P(1)\in[-2{,}11;-2{,}10]$, $P(-1)\in[2{,}18;2{,}19]$. Therefore, $P(0)$, $P(1)$, $P(-1)$, $P(2)$ are all distinct, hence $\Phi(P)=4$. Moreover, Hypothesis (G) is satisfied.

Now, let $f,g\in\mathcal{M}(\mathbb{K})$ (resp. let $f,g\in\mathcal{M}_u(d(a,R^-))$, resp. let $f,g\in\mathcal{M}(\mathbb{C})$) and let $\alpha\in\mathcal{M}(\mathbb{K})$ (resp. let $\alpha\in\mathcal{M}(d(a,R^-))$, resp. let $\alpha\in\mathcal{M}(\mathbb{C})$) be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

Particularly, when f, g are entire functions in $\mathbb{C}$ we can simplify the hypothesis:

Theorem 1.11. *Let P be a polynomial of uniqueness for $\mathcal{A}(\mathbb{C})$ with $l\geq 2$ and $k_i\geq k_{i+1}$, $1\leq i\leq l-1$ when $l>2$ and let $k=\sum_{i=2}^{l}k_i$, let u_5 be the biggest of the i such that $k_i>4$ and let $s_5=\max(0,u_5-3)$ and for each $m\in\mathbb{N}$, let u_m be the biggest of the i such that $k_i>m$ and let $s_m=\max(0,u_m-2)$. Suppose P satisfies the following conditions:*

(1) $k_1\geq k+2$ or P satisfies hypothesis (G),

(2) $k_1\geq 5+\max(0,5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\min(2l-3,\sum_{m=5}^{\infty}s_m)$.

Let $f,g\in\mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha\in\mathcal{A}_f(\mathbb{C})\cap\mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

By Proposition 1.2, we have Corollaries 1.10 and 1.11.

Corollary 1.10. Let $P\in\mathbb{C}[X]$, let $P'=b\prod_{i=1}^{l}(X-a_i)^{k_i}$ with $b\in\mathbb{C}^*$, $k_i\geq k_{i+1}$, $1\leq i\leq l-1$ and let $k=\sum_{i=2}^{l}k_i$, let u_5 be the biggest of the i such that $k_i>4$ and let $t_5=u_5-3$ and for each $m\in\mathbb{N}$, let u_m be the biggest of the i such that $k_i>m$ and let $t_m=\max(0,u_m-2)$. Suppose P satisfies the following conditions:

(1) either $k_1\geq k+2$ or P satisfies hypothesis (G),

(2) $k_1\geq 5+\max(0,5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\min(2l-3,\sum_{m=5}^{\infty}s_m)$.

Let $f,g\in\mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha\in\mathcal{M}_f(\mathbb{C})\cap\mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

Corollary 1.11. Let $P\in\mathbb{C}[X]$ and let $P'=b(X-a_1)^n(X-a)^k$ with $\min(k,n)\geq 2$ and $\max(n,k)\geq 3$. Suppose that P satisfies $n\geq 5+\max(0,5-k)$.

Let $f,g\in\mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha\in\mathcal{A}_f(\mathbb{C})\cap\mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f=g$.

Example 1.4. Let

$$P(X)=\frac{X^{11}}{11}+\frac{5X^{10}}{10}+\frac{10X^9}{9}+\frac{10X^8}{8}+\frac{5X^7}{7}+\frac{X^6}{6}.$$

Then $P'(X)=X^5(X+1)^5$. We can apply Corollary 1.11 given $f,g\in\mathcal{A}(\mathbb{C})$ transcendental such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha\in\mathcal{M}(\mathbb{C})$ C.M., we have $f=g$.

2 The proofs

Notation 2.1. As usual, given a function $f\in\mathcal{M}(\mathbb{E})$ (resp. $\mathcal{M}(d(0,R^-))$), we denote by $S_f(r)$ a function of r defined in $[0,+\infty]$ (resp. in $[0,R]$), such that

$$\lim_{r\to+\infty}\frac{S_f(r)}{T(r,f)}=0,\quad \text{resp. }\lim_{r\to R}\frac{S_f(r)}{T(r,f)}=0.$$

We must recall the classical Nevanlinna Main Theorem:

Theorem 2.1 (see [7,12]). *Let $a_1,\cdots,a_n\in\mathbb{K}$ (resp. $a_1,\cdots,a_n\in\mathbb{K}$, resp. $a_1,\cdots,a_n\in\mathbb{C}$) with $n\geq 2$, $n\in\mathbb{N}$, and let $f\in\mathcal{M}(\mathbb{K})$ (resp. let $f\in\mathcal{M}(d(0,R^-))$, resp. let $f\in\mathcal{M}(\mathbb{C})$). Let $S=\{a_1,\cdots,a_n\}$. Then, for $r>0$ we have*

$$(n-1)T(r,f)\leq\sum_{j=1}^{n}\overline{Z}(r,f-a_j)+\overline{N}(r,f)-\log r+\mathcal{O}(1),$$

resp.

$$(n-1)T(r,f)\leq\sum_{j=1}^{n}\overline{Z}(r,f-a_j)+\overline{N}(r,f)+\mathcal{O}(1),$$

resp.

$$(n-1)T(r,f)\leq\sum_{j=1}^{n}\overline{Z}(r,f-a_j)+\overline{N}(r,f)+S_f(r).$$

Let us recall the following corollary of the Nevanlinna Second Main Theorem on three small function:

Theorem 2.2. *Let $f\in\mathcal{A}(\mathbb{K})$ (resp. let $f\in\mathcal{A}(d(0,R^-))$, resp. let $f\in\mathcal{A}(\mathbb{C})$) and let $u\in f\in\mathcal{A}_f(\mathbb{K})$ (resp. let $u\in\mathcal{A}_f(d(0,R^-))$, resp. $u\in f\in\mathcal{A}_f(\mathbb{C})$). Then $T(r,f)\leq\overline{Z}(r,f)+\overline{Z}(r,f-u)+S_f(r)$.*

In order to prove Theorem 2.3, we need additional lemmas:

Notation 2.2. Let $f\in\mathcal{M}(d(a,R^-))$, and let $r\in[0,R]$. By classical results [8, 10] we know that $|f(x)|$ has a limit when $|x|$ tends to r, while being different from r.

We set $|f|(r)=\lim_{|x|\to r,|x|\neq r}|f(x)|$.

Lemma 2.1. *For every $r\in[0,R]$, the mapping $|\cdot|(r)$ is an ultrametric multiplicative norm on $\mathcal{M}(d(0,R^-))$.*

The following Lemma 2.2 is the p-adic Schwarz formula:

Lemma 2.2. *Let $f\in\mathcal{A}(\mathbb{K})$ (resp. $f\in\mathcal{A}(d(0,R^-))$) and let $r',r''\in[0,+\infty]$ (resp. let $r',r''\in[0,R]$) satisfy $r'<r''$. Then $\log(|f|(r''))-\log(|f|(r'))=Z(r'',f)-Z(r',f)$.*

Lemma 2.3. *Let $f\in\mathcal{M}(\mathbb{K})$ (resp. $f\in\mathcal{M}(d(0,R^-))$). Suppose that there exists $a\in\mathbb{K}$ and a sequence of intervals $I_n=[u_n,v_n]$ such that $u_n<v_n<u_{n+1}$, $\lim_{n\to+\infty}u_n=+\infty$ (resp. $\lim_{n\to+\infty}u_n=R$) and $\lim_{n\to+\infty}\inf_{r\in I_n}qT(r,f)-Z(r,f-a)=+\infty$. Set $L=\bigcup_{n=0}^{+\infty}I_n$. Let $b\in\mathbb{K}$, $b\neq a$. Then $Z(r,f-b)=T(r,f)+\mathcal{O}(1)$, $\forall r\in L$.*

Proof. We know that the Nevanlinna functions of a meromorphic function f are the same in $\mathbb{K}$ and in an algebraically closed complete extension of $\mathbb{K}$ whose absolute value extends that of $\mathbb{K}$. Consequently, without loss of generality, we can suppose that $\mathbb{K}$ is spherically complete because we know that such a field does admit a spherically complete algeberaically closed extension whose absolute value expands that of $\mathbb{K}$. If f belongs to $\mathcal{M}(\mathbb{K})$, we can obviously set it in the form g/h, where g, h belong to $\mathcal{A}(\mathbb{K})$ and have no common zero. Next, since $\mathbb{K}$ is supposed to be spherically complete, if f belongs to $\mathcal{M}(d(0,R^-))$ we can also set it in the form g/h where g,h belong to $\mathcal{A}(d(0,R^-))$ and have no common zero [8, 10]. Consequently, we have $T(r,f)=\max(Z(r,g),Z(r,h))$.

By hypothesis we have

$$\lim_{n\to+\infty}\Big(\inf_{r\in I_n}T(r,f)-Z(r,f-a)\Big)=+\infty,$$

i.e.,

$$\lim_{n\to+\infty}\Big(\inf_{r\in I_n}T(r,f)-Z(r,f-a)\Big)=+\infty,$$

i.e.,

$$\lim_{n\to+\infty}\Big(\inf_{r\in I_n}\max(Z(r,g),Z(r,h))-Z(r,g-ah)\Big)=+\infty.$$

Set

$$B_n=\inf_{r\in I_n}\max(Z(r,g),Z(r,h))-Z(r,g-ah)).$$

Since the sequence B_n tends to $+\infty$, clearly, by Lemma 2.2, the sequence (D_n) defined as

$$D_n=\sup_{r\in I_n}\Big(\frac{|g-ah|(r)}{\max(|g|(r),|h|(r))}\Big)$$

tends to zero. Therefore, by Lemma 2.1, we have $|g|(r)=|ah|(r)$ in I_n when n is big enough. Consequently, by Lemma 2.2, we have $Z(r,g)=Z(r,ah)+\mathcal{O}(1)$, $\forall r\in L$ and hence $T(r,f)=Z(r,h)+\mathcal{O}(1)=Z(r,g)+\mathcal{O}(1)$, $\forall r\in L$.

Now, consider $g-bh=g-ah+(a-b)h$. By hypothesis we have

$$\lim_{n\to+\infty}\Big(\inf_{r\in I_n} Z(r,h)-Z(r,g-ah)\Big)=+\infty.$$

On the other hand, of course $Z(r,(a-b)h)=Z(r,h)+\mathcal{O}(1)$. Consequently, since $Z(r,g-bh)=Z(r,g-ah+(a-b)h)$, we have

$$\lim_{n\to+\infty}\Big(\inf_{r\in I_n}(Z(r,(a-b)h)-Z(r,g-ah)\Big)=+\infty.$$

Consider now the sequence (E_n) defined as

$$E_n=\sup_{r\in I_n}\Big(\frac{|g-ah|(r)}{|(a-b)h|(r)}\Big).$$

By Lemma 2.2, that sequence tends to zero and hence, when r is big enough in L, by Lemma 2.1 we have $|g-bh|(r)=|a-bh|(r)$. Consequently, when r is big enough in L, we have $Z(r,g-bh)=Z(r,bh)=Z(r,h)+\mathcal{O}(1)$. Moreover, we have seen that $Z(r,g)=Z(r,h)+\mathcal{O}(1)$ in L, hence $\max(Z(r,g),Z(r,h))=Z(,g-bh)\mathcal{O}(1)=\max(Z(r,g-bh),Z(r,h)+\mathcal{O}(1)$, i.e., $T(r,f)=T(r,f-b)+\mathcal{O}(1)$ in L. □

The second Main Theorem is well known in complex and p-adic analysis and is recalled below. But first, we can give here a new theorem of that kind which will be efficient in Theorems 1.8-1.10.

Theorem 2.3. *Let $f\in\mathcal{M}(\mathbb{K})$ (resp. $f\in\mathcal{M}(d(0,R^-))$) and let $a_1,\cdots,a_q\in\mathbb{K}$ be distinct. Then $(q-1)T(r,f)\le\sum_{j=1}^q Z(r,f-a_j)\mathcal{O}(1)$.*

Proof. Suppose the theorem is wrong. There exists $f\in\mathcal{M}(\mathbb{K})$ (resp. $f\in\mathcal{M}(d(0,R^-))$) and $a_1,\cdots,a_q$ such that $(q-1)T(r,f)-\sum_{j=1}^q Z(r,f-a_j)$ admits no superior bound in $[0,+\infty]$. So, there exists a sequence of intervals $J_s=[w_s,y_s]$ such that $w_s<y_s<w_{s+1}$, $\lim_{s\to+\infty}w_s=+\infty$ (resp. $\lim_{s\to+\infty}w_s=R$) and

$$\lim_{s\to+\infty}\Big(\inf_{r\in J_s}(q-1)T(r,f)-\sum_{j=1}^q Z(r,f-a_j)\Big)=+\infty. \tag{2.1}$$

Let $M=\bigcup_{s=0}^\infty J_s$. For each $j=1,\cdots,q$, we have $Z(r,f-a_i)\le T(r,f)+\mathcal{O}(1)$ in $\mathbb{R}_+$ and hence (2.4) implies that there exists an index t and a sequence of intervals $I_n=[u_n,v_n]$ included in M, such that $u_n<v_n<u_{n+1}$, $\lim_{n\to+\infty}u_n=+\infty$ (resp. $\lim_{n\to+\infty}u_n=R$) and

$$\lim_{n\to+\infty}\Big(\inf_{r\in I_n}(T(r,f)-Z(r,f-a_t))\Big)=+\infty. \tag{2.2}$$

Let $L=\bigcup_{n=1}^{\infty} I_n$. Then by Lemma 2.3, in L we have $Z(r,g-a_k h)=T(rf)+\mathcal{O}(1)$, $\forall k\neq t$. Therefore $\sum_{j=1}^{q} Z(r,f-a_j)\geq (q-1)T(r,f)+\mathcal{O}(1)$ in L, a contradiction to (2.4). Consequently, the Theorem is not wrong. ☐

Remark 2.1. Theorem 2.3 is trivial for analytic functions since by definition, for a function $f\in\mathcal{A}(\mathbb{K})$ or $\mathcal{A}(d(0,R^-))$ we have $T(r,f)=Z(r,f)$. On the other hand, the theorem does not apply to meromorphic functions in $\mathbb{C}$. Indeed, consider a meromorphic function f on $\mathbb{C}$ omitting two values a and b. We have $Z(r,f-a)+Z(r,f-b)=0$.

In the proof of Theorems 1.7-1.11 will need the following Lemmas:

Lemma 2.4. *Let $Q\in\mathbb{K}[x]$ (resp. $Q\in\mathbb{K}[X]$, resp. $Q\in\mathbb{C}[x]$) be of degree n and let $f\in\mathcal{M}(\mathbb{K})$, (resp. $f\in\mathcal{M}(d(0,R^-))$, resp. $f\in\mathcal{M}(\mathbb{C})$) be transcendental. Then*

$$N(r,f')=N(r,f)+\overline{N}(r,f),\quad Z(r,f')\leq Z(r,f)+\overline{N}(r,f)+\mathcal{O}(1),$$
$$nT(r,f)\leq T(r,f'Q(f))\leq (n+2)T(r,f)-\log r+\mathcal{O}(1),$$

resp.

$$nT(r,f)\leq T(r,f'Q(f))\leq (n+2)T(r,f)+\mathcal{O}(1),$$

resp.

$$nT(r,f)\leq T(r,f'Q(f))+m(r,1/f')\leq (n+2)T(r,f)+S_f(r)).$$

Particularly, if $f\in\mathcal{A}(\mathbb{K})$, resp.

$$f\in\mathcal{A}(d(0,R^-)),$$

then

$$nT(r,f)\leq T(r,f'Q(f))\leq (n+1)T(r,f)-\log r+\mathcal{O}(1),$$

resp.

$$nT(r,f)\leq T(r,f'Q(f))\leq (n+1)T(r,f)+\mathcal{O}(1).$$

Let us recall the following corollary of the Nevanlinna Second Main Theorem on three small function:

Lemma 2.5. *Let $Q(X)\in\mathbb{K}[X]$ and let $f,g\in\mathcal{A}(\mathbb{K})$ (resp. let $f,g\in\mathcal{A}_u(d(0,R^-))$) be such that $Q(f)-Q(g)$ is bounded. Then $f=g$.*

Proof. The polynomial $Q(X)-Q(Y)$ factorizes in the form $(X-Y)F(X,Y)$ with $F(X,Y)\in\mathbb{K}[X,Y]$. Since $Q(f)-Q(g)$ is bounded, so are both factors because the semi-norm $|\cdot|(r)$ is multiplicative on $\mathcal{A}(\mathbb{K})$ (resp. on $\mathcal{A}_u(d(0,R^-))$). Consequently, $f-g$ is a constant c (resp. is a bounded function $u\in\mathcal{A}_b(d(0,R^-))$). Therefore $F(f,g)=F(f,f+c)$ (resp. $F(f,g)=F(f;f+u)$). Let $n=\deg(Q)$. Then we can check that $F(X,X+c)$ is a polynomial in X of degree $n-1$. Consequently, if $f\in\mathcal{A}(\mathbb{K})$, $F(f,f+c)$ is a non-constant entire function and therefore is unbounded in $\mathbb{K}$. Similarly,$f\in(d(0,R^-))$, $F(X,X+u)$ is a polynomial in X of degree $n-1$ with coefficients in $\mathcal{A}(d(0,R^-))$ and therefore $F(f,f+u)$ is unbounded in $d(0,R^-)$, which ends the proof. ☐

Proof of Theorem 1.6. Without loss of generality, we may assume that $b=1$. Put $F=f'\prod_{j=1}^{l}(f-a_j)^{k_j}$ and $G=g'\prod_{j=1}^{l}(g-a_j)^{k_j}$. Since $f,g\in\mathcal{A}(\mathbb{K})$ and since F and G share α C.M., then $(F-\alpha)/(G-\alpha)$ is a meromorphic function having no zeros and no pole in $\mathbb{K}$ (resp. in $d(0,R^-)$), hence it is a constant u in $\mathbb{K}\setminus\{0\}$ (resp. it is an invertible function $u\in\mathcal{A}(d(0,R^-))$).

Suppose $u\neq 1$. Then,

$$F=uG+\alpha(1-c). \tag{2.3}$$

Let $r>0$. Since $\alpha(1-u)\in\mathcal{A}_f(\mathbb{K})$ (resp. $\alpha(1-u)\in\mathcal{A}_f(d(0,R^))$), $\alpha(1-u)$ obviously belongs to $\mathcal{A}_F(\mathbb{K})$ (resp. to $\mathcal{A}_F(d(0,R^-))$). So, applying Theorem 2.2 to F, we obtain

$$\begin{aligned}T(r,F)&\leq\overline{Z}(r,F)+\overline{Z}\big(r,F-\alpha(1-c)\big)+S_F(r)=\overline{Z}(r,F)+\overline{Z}(G)+S_F(r)\\&=\sum_{j=1}^{l}\overline{Z}\big(r,(f-a_j)^k\big)+\overline{Z}(r,f')+\sum_{j=1}^{l}\overline{Z}\big(r,(g-a_j)^k\big)+\overline{Z}(r,g')+S_f(r)\\&\leq l(T(r,f)+T(r,g))+Z(r,f')+Z(r,g')+S_f(r).\end{aligned}$$

We also notice that if $f,g\in\mathcal{A}(\mathbb{K})$ and if $\alpha\in\mathbb{K}$, we have

$$T(r,F)\leq\overline{Z}(r,F)+\overline{Z}\big(r,F-\alpha(1-c)\big)-\log r+\mathcal{O}(1)$$

and therefore we obtain

$$T(r,F)\leq l(T(r,f)+T(r,g))+Z(r,f')+Z(r,g')-\log r+\mathcal{O}(1).$$

Now, let us go back to the general case. Since f is entire, by Lemma 2.4 we have,

$$T(r,F)=\Big(\sum_{j=1}^{l}k_j\Big)T(r,f)+Z(r,f')+\mathcal{O}(1).$$

Consequently,

$$\Big(\sum_{j=1}^{l}k_j\Big)T(r,f)\leq l(T(r,f)+T(r,g))+Z(r,g')+S_f(r).$$

Similarly,

$$\Big(\sum_{j=1}^{l}k_j\Big)T(r,g)\leq l(T(r,g)+T(r,g))+Z(r,f')+S_f(r).$$

Therefore

$$\begin{aligned}\Big(\sum_{j=1}^{l}k_j\Big)(T(r,f)+T(r,g))&\leq 2l(T(r,f)+T(r,g))+Z(r,f')+Z(r,g')+S_f(r)\\&\leq(2l+1)(T(r,f)+T(r,g))+S_f(r).\end{aligned}$$

So,

$$\sum_{j=1}^{l} k_j \leq 2l+1.$$

Thus, since $\sum_{j=1}^{l} k_j > 2l+1$, we have $u=1$.

And if $\alpha \in \mathbb{K}$, we obtain,

$$\begin{aligned}\sum_{j=1}^{l} k_j(T(r,f)+T(r,g)) &\leq 2l(T(r,f)+T(r,g))+Z(r,f')+Z(r,g')+S_f(r)\\ &\leq (2l+1)(T(r,f)+T(r,g))-3\log r+\mathcal{O}(1),\end{aligned}$$

because $T(r,f') \leq T(r,f)-\log r+\mathcal{O}(1)$, hence $\sum_{j=1}^{l} k_j \leq 2l$ which also contradicts the hypothesis $c \neq 1$ whenever $\sum_{j=1}^{l} k_j > 2l$.

Consequently, in the general case, whenever $\sum_{j=1}^{l} k_j > 2l+1$, we have $u=1$ and therefore $f'P'(f)=g'P'(g)$ hence $P(f)-P(g)$ is a constant D. But then by Lemma 2.5, we have $P(f)=P(g)$. And since P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$ (resp. for $\mathcal{A}(d(0,R^-))$), we can conclude $f=g$. Similarly, if $f,g \in \mathcal{A}(\mathbb{K})$ and if α is a non-zero constant, we have have $u=1$ whenever $\sum_{j=1}^{l} k_j > 2l$ and we conclude in the same way. □

On $\mathbb{K}$, we have this theorem from results of [4]:

Theorem 2.4. *Let $P,Q \in \mathbb{K}[x]$ satisfy one of the following two statements:*

$$\sum_{a_i \in F'} k_i \geq s-m+2 \ \ (resp. \sum_{a_i \in \Delta} k_i \geq s-m+3),$$
$$\sum_{b_j \in F''} q_j \geq 2 \ \ (resp. \sum_{b_i \in \Lambda} q_j \geq 3).$$

If two meromorphic functions $f,g \in \mathcal{M}(\mathbb{K})$ (resp. $f,g \in \mathcal{M}(d(a,R^-))$)) satisfy $P(f(x))=Q(g(x))$, $x \in \mathbb{K}$, (resp. $x \in d(a,R^-)$) then both f and g are constant (resp. belong to $\mathcal{M}_b(d(a,R^-))$)).

And on $\mathbb{C}$, we have this theorem from results of [3]:

Theorem 2.5. *Let $P,Q \in \mathbb{C}[X]$ satisfy one of the following two conditions:*

$$\sum_{a_i \in F'} k_j \geq s-m+3, \quad \sum_{b_j \in F''} q_i \geq 3,$$

and if the polynomial $P(X)-Q(Y)$ has no factor of degree 1, then there is no non-constant function $f,g \in \mathcal{M}(\mathbb{C})$ such that $P(f(x))-Q(g(x))=0$, $\forall x \in \mathbb{C}$.

From Theorem 2.5 we can derive the following Theorem 2.6:

Theorem 2.6. *Let $P,Q \in \mathbb{C}[X]$ satisfy one of the following two conditions:*

$$\sum_{a_i \in F'} k_i \geq s-m+3, \quad \sum_{b_j \in F''} q_j \geq 3.$$

Then there is no non-constant function $f,g \in \mathcal{M}(\mathbb{C})$ such that $P(f(x))-Q(g(x))=0$, $\forall x \in \mathbb{C}$.

Proof. Let $F(X,Y)=P(X)-Q(Y)$. Since $\mathbb{C}$ is algebraically isomorphic to an ultrametric field such as $\mathbb{C}_p$ (with p any prime integer), without loss of generality we can transfer the problem onto the field $\mathbb{C}_p$. So, the image of the polynomial F in $\mathbb{C}_p[X,Y]$ is a polynomial $\widetilde{F}(X,Y)$.

Thus, the hypothesis $\sum_{a_i\in F'}k_i\geq s-m+3$ still holds in $\mathbb{C}_p$ and similarly, for the hypothesis $\sum_{b_j\in F''}q_j\geq 3$. Suppose for instance $\sum_{a_i\in F'}k_i\geq s-m+3$. By Theorem 2.5, there is no pair of non-constant functions $f,g\in\mathcal{M}(\mathbb{C}_p)$ such that $P(f(x))-Q(g(x))=0$. Particularly, $\widetilde{F}(X,Y)$ admits no factor of degree 1 in $\mathbb{C}_p[X,Y]$. But then, $F(X,Y)$ does not admit a factor of degree 1 in $\mathbb{C}[X,Y]$ either, because the factorization is conserved by a transfer. Now, we can apply Theorem 2.5 proving that when two functions $f,g\in\mathcal{M}(\mathbb{C})$ satisfy $P(f(x))=Q(g(x))$, $\forall x\in\mathbb{C}$, then they are constant. □

Proof of Proposition 1.1. Suppose that two functions $f,g\in\mathcal{M}(\mathbb{K})$ (resp. $f,g\in\mathcal{M}(d(a,R^-))$) satisfy $P(f(x))=P(g(x))+C$ $(C\in\mathbb{K})$, $\forall x\in\mathbb{K}$ (resp. $\forall x\in d(a,R^-)$). We can apply Theorem 2.4 by putting $Q(X)=P(X)+C$. So, we have $h=l$ and $b_i=a_i$, $i=1,\cdots,l$. Let Γ be the curve of equation $P(X)-P(Y)=C$. By hypothesis we have $n\geq 2$, hence $\deg(P)\geq 3$, so Γ is of degree ≥ 3. Therefore, if Γ has no singular point, it is of genus ≥ 1 and hence, by Picard-Berkovich Theorem, the conclusion is immediate. Consequently, we can assume that Γ has a singular point (α,β). But then $P'(\alpha)=P'(\beta)=0$ and hence (α,β) is of the form (a_h,a_k). Consequently, $C=P(a_h)-P(a_k)$ and since $C\neq 0$, we have $h\neq k$. We will prove that either $a_1\in F'$, or $a_1\in F''$.

Suppose first that $a_1\notin F'\cup F''$. Since $a_1\notin F'$, there exists $i\in\{2,\cdots,l\}$ such that $P(a_1)=P(a_i)+C$. Now since $1\notin F''$, there exists $j\in\{2,\cdots,l\}$ such that $P(a_1)+C=P(a_i)$. But since $C=-P(a_i)$, we have $P(a_j)=-P(a_i)$, therefore $P(a_i)+P(a_j)=0$. Since P satisfies (G), we have $i=j$, hence $P(a_i)=0$. But then $C=0$, a contradiction. Therefore, we have proven that $a_1\in F'\cup F''$. Now, by Theorem 2.4, f and g are constant (resp. f and g belong to $\mathcal{M}_b(d(a,R^-))$). □

Proof of Proposition 1.2. Suppose that two functions $f,g\in\mathcal{M}(\mathbb{C})$ satisfy $P(f(x))=P(g(x))+C$ $(C\in\mathbb{C})$, $\forall x\in\mathbb{C}$. We will apply Theorem 2.6 by putting $Q(X)=P(X)+C$. Since $n\geq 3$, we have $\deg(P)\geq 4$ and hence Γ is of degree ≥ 4. Consequently, if Γ has no singular point, it has genus ≥ 2 and hence, by Picard's Theorem, there exist no functions $f,g\in\mathcal{M}(\mathbb{C})$ such that $P(f(x))=P(g(x))+C, x\in\mathbb{C}$. Consequently, we can assume that Γ admits a singular point (a_h,a_k). The proof is then similar to that of Proposition 1.1. □

Notation 2.3. Let $f\in\mathcal{M}(\mathbb{C})$ be such that $f(0)\neq 0,\infty$. We denote by $Z_{[2]}(r,f)$ the counting function of the zeros of f each being counted with multiplicity when it is at most 2 and with multiplicity 2 when it is bigger.

The following basic lemma applies to both complex and meromorphic functions. A proof is given in [5] for p-adic meromorphic functions and in [6] for complex meromorphic functions.

The following Theorem 2.7 is indispensable in the proof of theorems:

Theorem 2.7. *Let $P(x)=(x-a_1)^n\prod_{i=2}^{l}(x-a_i)^{k_i}\in\mathbb{E}[x]$ ($a_i\neq a_j$, $\forall i\neq j$) with $l\geq 2$ and $n\geq\max\{k_2,\cdots,k_l\}$ and let $k=\sum_{i=2}^{l}k_i$. Let $f,g\in\mathcal{M}(\mathbb{E})$ be transcendental (resp. let $f,g\in\mathcal{M}(d(a,R^-))$) and let $\theta=P(f)f'P(g)g'$. If θ belongs to $\mathcal{M}_f(\mathbb{E})\cap\mathcal{M}_g(\mathbb{E})$, (resp. if θ belongs to $\mathcal{M}_f(d(a,R^-))\cap\mathcal{M}_g(d(a,R^-))$) then we have the following:*

(a) if $l=2$ then n belongs to $\{k,k+1,2k,2k+1,3k+1\}$,

(b) if $l=3$ then n belongs to $\{k/2,k+1,2k+1,3k_2-k,3k_3-k\}$,

(c) if $l\geq 4$ then $n=k+1$.

Moreover, if f,g belong to $\mathcal{M}(\mathbb{K})$ and if θ is a constant, then $n=k+1$. Further, if f,g belong to $\mathcal{A}(\mathbb{E})$, then θ does not belong to $\mathcal{A}_f(\mathbb{E})$.

Lemma 2.6. *Let $f\in\mathcal{M}(\mathbb{K})$, (resp. $f\in\mathcal{M}(d'0,R^-)$), resp. $f\in\mathcal{M}(\mathbb{C})$). Then*

$$T(r,f)-Z(r,f)\leq T(r,f')-Z(r,f')+\mathcal{O}(1).$$

Now, we can extract the following Lemma 2.7 from a result that is proven in several papers and particularly in Lemma 3 [14] when $\mathbb{E}=\mathbb{C}$ and, with precisions in Lemma 11 [5] when $\mathbb{E}=\mathbb{K}$. We put

$$\Psi_{F,G}=\frac{F''}{F'}-\frac{2F'}{F-1}-\frac{G''}{G'}+\frac{2G'}{G-1}.$$

Lemma 2.7. *Let $f,g\in\mathcal{M}(\mathbb{C})$ (resp. $f,g\in\mathcal{M}(\mathbb{K})$) share the value 1 CM. If $\Psi_{f,g}$ is not identically zero, then,*

$$\max(T(r,f),T(r,g))\leq N_{[2]}(r,f)+Z_{[2]}(r,f)+N_{[2]}(r,g)+Z_{[2]}(r,g)+S_f(r)+S_g(r),$$

resp.

$$\max(T(r,f),T(r,g))\leq N_{[2]}(r,f)+Z_{[2]}(r,f)+N_{[2]}(r,g)+Z_{[2]}(r,g)-6\log r.$$

We will need the following Lemma 2.8:

Lemma 2.8. *Let $f,g\in\mathcal{M}(\mathbb{K})$ be transcendental (resp. $f,g\in\mathcal{M}_u(d(0,R^-))$, resp. $f,g\in\mathcal{M}(\mathbb{C})$). Let $P(x)=x^{n+1}Q(x)$ be a polynomial such that $n\geq deg(Q)+2$ (resp. $n\geq deg(Q)+3$, resp. $n\geq deg(Q)+3$). If $P'(f)f'=P'(g)g'$ then $P(f)=P(g)$.*

For simplicity, we can assume $a_1=0$. Set $F=f'P'(f)/\alpha$ and $G=g'P'(g)/\alpha$. Clearly F and G share the value 1 C.M..

Since f,g are transcendental, we notice that so are F and G. We will prove that under the hypotheses of Theorems, $\Psi_{F,G}$ is identically zero.

The following lemma holds in the same way in p-adic analysis and in complex analysis. It is proven in [5] for the p-adic version and in [21] for the complex version.

Lemma 2.9. *Let $f,g\in\mathcal{M}(\mathbb{E})$ (resp. let $f,g\in\mathcal{M}(d(0,R^-))$ be non-constant and sharing the value 1 C.M.. Suppose that $\Psi_{f,g}=0$ and that*

$$\limsup_{r\to+\infty}\Big(\frac{\overline{Z}(r,f)+\overline{Z}(r,g)+\overline{N}(r,f)+\overline{N}(r,g)}{\max(T(r,f),T(r,g))}\Big)<1,$$

resp.

$$\limsup_{r\to R^-}\Big(\frac{\overline{Z}(r,f)+\overline{Z}(r,g)+\overline{N}(r,f)+\overline{N}(r,g)}{\max(T(r,f),T(r,g))}\Big)<1.$$

Then either $f=g$ or $fg=1$.

Proofs of Theorems 1.7-1.11. For simplicity, now we set $n=k_1$. Set $F=f'P'(f)/\alpha$, $G=g'P'(g)/\alpha$ and $\widehat{F}=P(f)$, $\widehat{G}=P(g)$. Suppose $F\neq G$. We notice that $P(x)$ is of the form $x^{n+1}Q(x)$ with $Q\in K[x]$ of degree k. Now, with help of Lemma 2.6, we can check that we have

$$T(r,\widehat{F})-Z(r,\widehat{F})\leq T(r,\widehat{F}')-Z(r,\widehat{F}')+\mathcal{O}(1).$$

Consequently, since $(\widehat{F})'=\alpha F$, we have

$$T(r,\widehat{F})\leq T(r,F)+Z(r,\widehat{F})-Z(r,F)+T(r,\alpha)+\mathcal{O}(1), \tag{2.4}$$

hence, by (2.4), we obtain

$$\begin{aligned}T(r,\widehat{F})\leq &T(r,F)+(n+1)Z(r,f)+Z\big(r,Q(f)\big)-nZ(r,f)\\&-\sum_{i=2}^{l}k_iZ(r,f-a_i)-Z(r,f')+T(r,\alpha)+\mathcal{O}(1),\end{aligned}$$

i.e.,

$$T(r,\widehat{F})\leq T(r,F)+Z(r,f)+Z\big(r,Q(f)\big)-\sum_{i=2}^{l}k_iZ(r,f-a_i)-Z(r,f')+T(r,\alpha)+\mathcal{O}(1), \tag{2.5}$$

and similarly,

$$T(r,\widehat{G})\leq T(r,G)+Z(r,g)+Z\big(r,Q(g)\big)-\sum_{i=2}^{l}k_iZ(r,g-a_i)-Z(r,g')+T(r,\alpha)+\mathcal{O}(1). \tag{2.6}$$

Now, it follows from the definition of F and G that

$$Z_{[2]}(r,F)+N_{[2]}(r,F)\leq 2Z(r,f)+2\sum_{i=2}^{l}Z(r,f-a_i)+Z(r,f')+2\overline{N}(r,f)+T(r,\alpha)+\mathcal{O}(1), \tag{2.7}$$

and similarly

$$Z_{[2]}(r,G)+N_{[2]}(r,G)\leq 2Z(r,g)+2\sum_{i=2}^{l}Z(r,g-a_i)+Z(r,g')+2\overline{N}(r,g)+T(r,\alpha)+\mathcal{O}(1). \tag{2.8}$$

And particularly, if $k_i=1$, $\forall i\in\{2,\cdots,l\}$, then

$$Z_{[2]}(r,F)+N_{[2]}(r,F)\le 2Z(r,f)+\sum_{i=2}^{l}Z(r,f-a_i)+Z(r,f')+2\overline{N}(r,f)+T(r,\alpha)+\mathcal{O}(1), \tag{2.9}$$

and similarly

$$Z_{[2]}(r,G)+N_{[2]}(r,G)\le 2Z(r,g)+\sum_{i=2}^{l}Z(r,g-a_i)+Z(r,g')+2\overline{N}(r,g)+T(r,\alpha)+\mathcal{O}(1). \tag{2.10}$$

Suppose now that $\Psi_{F,G}$ is not identically zero. Let us place us in the p-adic context: $\mathbb{E}=\mathbb{K}$. By Lemma 2.7, we have

$$T(r,F)\le Z_{[2]}(r,F)+N_{[2]}(r,F)+Z_{[2]}(r,G)+N_{[2]}(r,G)-3\log r,$$

hence by (2.5), we obtain

$$\begin{aligned}T(r,\widehat{F})\le & Z_{[2]}(r,F)+N_{[2]}(r,F)+Z_{[2]}(r,G)+N_{[2]}(r,G)+Z(r,f)+Z(r,Q(f))\\ &-\sum_{i=2}^{l}k_iZ(r,f-a_i)-Z(r,f')+T(r,\alpha)-3\log r+\mathcal{O}(1),\end{aligned}$$

and hence by (2.7) and (2.8)

$$\begin{aligned}T(r,\widehat{F})\le & 2Z(r,f)+2\sum_{i=2}^{l}Z(r,f-a_i)+Z(r,f')+2\overline{N}(r,f)+2Z(r,g)\\ &+2\sum_{i=2}^{l}Z(r,g-a_i)+Z(r,g')+2\overline{N}(r,g)+Z(r,f)+Z(r,Q(f))\\ &-\sum_{i=2}^{l}k_iZ(r,f-a_i)-Z(r,f')+2T(r,\alpha)-3\log r+\mathcal{O}(1),\end{aligned} \tag{2.11}$$

and similarly,

$$\begin{aligned}T(r,\widehat{G})\le & 2Z(r,g)+2\sum_{i=2}^{l}Z(r,g-a_i)+Z(r,g')+2\overline{N}(r,g)+2Z(r,f)\\ &+2\sum_{i=2}^{l}Z(r,f-a_i)+Z(r,f')+2\overline{N}(r,f)+Z(r,g)+Z(r,Q(g))\\ &-\sum_{i=2}^{l}k_iZ(r,g-a_i)-Z(r,g')+2T(r,\alpha)-3\log r+\mathcal{O}(1).\end{aligned} \tag{2.12}$$

Consequently,

$$
\begin{aligned}
T(r,\widehat{F})+T(r,\widehat{G}) \leq & 5(Z(r,f)+Z(r,g))+\sum_{i=2}^{l}(4-k_i)(Z(r,f-a_i)+Z(r,g-a_i)) \\
& +(Z(r,f')+Z(r,g'))+4(\overline{N}(r,f)+\overline{N}(r,g))+(Z(r,Q(f))+Z(r,Q(g))) \\
& +4T(r,\alpha)-6\log r+\mathcal{O}(1).
\end{aligned} \tag{2.13}
$$

Moreover, if $k_i=1$, $\forall i\in\{2,\cdots,l\}$, then by (2.9) and (2.10) we have

$$
\begin{aligned}
T(r,\widehat{F}) \leq & 2Z(r,f)+\sum_{i=2}^{l}Z(r,f-a_i)+Z(r,f')+2\overline{N}(r,f)+2Z(r,g) \\
& +\sum_{i=2}^{l}Z(r,g-a_i)+Z(r,g')+2\overline{N}(r,g)+Z(r,f)+Z(r,Q(f)) \\
& -\sum_{i=2}^{l}Z(r,f-a_i)-Z(r,f')+2T(r,\alpha)-3\log r+\mathcal{O}(1),
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
T(r,\widehat{G}) \leq & 2Z(r,g)+\sum_{i=2}^{l}Z(r,g-a_i)+Z(r,g')+2\overline{N}(r,g)+2Z(r,f) \\
& +\sum_{i=2}^{l}Z(r,f-a_i)+Z(r,f')+2\overline{N}(r,f)+Z(r,g)+Z(r,Q(g))) \\
& -\sum_{i=2}^{l}Z(r,g-a_i)-Z(r,g')+2T(r,\alpha)-3\log r+\mathcal{O}(1).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
T(r,\widehat{F})+T(r,\widehat{G}) \leq & 5(Z(r,f)+Z(r,g))+\sum_{i=2}^{l}(Z(r,f-a_i)+Z(r,g-a_i)) \\
& +Z(r,Q(f))+Z(r,Q(g))+(Z(r,f')+Z(r,g')) \\
& +4(\overline{N}(r,f)+\overline{N}(r,g))+4T(r,\alpha)-6\log r+\mathcal{O}(1).
\end{aligned} \tag{2.14}
$$

Now, let us go back to the general case. By Lemma 2.4, we can write $Z(r,f')+Z(r,g')\leq Z(r,f-a_2)+Z(r,g-a_2)+\overline{N}(r,f)+\overline{N}(r,g)-2\log r$. Hence, in general, by (2.13) we obtain

$$
\begin{aligned}
T(r,\widehat{F})+T(r,\widehat{G}) \leq & 5(Z(r,f)+Z(r,g))+\sum_{i=3}^{l}(4-k_i)\big((Z(r,f-a_i)+Z(r,g-a_i))\big) \\
& +(5-k_2)\big((Z(r,f-a_2)+Z(r,g-a_2))+5(\overline{N}(r,f)+\overline{N}(r,g)) \\
& +(Z(r,Q(f))+Z(r,Q(g)))+4T(r,\alpha)-8\log r+\mathcal{O}(1),
\end{aligned}
$$

and hence, since $T(r,Q(f))=kT(r,f)+\mathcal{O}(1)$ and $T(r,Q(g))=kT(r,g)+\mathcal{O}(1)$,

$$\begin{aligned}T(r,\widehat{F})+T(r,\widehat{G})\le &5(T(r,f)+T(r,g))+\sum_{i=3}^{l}(4-k_i)\big((Z(r,f-a_i)+Z(r,g-a_i))\big)\\&+(5-k_2)\big((Z(r,f-a_2)+Z(r,g-a_2))+5(\overline{N}(r,f)+\overline{N}(r,g))\\&+k(T(r,f)+T(r,g))+4T(r,\alpha)-8\log r+\mathcal{O}(1).\end{aligned}\tag{2.15}$$

Since $\widehat{F}$ is a polynomial in f of degree $n+k+1$, we have $T(r,\widehat{F})=(n+k+1)T(r,f)+\mathcal{O}(1)$ and similarly, $T(r,\widehat{G})=(n+k+1)T(r,g)+\mathcal{O}(1)$, hence by (2.15) we can derive

$$\begin{aligned}(n+k+1)(T(r,f)+T(r,g))\le &5(T(r,f)+T(r,g))+(5-k_2)(Z(r,f-a_2)+Z(r,g-a_2))\\&+\sum_{i=3}^{l}(4-k_i)\big((Z(r,f-a_i)+Z(r,g-a_i))\big)+5(\overline{N}(r,f)+\overline{N}(r,g))\\&+k(T(r,f)+T(r,g))+4T(r,\alpha)-8\log r+\mathcal{O}(1),\end{aligned}\tag{2.16}$$

hence

$$\begin{aligned}(n+k+1)(Tr,f)+T(r,g))\le &10(T(r,f)+T(r,g))+\sum_{i=3}^{l}(4-k_i)\big((Z(r,f-a_i)+Z(r,g-a_i))\big)\\&+(5-k_2)\big((Z(r,f-a_2)+Z(r,g-a_2))+k(T(r,f)+T(r,g))\\&+4T(r,\alpha)-8\log r+\mathcal{O}(1)),\end{aligned}$$

hence

$$\begin{aligned}n(Tr,f)+T(r,g))\le &9(T(r,f)+T(r,g))+(5-k_2)\big((Z(r,f-a_2)+Z(r,g-a_2))\\&+\sum_{i=3}^{l}(4-k_i)\big((Z(r,f-a_i)+Z(r,g-a_i))\big)\\&+4T(r,\alpha)-8\log r+\mathcal{O}(1)).\end{aligned}\tag{2.17}$$

Then $(5-k_2)(Z(r,f-a_2)+Z(r,g-a_2))\le\max(0,5-k_2)(T(r,f)+T(r,g))+\mathcal{O}(1)$ and at least, for each $i=3,\cdots,l$, we have

$$(4-k_i)(Z(r,f-a_i)+Z(r,g-a_i))\le\max(0,4-k_i)(T(r,f)+T(r,g))+\mathcal{O}(1).$$

Now suppose $s_5>0$. That means that $k_i\ge 5$, $\forall i=3,\cdots,u_5$ with $l\ge 5$. We notice that the number of indicies i superior or equal to 2 such that $k_i\ge 5$ is u_5-2. Similarly, for each $m>5$, the number of indicies superior or equal to 1 such that $k_i\ge m$ is u_m-1.

Suppose first $\mathbb{E}=\mathbb{K}$. then we can apply Theorem 2.3 and then we obtain $\sum_{i=3}^{u_5}Z(r,f-a_i)\ge(u_5-3)T(r,f)-\log r+\mathcal{O}(1)$ and for each $m\ge 6$, $\sum_{i=3}^{u_m}Z(r,g-a_i)\ge(u_m-2)T(r,g)-\log r+\mathcal{O}(1)$, i.e., $\sum_{i=3}^{u_5}Z(r,f-a_i)\ge s_5T(r,f)-\log r+\mathcal{O}(1)$, i.e., $\sum_{i=3}^{u_m}Z(r,g-a_i)\ge s_mT(r,g)-\log r+\mathcal{O}(1)$ in Theorems 1.7, 1.9, 1.10.

Consequently, by (2.17), we obtain

$$\begin{aligned} n(Tr,f)+T(r,g)) \le & 9(T(r,f)+T(r,g))+\max(0,5-k_2)(Z(r,f-a_2)+Z(r,g-a_2)) \\ & +\sum_{i=3}^{l}\max(0,4-k_i)\big(Z(r,f-a_i)+Z(r,g-a_i)\big)-\sum_{m=5}^{\infty}s_m(T(r,f)+T(r,g)) \\ & +4T(r,\alpha)-8\log r+\mathcal{O}(1)), \end{aligned} \tag{2.18}$$

therefore

$$n\le 9+\max(5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\sum_{j=5}^{\infty}s_j,$$

a contradiction to the hypotheses of Theorem 1.7.

Consider now the situation in Theorems 1.9 and 1.10. In Theorem 1.9, we have $T(r,\alpha)\le \log r+\mathcal{O}(1)$ and in Theorem 1.10, $T(r,\alpha)=0$. Consequently, Relation (2.18) now implies

$$\begin{aligned} n(Tr,f)+T(r,g)) \le & 9(T(r,f)+T(r,g))+\max(0,5-k_2)(Z(r,f-a_2)+Z(r,g-a_2)) \\ & +\sum_{i=3}^{l}\max(0,4-k_i)\big(Z(r,f-a_i)+Z(r,g-a_i)\big)-\sum_{m=5}^{\infty}s_m(T(r,f)+T(r,g)) \\ & -4\log r+\mathcal{O}(1)), \end{aligned}$$

therefore

$$n<9+\max(0,5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\sum_{m=5}^{\infty}s_m,$$

but this is incompatible with the hypothesis

$$n\ge 9+\max(5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\min\Big(2l,\sum_{j=5}^{\infty}s_j\Big).$$

Now, let us consider the complex context: $\mathbb{E}=\mathbb{C}$. All inequalities above hold just by replacing each expression $-q\log r$ by $S_f(r)+S_g(r)$. However, we cannot apply Theorem 2.3 here but only Theorem 2.1. Therefore we obtain

$$\begin{aligned} &\sum_{i=3}^{u_5}(Z(r,f-a_i)+Z(r,g-a_i)\ge(u_5-4)(T(r,f)+T(r,g))=t_5(T(r,f)+T(r,g)), \\ &\sum_{i=3}^{U_m}(Z(r,f-a_i)+Z(r,g-a_i)\ge(u_m-3)(T(r,f)+T(r,g))=t_m(T(r,f)+T(r,g)). \end{aligned}$$

Therefore we obtain

$$n\le 9+\max(5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\sum_{m=5}^{\infty}t_m$$

a contradiction to the hypothesis of Theorem 1.8.

Finally, consider the situation in Theorem 1.11. Since $N(r,f)=N(r,g)=0$, Relation (2.16) gets

$$\begin{aligned}(n+k+1)(T(r,f)+T(r,g)) \le & 5(T(r,f)+T(r,g))+(5-k_2)(Z(r,f-a_2)+Z(r,g-a_2))\\ &+\sum_{i=3}^{l}(4-k_i)\big((Z(r,f-a_i)+Z(r,g-a_i))\big)\\ &+k(T(r,f)+T(r,g))+4T(r,\alpha)+S_f(r)+S_g(r)).\end{aligned}$$

On the other hand, by applying Theorem 2.1 to f and g, which now are entire functions, we have

$$\begin{aligned}&\sum_{i=3}^{u_5} Z(r,f-a_i)\ge(u_5-3)T(r,f)=s_5T(r,f), \quad \sum_{i=3}^{u_5} Z(r,g-a_i)\ge(u_5-3)T(r,g)=s_5T(r,g),\\ &\sum_{i=3}^{u_m} Z(r,f-a_i)\ge(u_m-2)T(r,f)=s_mT(r,f), \quad \sum_{i=3}^{u_m} Z(r,g-a_i)\ge(u_m-2)T(r,g)=s_mT(r,g).\end{aligned}$$

Consequently,

$$n+k+1\le 5+k+\max(0,5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\sum_{m=1}^{\infty}s_m,$$

and therefore

$$n\le 4+\max(0,5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\sum_{m=1}^{\infty}s_m,$$

a contradiction to the hypotheses of Theorem 1.11.

Thus, in the hypotheses of Theorems 1.7-1.11 we have proven that $\Psi_{F,G}$ is identically zero. Henceforth, we can assume that $\Psi_{F,G}=0$ in each theorem.

Note that we can write

$$\Psi_{F,G}=\frac{\phi'}{\phi} \quad \text{with } \phi=\Big(\frac{F'}{(F-1)^2}\Big)\Big(\frac{(G-1)^2}{G'}\Big).$$

Since $\Psi_{F,G}=0$, there exist $A,B\in\mathbb{E}$ such that

$$\frac{1}{G-1}=\frac{A}{F-1}+B, \tag{2.19}$$

and $A\neq 0$.

We notice that $\overline{Z}(r,f)\le T(r,f)$,

$$\overline{N}(r,f)\le T(r,f)\overline{Z}(r,f-a_i)\le T(r,f-a_i)\le T(r,f)+\mathcal{O}(1), \quad i=2,\cdots,l,$$

and $\overline{Z}(r,f')\le T(r,f')\le 2T(r,f)+\mathcal{O}(1)$. Similarly for g and g'. Moreover, if $\mathbb{E}=\mathbb{K}$ by Lemma 2.4 we have

$$T(r,F)\ge (n+k)T(r,f), \tag{2.20}$$

and if $\mathbb{E}=\mathbb{C}$, we have

$$T(r,F)\ge (n+k)T(r,f)-m\Big(r,\frac{1}{f'}\Big)+S_f(r). \tag{2.21}$$

We will show that $F=G$ in each therorem. We first notice that according to all hypotheses in Theorems 1.7-1.10 we have

$$n+k\ge 2l+7 \tag{2.22}$$

and in Theorem 1.11, we have

$$n+k\ge 2l+5. \tag{2.23}$$

We will consider the following two cases: $B=0$ and $B\neq 0$.

Case 1: $B=0$.

Suppose $A\neq 1$. Then, by (2.19), we have $F=AG+(1-A)$. Suppose first $\mathbb{E}=\mathbb{K}$. Applying Theorem 2.1 to F, we obtain

$$\begin{aligned}T(r,F)\le&\overline{Z}(r,F)+\overline{Z}(r,F-(1-A))+\overline{N}(r,F)-\log r+\mathcal{O}(1)\\ \le&\overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,f')+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)+\overline{Z}(r,g')\\ &+\overline{N}(r,f)-\log r+\mathcal{O}(1).\end{aligned} \tag{2.24}$$

By (2.20) and (2.24), we obtain

$$\begin{aligned}(n+k)T(r,f)\le&\overline{Z}(r,F)+\overline{Z}\big(r,F-(1-A)\big)+\overline{N}(r,F)+-\log r+\mathcal{O}(1)\\ \le&\overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,f')+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)+\overline{Z}(r,g')+\overline{N}(r,f)\\ &-\log r+\mathcal{O}(1).\end{aligned} \tag{2.25}$$

By (2.25), we have

$$\begin{aligned}(n+k)T(r,f)\le&\overline{Z}(r,F)+\overline{Z}\big(r,F-(1-A)\big)+\overline{N}(r,F)-\log r+\mathcal{O}(1)\\ \le&\overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,f')+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)\\ &+\overline{Z}(r,g')+\overline{N}(r,f)-\log r+\mathcal{O}(1)),\end{aligned}$$

hence

$$(n+k)T(r,f)\le\overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)$$
$$+\overline{N}(r,f)+\overline{Z}(r,g')+\overline{Z}(r,f')-\log r+\mathcal{O}(1). \tag{2.26}$$

Then, considering all the previous inequalities, by Lemma 2.4 we can derive the following from (2.26)

$$(n+k)T(r,f)\le(l+3)T(r,f)+(l+2)T(r,g)-3\log r+\mathcal{O}(1). \tag{2.27}$$

Since f and g satisfy the same hypothesis, we also have

$$(n+k)T(r,g)\le(l+3)T(r,g)+(l+2)T(r,f)-3\log r+\mathcal{O}(1). \tag{2.28}$$

Hence, adding (2.27) and (2.28), we have

$$(n+k)\big[T(r,f)+T(r,g)\big]\le(2l+5)\big[T(r,f)+T(r,g)\big]-6\log r+\mathcal{O}(1),$$

therefore

$$n+k<2l+5. \tag{2.29}$$

A contradiction to (2.23) proving that $A\neq 1$ is impossible whenever $B=0$, in Theorems 1.7, 1.9 and 1.10.

Suppose now $\mathbb{E}=\mathbb{C}$. By (2.21), we have

$$\begin{aligned}(n+k)T(r,f)\le&\overline{Z}(r,F)+\overline{Z}(r,F-(1-A))+\overline{N}(r,F)+m\Big(r,\frac{1}{f'}\Big)+S_F(r)\\ \le&\overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,f')+m\Big(r,\frac{1}{f'}\Big)+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)\\ &+\overline{Z}(r,g')+\overline{N}(r,f)+S_f(r)+S_g(r).\end{aligned}$$

Here we notice that $\overline{Z}(r,f')+m(r,1/f')\le T(r,1/f')=T(r,f')+\mathcal{O}(1)$, hence

$$(n+k)T(r,f)\le\overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)$$
$$+\overline{N}(r,f)+\overline{Z}(r,g')+T(r,f')+S_f(r)+S_g(r). \tag{2.30}$$

Then, considering all the previous inequalities in (2.30), similarly we can derive

$$(n+k)T(r,f)\le(l+3)T(r,f)+(l+2)T(r,g)+S_f(r)+S_g(r). \tag{2.31}$$

Since f and g satisfy the same hypothesis, we also have

$$(n+k)T(r,g)\le(l+3)T(r,g)+(l+2)T(r,f)+S_f(r)+S_g(r). \tag{2.32}$$

Hence, adding (2.31) and (2.32), we have

$$(n+k)\big[T(r,f)+T(r,g)\big]\le(2l+5)\big[T(r,f)+T(r,g)\big]+S_f(r)+S_g(r),$$

therefore $n+k\le 2l+5$, a contradiction to (2.23) proving that $A\neq 1$ is impossible whenever $B=0$, in Theorem 1.8.

Consider now the situation in Theorem 1.11. By hypothesis we have

$$k_1\ge 5+\max(0,5-k_2)+\sum_{i=3}^{l}\max(0,4-k_i)-\min\Big(2l,\sum_{m=5}^{\infty}s_m\Big),$$

hence

$$n+k\ge 10+4(l-2)-\sum_{m=5}^{\infty}s_m=4l+2-\sum_{m=5}^{\infty}s_m.$$

Since $N(r,f)=N(r,g)=0$, we can use Theorem 2.1, for entire functions and we obtain

$$\sum_{i=3}^{u_5}Z(r,f-a_i)\ge(u_5-3)T(r,f)+S_f(r)+S_g(r),$$

and for each $m\ge 6$,

$$\sum_{i=3}^{u_m}Z(r,g-a_i)\ge(u_m-2)T(r,g)+S_f(r))+S_g(r),$$

i.e.,

$$\sum_{i=3}^{u_5}Z(r,f-a_i)\ge s_5T(r,f)+S_f(r)+S_g(r)$$

and

$$\sum_{i=3}^{u_m}Z(r,g-a_i)\ge s_mT(r,g)+S_f(r)+S_g(r).$$

Now, Relation (2.16) now gets

$$\begin{aligned}&(n+k+1)(T(r,f)+T(r,g))\\ \le&5(T(r,f)+T(r,g))+(5-k_2)(Z(r,f-a_2)+Z(r,g-a_2))\\ &+\sum_{i=3}^{l}(4-k_i)\big((Z(r,f-a_i)+Z(r,g-a_i))\big)+k(T(r,f)+T(r,g))+S_f(r)+S_g(r),\end{aligned}$$

therefore

$$n+k\leq 9+4(l-2)-\sum_{j=5}^{\infty}s_j=2l+1-\sum_{m=5}^{\infty}s_m$$

a contradiction to the hypothesis $n+k\geq 2l+5$ of Theorem 1.11. Consequently, the hypothesis $A\neq 1$ does not hold when $B=0$. Henceforth we suppose $B\neq 0$.

Case 2: $B\neq 0$.

Consider first the situation when $\mathbb{E}=\mathbb{K}$, i.e., in Theorems 1.7 and in Theorems 1.9 and 1.10. By (2.20) we have Immediately,

$$\begin{aligned}
&\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G)\\
\leq&\overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,f')+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)\\
&+\overline{Z}(r,g')+\overline{N}(r,f)+\overline{N}(r,g)+4T(r,\alpha)+\mathcal{O}(1)\\
\leq&(l+1)\big[T(r,f)+T(r,g)\big]+T(r,f')+T(r,g')+4T(r,\alpha)+\mathcal{O}(1)\\
\leq&(l+3)(T(r,f)+T(r,g))+4T(r,\alpha)-2\log r,
\end{aligned}$$

hence by Lemma 2.4,

$$\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G)\leq (l+3)(T(r,f)+4T(r,\alpha)-2\log r+\mathcal{O}(1)). \tag{2.33}$$

Moreover, by (2.19), $T(r,F)=T(r,G)+\mathcal{O}(1)$ and by Lemma 2.4, we have

$$T(r,f)\leq\frac{1}{n+k}(T(r,F)+T(r,\alpha))+\mathcal{O}(1)\quad\text{and}\quad T(r,g)\leq\frac{1}{n+k}(T(r,G)+T(r,\alpha))+\mathcal{O}(1).$$

Consequently,

$$T(r,f)+T(r,g)\leq 2\Big[\frac{1}{n+k}(T(r,F)+T(r,\alpha))\Big]+\mathcal{O}(1), \tag{2.34a}$$

$$\begin{aligned}
&\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G)\\
&\leq\frac{2l+6}{n+k}T(r,F)+\Big(\frac{2l+6}{n+k}+4\Big)T(r,\alpha)-2\log r+\mathcal{O}(1).
\end{aligned} \tag{2.34b}$$

Now, by Hypotheses, in Theorems 1.7, 1.9, 1.10 by (2.22), we have $n+k\geq 2l+7$. Consequently, by relation (2.34b) we obtain

$$\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G)\leq\frac{2l+6}{2l+7}T(r,F)+\Big(\frac{2l+6}{2l+7}+4\Big)T(r,\alpha)+\mathcal{O}(1), \tag{2.35}$$

and similarly,

$$\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G)\leq\frac{2l+6}{2l+7}T(r,G)+\Big(\frac{2l+6}{2l+7}+4\Big)T(r,\alpha)+\mathcal{O}(1), \tag{2.36}$$

hence

$$\limsup_{r\to+\infty}\Big(\frac{\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G)}{\max(T(r,F),T(r,G))}\Big)<1.$$

Therefore, by Lemma 2.9, and Theorems 1.7, 1.9, 1.10, we have either $F=G$, or $FG=1$.

Suppose $FG=1$. Then $f'P'(f)g'P'(g)=\alpha^2$. But in Theorems 1.7, 1.9, 1.10, we have assumed that $n\neq k+1$ and if $l=2$, then $n\neq 2k,2k+1,3k+1$ and if $l=3$ then $n\neq k,3k_2-k,3k_3-k$. Consequently, we have a contradiction to Theorem 2.7. Thus, the hypothesis $FG=1$ is impossible and therefore we have $F=G$.

Consider now the situation when $\mathbb{E}=\mathbb{C}$, i.e., in Theorems 1.8 and 1.11. The proof is very similar to that in the case when $\mathbb{E}=\mathbb{K}$. We have

$$\begin{aligned}&\overline{Z}(r,F)\leq\overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,f')+S_f(r),\\&\overline{N}(r,F)\leq\overline{N}(r,f)+S_f(r),\end{aligned}$$

and similarly for G, so we can derive

$$\begin{aligned}&\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G)\\\leq&\overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,f')+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)\\&+\overline{Z}(r,g')+\overline{N}(r,f)+\overline{N}(r,g)+S_f(r)+S_g(r)\\\leq&(l+2)\big[T(r,f)+T(r,g)\big]+S_f(r)+S_g(r).\end{aligned}\tag{2.37}$$

Moreover, by (2.19), $T(r,F)=T(r,G)+\mathcal{O}(1)$ and, by Lemma 2.4, we have

$$T(r,f)\leq\frac{1}{n+k}T(r,F)+S_f(r)\quad\text{and}\quad T(r,g)\leq\frac{1}{n+k}T(r,G)+S_g(r).$$

Consequently,

$$T(r,f)+T(r,g)\leq\frac{2}{n+k}T(r,F)+S_f(r)+S_g(r).$$

Thus, (2.37) implies

$$\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G)\leq\frac{2l+6}{n+k}T(r,F)+S_f(r)+S_g(r).$$

Now, as in Theorems 1.7, 1.9, 1.10, we can check that $n+k\geq 2l+7$ in Theorem 1.8. Consequently, the previous inequality implies

$$\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G)\leq\frac{2l+6}{2l+7}T(r,F)+S_f(r)+S_g(r)$$

and similarly,

$$\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G)\leq\frac{2l+6}{2l+7}T(r,G)+S_f(r)+S_g(r),$$

hence by Lemma 2.9 again, we have $F=G$ or $FG=1$. Then, by Theorem 2.7 as in Theorems 1.7, 1.9, 1.10, the hypotheses of Theorem 1.8 prevent the case $FG=1$ and therefore $F=G$.

Consider now the situation in Theorem 1.11. Relation (2.37) implies

$$\overline{Z}(r,F)+\overline{Z}(r,G)\leq(l+2)\big[T(r,f)+T(r,g)\big]+S_f(r)+S_g(r). \tag{2.38}$$

Moreover, by (16), $T(r,F)=T(r,G)+\mathcal{O}(1)$ and, by Lemma 2.4, we have

$$T(r,f)\leq\frac{1}{n+k}T(r,F)+S_f(r)\quad\text{and}\quad T(r,g)\leq\frac{1}{n+k}T(r,G)+S_g(r).$$

Consequently,

$$T(r,f)+T(r,g)\leq\frac{2}{n+k}T(r,F)+S_f(r)+S_g(r).$$

Thus, (2.37) implies

$$\begin{aligned}\overline{Z}(r,F)+\overline{Z}(r,G)\leq&\overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,f')+\overline{Z}(r,g)\\&+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)+\overline{Z}(r,g')+S_f(r)+S_g(r)\\\leq&4\big[T(r,f)+T(r,g)\big]+S_f(r)+S_g(r).\end{aligned}$$

Therefore,

$$\overline{Z}(r,F)+\overline{Z}(r,G)\leq\frac{2l+4}{n+k}T(r,F)+S_f(r)+S_g(r),$$

hence by (2.23) we have

$$\overline{Z}(r,F)+\overline{Z}(r,G)\leq\frac{2l+4}{2l+5}T(r,F)+S_f(r)+S_g(r).$$

In the same way, this proves that either $F=G$ of $FG=1$. But by Theorem 2.7, $FG=1$ is impossible. Hence $F=G$.

Thus, in Theorems 1.7-1.11, we have proven that $F=G$, i.e., $f'P'(f)=g'P'(g)$. Consequently, $P(f)-P(g)$ is a constant C. Then, by Lemma 2.8 and Proposition 1.1, in Theorems 1.7, 1.9, 1.10, we have $P(f)=P(g)$ and by Lemma 2.8 and Proposition 1.2, we have $P(f)=P(g)$ in Theorems 1.8 and 1.11. Finally, in each theorem, P is a polynomial of uniqueness for the family of functions we consider. Consequently, $f=g$. □

Acknowledgements

Partially funded by the research project CONICYT (Inserción de nuevos investigadores en la academia, NO. 79090014) from the Chilean Government.

References

[1] T. T. H. An, J. T. Y. Wang and P. M. Wong, Unique range sets and uniqueness polynomials in positive characteristic II, Acta Arithmetica, (2005), 115–143.

[2] T. T. H. An, J. T. Y. Wang and P. M. Wong, Strong uniqueness polynomials: the complex case, Complex Variables, 49(1) (2004), 25–54.

[3] T. T. H. An and N. T. N. Diep, Genus one factors of curves dened by separated variable polynomials, J. Number. Theory, 133 (2013), 2616-2634.

[4] T. T. H. An and A. Escassut, Meromorphic solutions of equations over non-archimedean fields, Ramanujan J., 15(3) (2008), 415–433.

[5] K. Boussaf, A. Escassut and J. Ojeda, p-adic meromorphic functions $f'P'(f),g'P'(g)$ sharing a small function, Bulletin des Sciences Mathématiques, 136(2) (2012), 172–200.

[6] K. Boussaf, A. Escassut and J. Ojeda, Complex meromorphic functions $f'P'(f)$, $g'P'(g)$ sharing a small function, Indagationes, 24(1) (2013), 15–41.

[7] A. Boutabaa, Théorie de Nevanlinna p-adique, Manuscripta Math., 67 (1990), 251–269.

[8] A. Escassut, Analytic Elements in p-Adic Analysis, World Scientific Publishing Co. Pte. Ltd. Singapore, 1995.

[9] A. Escassut, Meromorphic functions of uniqueness, Bulletin des Sciences Mathématiques, 131(3) (2007), 219–241.

[10] A. Escassut, p-adic value distribution, Some Topics on Value Distribution and Differentability in Complex and P-Adic Analysis, 42–138. Mathematics Monograph, Series 11, Science Press, Beijing, 2008.

[11] H. Fujimoto, On uniqueness of meromorphic functions sharing finite sets, Amer. J. Math., 122(6) (2000), 1175–1203.

[12] W. K. Hayman, Meromorphic Functions, Oxford University Press, 1975.

[13] P. C. Hu and C. C. Yang, Meromorphic Functions over Non-Archimedean Fields, Kluwer Academic Publishers, 2000.

[14] X. Hua and C. C. Yang, Uniqueness and value-sharing of meromorphic functions, Ann. Acad. Sci. Fenn. Math., 22 (1997), 395–406.

[15] W. Lin and H. Yi, Uniqueness theorems for meromorphic functions concerning fixed-points, Complex Var. Theory Appl., 49(11) (2004), 793–806.

[16] R. Nevanlinna, Le Théorème de Picard-Borel et la Théorie des Fonctions Méromorphes, Gauthiers-Villars, Paris, 1929.

[17] J. Ojeda, Uniqueness for ultrametric analytic functions, Bulletin Mathématique des Sciences Mathématiques de Roumanie, 54.

[18] J. T. Y. Wang, Uniqueness polynomials and bi-unique range sets, Acta Arithmetica, 104 (2002), 183–200.

[19] C. C. Yang and P. Li, Some further results on the functional equation $P(f)=Q(g)$, Series Advances in Complex Analysis and Applications, Value Distribution Theory and Related Topics, 219–231, Kluwer Academic Publishers, Dordrecht-Boston-London, 2004.

[20] C. C. Yang and X. Hua, Unique polynomials of entire and meromorphic functions, Matematicheskaia Fizika Analys Geometriye, 4(3) (1997), 391–398.
[21] H. X. Yi, Meromorphic functions that share one or two values, Complex Variables and Applications, 28 (1995), 1–11.