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Coefficient Estimates for Certain Subclasses of Bi-Univalent Ma-Minda Mocanu-Convex Functions

C. Selvaraj[1], O. S. Babu[2] and G. Murugusundaramoorthy[3,*]

[1] *Department of Mathematics, Presidency College (Autonomous), Chennai–600005, India*
[2] *Department of Mathematics, Dr. Ambedkar Govt. Arts College, Chennai–600039, India*
[3] *School of Advanced Sciences, VIT University, Vellore–632 014, India*

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Abstract. In this paper, we introduce and investigate a new subclass of the function class Σ of bi-univalent functions of the Mocanu-convex type defined in the open unit disk, satisfy Ma and Minda subordination conditions. Furthermore, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the new subclass introduced here. Further Application of Hohlov operator to this class is obtained. Several (known or new) consequences of the results are also pointed out.

Key Words: Analytic functions, univalent functions, bi-univalent functions, bi-starlike functions, bi-convex functions, bi-Mocanu-convex functions, subordination, Hohlov operator.

AMS Subject Classifications: 30C45

1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

normalized by the conditions $f(0)=0=f'(0)-1$ defined in the open unit disk $\Delta=\{z\in\mathbb{C}:|z|<1\}$. A function $f\in\mathcal{A}$ is said to be bi-univalent in Δ if both f and f^{-1} are univalent in Δ. Let Σ denote the class of bi-univalent functions defined in the unit disk Δ. Since $f\in\Sigma$

*Corresponding author. *Email addresses:* `pamc9439@yahoo.co.in` (C. Selvaraj), `osbabu1009@gmail.com` (O. S. Babu), `gmsmoorthy@yahoo.com` (G. Murugusundaramoorthy)

has the Maclaurin series given by (1.1), a computation shows that its inverse $g=f^{-1}$ has the expansion

$$g(w)=f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3+\cdots. \tag{1.2}$$

An analytic function f is subordinate to an analytic function g, written as $f(z)\prec g(z)$, provided there is an analytic function w defined on Δ with $w(0)=0$ and $|w(z)|<1$ satisfying $f(z)=g(w(z))$. Ma and Minda [5] unified various subclasses of starlike and convex functions for which either of the quantity $zf'(z)/f(z)$ or $1+zf''(z)/f'(z)$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function φ with positive real part in the unit disk Δ, $\varphi(0)=1$, $\varphi'(0)>0$, and φ maps Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f\in\mathcal{A}$ satisfying the subordination $zf'(z)/f(z)\prec\varphi(z)$. Similarly, the class of Ma-Minda convex functions of functions $f\in\mathcal{A}$ satisfying the subordination $1+zf''(z)/f'(z)\prec\varphi(z)$. A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are respectively Ma-Minda starlike or convex. These classes are denoted respectively by $\mathcal{S}^*_\Sigma(\varphi)$ and $\mathcal{K}_\Sigma(\varphi)$. Also denote by $\mathcal{M}_\Sigma(\lambda,\phi)$ the class of Ma-Minda Mocanu-convex functions consists of functions $f\in\mathcal{A}$ satisfying the subordination

$$(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\Big(1+\frac{zf''(z)}{f'(z)}\Big)\prec\phi(z),\quad \lambda\geq 0.$$

In the sequel, it is assumed that φ is an analytic function with positive real part in the unit disk Δ, satisfying $\varphi(0)=1$, $\varphi'(0)>0$, and $\varphi(\Delta)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form

$$\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots,\quad B_1>0. \tag{1.3}$$

Recently there has been triggering interest to study bi-univalent functions (see [7,9,10]. Motivated by the works of Ali et al. [1] and Goyal and Goswami [3], in this paper we introduce a new subclass $\mathcal{SP}^\gamma_\Sigma(\lambda,h)$ of bi-univalent functions to estimate the coefficients $|a_2|$ and $|a_3|$ for the functions in the class $\mathcal{SP}^\gamma_\Sigma(\lambda,h)$.

Definition 1.1. Let $h:\Delta\to\mathbb{C}$ be a convex univalent function such that $h(0)=1$ and $\Re(h(z))>0$, $z\in\Delta$. A function $f(z)$ is said to be in the class $\mathcal{SP}^\gamma_\Sigma(\lambda,h)$ if the following conditions are satisfied:

$$e^{i\gamma}\Big[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\Big(1+\frac{zf''(z)}{f'(z)}\Big)\Big]\prec h(z)\cos\gamma+i\sin\gamma,\quad f\in\Sigma,\quad z\in\Delta, \tag{1.4}$$

and

$$e^{i\gamma}\Big[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\Big(1+\frac{wg''(w)}{g'(w)}\Big)\Big]\prec h(w)\cos\gamma+i\sin\gamma,\quad w\in\Delta, \tag{1.5}$$

where $\gamma\in(-\pi/2,\pi/2)$, $\lambda\geq 0$ and $g=f^{-1}$.

Remark 1.1. If we set $h(z)=(1+Az)/(1+Bz)$, $-1\leq B<A\leq 1$, then the class $\mathcal{SP}_{\Sigma}^{\gamma}(\lambda,h)$ reduces to $\mathcal{SP}_{\Sigma}^{\gamma}(\lambda,A,B)$ which is defined as

$$e^{i\gamma}\Big[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\Big(1+\frac{zf''(z)}{f'(z)}\Big)\Big]\prec\frac{1+Az}{1+Bz}\cos\gamma+i\sin\gamma,\quad f\in\Sigma,\quad z\in\Delta, \tag{1.6}$$

and

$$e^{i\gamma}\Big[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\Big(1+\frac{wg''(w)}{g'(w)}\Big)\Big]\prec\frac{1+Aw}{1+Bw}\cos\gamma+i\sin\gamma,\quad w\in\Delta, \tag{1.7}$$

where $\gamma\in(-\pi/2,\pi/2)$, $\lambda\geq 0$ and $g=f^{-1}$.

In particular, by setting

$$h(z)=\frac{1+(1-2\beta)z}{1-z},\quad 0\leq\beta<1,$$

the class $\mathcal{SP}_{\Sigma}^{\gamma}(\lambda,h)$ reduces to $\mathcal{SP}_{\Sigma}^{\gamma}(\lambda,1-2\beta,-1)\equiv\mathcal{SP}_{\Sigma}^{\gamma}(\lambda,\beta)$.

Remark 1.2. Taking $\lambda=0$ in the class $\mathcal{SP}_{\Sigma}^{\gamma}(\lambda,A,B)$ we have $\mathcal{SP}_{\Sigma}^{\gamma}(0,A,B)$ and if $f\in\mathcal{SP}_{\Sigma}^{\gamma}(0,A,B)$, then

$$e^{i\gamma}\frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz}\cos\gamma+i\sin\gamma,\quad f\in\Sigma,\quad z\in\Delta, \tag{1.8}$$

and

$$e^{i\gamma}\frac{wg'(w)}{g(w)}\prec\frac{1+Aw}{1+Bw}\cos\gamma+i\sin\gamma,\quad w\in\Delta, \tag{1.9}$$

where $\gamma\in(-\pi/2,\pi/2)$ and $g=f^{-1}$.

We note that $\mathcal{SP}_{\Sigma}^{0}(0,1-2\beta,-1)\equiv\mathcal{S}_{\Sigma}^{*}(\beta)$ [2].

Remark 1.3. Taking $\lambda=1$ in the class $\mathcal{SP}_{\Sigma}^{\gamma}(\lambda,A,B)$, we have $\mathcal{SP}_{\Sigma}^{\gamma}(1,A,B)$ and if $f\in\mathcal{SP}_{\Sigma}^{\gamma}(1,A,B)$, then

$$e^{i\gamma}\Big(1+\frac{zf''(z)}{f'(z)}\Big)\prec\frac{1+Az}{1+Bz}\cos\gamma+i\sin\gamma,\quad f\in\Sigma,\quad z\in\Delta, \tag{1.10}$$

and

$$e^{i\gamma}\Big(1+\frac{wg''(w)}{g(w)}\Big)\prec\frac{1+Aw}{1+Bw}\cos\gamma+i\sin\gamma,\quad w\in\Delta, \tag{1.11}$$

where $\gamma\in(-\pi/2,\pi/2)$ and $g=f^{-1}$.

We also note that $\mathcal{SP}_{\Sigma}^{0}(1,1-2\beta,-1)\equiv\mathcal{K}_{\Sigma}(\beta)$ [2].

The object of the paper is to estimate the coefficients $|a_2|$ and $|a_3|$ for the functions in the class $\mathcal{SP}_{\Sigma}^{\gamma}(\lambda,h)$. Further we define a new generalized subclass of $\mathcal{SP}_{\Sigma}^{\gamma}(\lambda,h)$ involving Hohlov operator and obtained the Maclaurin coefficients for f in this new generalized class.

2 Coefficients estimates for the function class $\mathcal{SP}_{\Sigma}^{\gamma}(\lambda,h)$

In order to prove our main result for the functions class $\mathcal{SP}_{\Sigma}^{\gamma}(\lambda,h)$, we recall the following lemma.

Lemma 2.1. *Let the function $\phi(z)$ given by $\phi(z)=\sum_{n=1}^{\infty}B_n z^n$ be convex in Δ. Suppose also that the function $h(z)$ given by $h(z)=\sum_{n=1}^{\infty}h_n z^n$, is holomorphic in Δ. If $h(z)\prec\phi(z)$ $(z\in\Delta)$, then $|h_n|\le|B_1|$ $(n\in\mathbb{N})$.*

Theorem 2.1. *Let f given by* (1.1) *be in the class $\mathcal{SP}_{\Sigma}^{\gamma}(\lambda,h)$. Then*

$$|a_2|\le\sqrt{\frac{|B_1|\cos\gamma}{1+\lambda}}, \tag{2.1}$$

and

$$|a_3|\le\Big(\frac{1}{2(1+2\lambda)}+\frac{|B_1|\cos\gamma}{(1+\lambda)^2}\Big)|B_1|\cos\gamma, \tag{2.2}$$

where $\gamma\in(-\pi/2,\pi/2)$ and $\lambda\ge 0$.

Proof. Let $f\in\mathcal{SP}_{\Sigma}^{\gamma}(\lambda,h)$ and $g=f^{-1}$. Then from (1.4) and (1.5) we have

$$e^{i\gamma}\Big[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\Big(1+\frac{zf''(z)}{f'(z)}\Big)\Big]\prec p(z)\cos\gamma+i\sin\gamma,\quad z\in\Delta, \tag{2.3}$$

and

$$e^{i\gamma}\Big[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\Big(1+\frac{wg''(w)}{g'(w)}\Big)\Big]\prec q(w)\cos\gamma+i\sin\gamma,\quad w\in\Delta, \tag{2.4}$$

where $p(z)\prec h(z)$ and $q(w)\prec h(w)$ and have the following forms:

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots,\quad z\in\Delta, \tag{2.5}$$

and

$$q(w)=1+q_1w+q_2w^2+q_3w^3+\cdots,\quad w\in\Delta. \tag{2.6}$$

Now, equating the coefficients in (2.3) and (2.4), we get

$$e^{i\gamma}(1+\lambda)a_2=p_1\cos\gamma, \tag{2.7a}$$

$$e^{i\gamma}[-(1+3\lambda)a_2^2+2(1+2\lambda)a_3]=p_2\cos\gamma, \tag{2.7b}$$

$$-e^{i\gamma}(1+\lambda)a_2=q_1\cos\gamma, \tag{2.7c}$$

and

$$e^{i\gamma}[(3+5\lambda)a_2^2-2(1+2\lambda)a_3]=q_2\cos\gamma. \tag{2.8}$$

From (2.7a) and (2.7c), it follows that

$$p_1=-q_1, \tag{2.9}$$

and

$$2e^{2i\gamma}(1+\lambda)^2a_2^2=(p_1^2+q_1^2)\cos^2\gamma. \tag{2.10}$$

Adding (2.7b) and (2.8) it follows that

$$a_2^2=\frac{(p_2+q_2)}{2(1+\lambda)}e^{-i\gamma}\cos\gamma. \tag{2.11}$$

Since $p(z),q(z)\in h(\Delta)$, applying Lemma 2.1 for the coefficients p_2 and q_2, we get

$$|a_2|^2=\frac{|B_1|\cos\gamma}{(1+\lambda)}, \tag{2.12}$$

which gives the estimate on $|a_2|$ as asserted in (2.1).

Subtracting (2.8) from (2.7b) we get

$$a_3-a_2^2=\frac{(p_2-q_2)e^{-i\gamma}\cos\gamma}{4(1+2\lambda)}. \tag{2.13}$$

Substituting the value of a_2^2 from (2.10) in (2.13) we get

$$a_3=\frac{(p_2-q_2)e^{-i\gamma}\cos\gamma}{4(1+2\lambda)}+\frac{(p_1^2+q_1^2)e^{-2i\gamma}\cos^2\gamma}{2(1+\lambda)^2}.$$

Applying Lemma 2.1, once again for the coefficients p_1, p_2, q_1 and q_2, we get

$$|a_3|\le\frac{|B_1|\cos\beta}{2(1+2\lambda)}+\frac{|B_1|^2\cos^2\gamma}{(1+\lambda)^2},$$

which gives the estimate on $|a_3|$ as asserted in (2.2). □

By setting

$$h(z)=\frac{1+Az}{1+Bz},\quad -1\le B<A\le 1,$$

in Theorem 2.1, we get the following corollary:

Corollary 2.1. Let f given by (1.1) be in the class $\mathcal{SP}_\Sigma^\gamma(\lambda,A,B)$. Then

$$|a_2|\le\sqrt{\frac{(A-B)\cos\gamma}{1+\lambda}}, \tag{2.14}$$

and

$$|a_3|\le\Big(\frac{1}{2(1+2\lambda)}+\frac{(A-B)\cos\gamma}{(1+\lambda)^2}\Big)(A-B)\cos\gamma, \tag{2.15}$$

where $\gamma\in(-\pi/2,\pi/2)$ and $\lambda\ge 0$.

If we take $\lambda=0$ in corollary 2.1, we obtain

Corollary 2.2. Let f given by (1.1) be in the class $\mathcal{SP}_{\Sigma}^{\gamma}(0,A,B)$. Then

$$|a_2|\le\sqrt{(A-B)\cos\gamma},$$

and

$$|a_3|\le\Big(\frac{1}{2}+(A-B)\cos\gamma\Big)(A-B)\cos\gamma,$$

where $\gamma\in(-\pi/2,\pi/2)$.

If we take $\lambda=1$ in corollary 2.1 we obtain

Corollary 2.3. Let f given by (1.1) be in the class $\mathcal{SP}_{\Sigma}^{\gamma}(1,A,B)$. Then

$$|a_2|\le\sqrt{\frac{(A-B)\cos\gamma}{2}},$$

and

$$|a_3|\le\Big(\frac{1}{6}+\frac{(A-B)\cos\gamma}{4}\Big)(A-B)\cos\gamma,$$

where $\gamma\in(-\pi/2,\pi/2)$.

Further, by setting

$$h(z)=\frac{1+(1-2\beta)z}{1-z},\quad 0\le\beta<1,$$

in Theorem 2.1 we get the following corollary:

Corollary 2.4. Let f be given by (1.1) be in the class $\mathcal{SP}_{\Sigma}^{\gamma}(\lambda,\beta)$. Then

$$|a_2|\le\sqrt{\frac{2(1-\beta)\cos\gamma}{1+\lambda}},$$

and

$$|a_3|\le\Big(\frac{1}{(1+2\lambda)}+\frac{4(1-\beta)\cos\gamma}{(1+\lambda)^2}\Big)(1-\beta)\cos\gamma,$$

where $\gamma\in(-\pi/2,\pi/2)$ and $\lambda\ge0$.

If we take $\lambda=0$ in corollary 2.4 we obtain

Corollary 2.5. Let f given by (1.1) be in the class $\mathcal{SP}_{\Sigma}^{\gamma}(0,\beta)$. Then

$$|a_2|\le\sqrt{2(1-\beta)\cos\gamma},$$

and

$$|a_3|\le(1+4(1-\beta)\cos\gamma)(1-\beta)\cos\gamma,$$

where $\gamma\in(-\pi/2,\pi/2)$.

If we take $\lambda=1$ in corollary 2.4 we obtain

Corollary 2.6. Let f given by (1.1) be in the class $\mathcal{SP}_{\Sigma}^{\gamma}(1,\beta)$. Then

$$|a_2|\le\sqrt{(1-\beta)\cos\gamma},$$

and

$$|a_3|\le\Big(\frac{1}{3}+(1-\beta)\cos\gamma\Big)(1-\beta)\cos\gamma,$$

where $\gamma\in(-\pi/2,\pi/2)$.

Remark 2.1. On taking $\gamma=0$, we observe that the classes $\mathcal{SP}_{\Sigma}^{\gamma}(\lambda,h)$, $\mathcal{SP}_{\Sigma}^{\gamma}(0,h)$ and $\mathcal{SP}_{\Sigma}^{\gamma}(1,h)$ becomes the familiar classes $\mathcal{SP}_{\Sigma}^{0}(\lambda,h)\equiv\mathcal{M}_{\Sigma}(\lambda,h)$, $\mathcal{SP}_{\Sigma}^{0}(0,h)\equiv\mathcal{S}_{\Sigma}^{*}(h)$ and $\mathcal{SP}_{\Sigma}^{0}(1,h)\equiv\mathcal{K}_{\Sigma}(h)$, respectively. These classes were studied by Ali et al. [1] for the superordinate function $h(z)=\phi(z)=1+B_1z+B_2z^2+\cdots$, $(B_1>0)$.

In particular, Corollaries 2.4 and 2.5 reduces to Theorem 3.2 and Corollary 3.3 of [8] (also see Theorem 3.1 of [2]) respectively. Further Corollary 2.6 reduces to Theorem 4.1 of [2]. Also see Theorem 3.1, Corollary 3.2 and Corollary 3.3 [6].

3 Application of Hohlov operator

For the functions $f_1,f_2\in\mathcal{A}$ and given by the series

$$f_1(z)=\sum_{n=0}^{\infty}a_nz^n \quad\text{and}\quad f_2(z)=\sum_{n=0}^{\infty}b_nz^n,\quad z\in\Delta,$$

the Hadamard product (or convolution) of f_1 and f_2 denoted by $f_1\star f_2$ is defined as

$$(f_1\star f_2)(z)=\sum_{n=0}^{\infty}a_nb_nz^n=(f_2\star f_1)(z),\quad z\in\Delta.$$

By using the Hadamard product (or convolution), Hohlov [4] introduced and studied the linear operator $I_c^{a,b}:\mathcal{A}\to\mathcal{A}$ defined by

$$I_c^{a,b}f(z)=z\,{}_2F_1(a,b;c;z)\star f(z),\quad f\in\mathcal{A},\quad z\in\Delta,$$

where ${}_2F_1(z)$ is the Gaussian hypergeometric function defined by

$${}_2F_1(z)={}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n,\quad a,b\in\mathbb{C},\quad c\in\mathbb{C}\setminus\{0,-1,-2,\cdots\},\tag{3.1}$$

and $(\kappa)_n$ the Pochhammer symbol or shifted factorial, written in terms of the gamma function Γ by

$$(\kappa)_n=\frac{\Gamma(\kappa+n)}{\Gamma(\kappa)}=\begin{cases}1, & n=0,\\ \kappa(\kappa+1)\cdots(\kappa+n-1), & n\in\mathbb{N}:=\{1,2,3,\cdots\}.\end{cases}$$

Note that ${}_2F_1(z)$ is symmetric in a and b and that the series (3.1) terminates if at least one of the numerator parameter a and b is zero or a negative integer.

For the function f of the form (1.1), we observe that

$$\mathcal{I}_c^{a,b} f(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n = z + \sum_{n=2}^{\infty} \Gamma_n a_n z^n, \quad z \in \Delta, \tag{3.2}$$

where

$$\Gamma_n = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}. \tag{3.3}$$

The three parameter family of operators $\mathcal{I}_c^{a,b}$ contains as a special cases most of the known linear integral or differential operators. In particular, if $b=1$ in (3.2), then $\mathcal{I}_c^{a,1}$ reduces to the Carlson-Shaffer operator a special case of Hohlov operator. Similarly, it is straight forward to show that Hohlov operator is also a generalization of Ruscheweyh and Bernardi operators.

Definition 3.1. Let $h : \Delta \to \mathbb{C}$ be a convex univalent function such that $h(0) = 1$ and $\Re(h(z)) > 0$, $z \in \Delta$. A function $f(z)$ is said to be in the class $\mathcal{SP}_{c,\Sigma}^{a,b}(\gamma,\lambda,h)$, if the following conditions are satisfied:

$$e^{i\gamma}\Big[(1-\lambda)\frac{z(\mathcal{I}_c^{a,b} f(z))'}{\mathcal{I}_c^{a,b} f(z)} + \lambda\Big(1 + \frac{z(\mathcal{I}_c^{a,b} f(z))''}{(\mathcal{I}_c^{a,b} f(z))'}\Big)\Big] \prec h(z)\cos\gamma + i\sin\gamma, \quad f \in \Sigma, \quad z \in \Delta, \tag{3.4}$$

and

$$e^{i\gamma}\Big[(1-\lambda)\frac{w(\mathcal{I}_c^{a,b} g(w))'}{\mathcal{I}_c^{a,b} g(w)} + \lambda\Big(1 + \frac{w(\mathcal{I}_c^{a,b} g(w))''}{(\mathcal{I}_c^{a,b} f(z))'}\Big)\Big] \prec h(w)\cos\gamma + i\sin\gamma, \quad w \in \Delta, \tag{3.5}$$

where $\gamma \in (-\pi/2, \pi/2)$, $\lambda \geq 0$ and $g = f^{-1}$.

First we obtain the coefficients $|a_2|$ and $|a_3|$ for the functions in the class $\mathcal{SP}_{c,\Sigma}^{a,b}(\gamma,\lambda,h)$.

Theorem 3.1. *Let f given by* (1.1) *be in the class $\mathcal{SP}_{c,\Sigma}^{a,b}(\gamma,\lambda,h)$. Then*

$$|a_2| \leq \sqrt{\frac{|B_1|\cos\gamma}{2(1+2\lambda)\Gamma_3 - (1+3\lambda)\Gamma_2^2}}, \tag{3.6}$$

and

$$|a_3| \leq |B_1|\cos\gamma\Big(\frac{1}{2(1+2\lambda)\Gamma_3} + \frac{|B_1|\cos\gamma}{(1+\lambda)^2\Gamma_2^2}\Big), \tag{3.7}$$

where $\gamma \in (-\pi/2, \pi/2)$ and $\lambda \geq 0$.

Proof. Let $f\in\mathcal{SP}^{a,b}_{c,\Sigma}(\gamma,\lambda,h)$ and $g=f^{-1}$. Then from (3.4) and (3.5), we have

$$e^{i\gamma}\Big[(1-\lambda)\frac{z(\mathcal{J}^{a,b}_c f(z))'}{\mathcal{J}^{a,b}_c f(z)}+\lambda\Big(1+\frac{z(\mathcal{J}^{a,b}_c f(z))''}{(\mathcal{J}^{a,b}_c f(z))'}\Big)\Big]=p(z)\cos\gamma+i\sin\gamma,\quad z\in\Delta, \tag{3.8}$$

and

$$e^{i\gamma}\Big[(1-\lambda)\frac{w(\mathcal{J}^{a,b}_c g(w))'}{\mathcal{J}^{a,b}_c g(w)}+\lambda\Big(1+\frac{w(\mathcal{J}^{a,b}_c g(w))''}{(\mathcal{J}^{a,b}_c f(z))'}\Big)\Big]=q(w)\cos\gamma+i\sin\gamma,\quad w\in\Delta, \tag{3.9}$$

where $p(z)\prec h(z)$ and $q(w)\prec h(w)$ and have the forms as in (2.5) and (2.6) respectively.

Now equating the coefficients in (3.8) and (3.9), we get

$$e^{i\gamma}(1+\lambda)\Gamma_2 a_2=p_1\cos\gamma, \tag{3.10a}$$

$$e^{i\gamma}[-(1+3\lambda)\Gamma_2^2a_2^2+2(1+2\lambda)\Gamma_3a_3]=p_2\cos\gamma, \tag{3.10b}$$

$$-e^{i\gamma}(1+\lambda)\Gamma_2a_2=q_1\cos\gamma, \tag{3.10c}$$

and

$$e^{i\gamma}[\{4(1+2\lambda)\Gamma_3-(1+3\lambda)\Gamma_2^2\}a_2^2-2(1+2\lambda)\Gamma_3a_3]=q_2\cos\gamma. \tag{3.11}$$

Proceeding as in the proof of Theorem 2.1 we can complete the proof of this theorem. □

Remark 3.1. For $a=c$ and $b=1$, we get $\Gamma_n=1$ whence Theorem 3.1 reduces to Theorem 2.1. Consequently various other interesting corollaries of our main result (Theorem 2.1) can be derived similarly.

4 Concluding remarks

On specializing the parameters γ,λ and suitable choices of A and B, as mentioned in Remark 1.1 to Remark 1.3 we can deduce analogous new subclasses of the class $\mathcal{SP}^{a,b}_{c,\Sigma}(\gamma,\lambda,A,B)$, denoted by $\mathcal{SP}^{a,b}_{c,\Sigma}(\gamma,0,A,B)$, $\mathcal{SP}^{a,b}_{c,\Sigma}(\gamma,1,A,B)$, $\mathcal{SP}^{a,b}_{c,\Sigma}(\gamma,\lambda,\alpha)$, $\mathcal{SP}^{a,b}_{c,\Sigma}(\gamma,0,\alpha)$ and $\mathcal{SP}^{a,b}_{c,\Sigma}(\gamma,1,\alpha)$. Various other interesting corollaries and consequences of our main results (which are asserted by Theorems 2.1 and 3.1 above) can be derived similarly hence we omit the details.

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