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Weighted Lipschitz Estimate for Commutator of Bochner-Riesz Operators on Weighted Morrey Spaces

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Abstract. In this paper, we will use a method of sharp maximal function approach to show the boundedness of commutator $[b,T_R^\delta]$ by Bochner-Riesz operators and the function b on weighted Morrey spaces $L^{p,\kappa}(\omega)$ under appropriate conditions on the weight ω, where b belongs to Lipschitz space or weighted Lipschitz space.

Key Words: Bochner-Riesz operator, weighted Morrey space, Lipschitz function.

AMS Subject Classifications: 42B25

1 Introduction and definitions

The Bochner-Riesz operator T_R^δ in $\mathbb{R}^n$ is defined in terms of Fourier transform by

$$(T_R^\delta f)^\wedge(\xi)=\Big(1-\frac{|\xi|^2}{R^2}\Big)_+^\delta f^\wedge(\xi),\quad R>0,$$

where $\hat{f}$ denotes the Fourier transform of f. And the maximal Bochner-Riesz operator is defined by

$$(T_*^\delta)(x)=\sup_{R>0}|(T_R^\delta)(x)|.$$

It is well known that $T_R^\delta=(f*\phi_{1/R})(x)$ is a convolution operator with the kernel $\phi_{1/R}$ [1], where

$$\phi(x)=\pi^{-\delta}\Gamma(\delta+1)|x|^{-(\frac{n}{2})+\delta}J_{\frac{n}{2}+\delta}(2\pi|x|),\quad \phi_{1/R}=R^n\cdot\phi(Rx)$$

and $J_\mu(t)$ is the Bessel function,

$$J_\mu(t)=\frac{(\frac{t}{2})^\mu}{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^1 e^{its}(1-s^2)^{\mu-\frac{1}{2}}ds.$$

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The inequality

$$|\phi(x)|+|\nabla\phi(x)|\leq\frac{C}{(1+|x|)^{\delta+\frac{n+1}{2}}} \tag{1.1}$$

holds for ϕ^δ.

Bochner-Riesz operators have been investigated by many authors. Lee [16], Tao [17] and many others studied the so-called Bochner-Riesz conjecture, i.e., if $p>1$ and

$$\delta>\delta(p):=\max\Big\{n\Big|\frac{1}{p}-\frac{1}{2}\Big|-\frac{1}{2},0\Big\},$$

then T_R^δ is bounded on L^p. On the other hand, there are also many results concerning the weighted inequalities for them, see [18,19].

Let b be a locally integrable function and T_R^δ the Bochner-Riesz operator T_R^δ, we define the commutator operator by Bochner-Riesz operator

$$[b,T_R^\delta]f(x)=b(x)T_R^\delta f(x)-T_R^\delta(bf)(x).$$

Wang [18] proved that and $T_R^\delta(\delta)(\delta>(n-1)/2)$ is a bounded operator on the weighted Morrey spaces $L^{p,\kappa}(\omega)$ for $1<p<\infty$ and $0<\kappa<1$. In 2013, we proved the boundedness of the commutator of Bochner-Riesz operators and weighted BMO functions.

In 2009, Komori and Shirai [2] defined Morrey space $L^{p,\kappa}(\omega)$ and investigated the boundedness of classical operators in harmonic analysis, that is, the Hardy-Littlewood maximal operators, Calderón-Zygmund operators,the fractional integral operators, etc.

First we shall define the weighted Morrey space.

Definition 1.1 (see [2])**.** Let $1\leq p<\infty$, $0<\kappa<1$ and ω be a weight. then the weighted Morrey space is defined by

$$L^{p,\kappa}(\omega)=\{f\in L^p_{loc}(\omega):\|f\|_{L^{p,\kappa}(\omega)}<\infty\},$$

where

$$\|f\|_{L^{p,\kappa}(\omega)}=\sup_Q\Big(\frac{1}{\omega(Q)^\kappa}\int_Q|f(x)|^p\omega(x)dx\Big)^{\frac{1}{p}},$$

and the supermum is taken over all balls Q in $\mathbb{R}^n$.

Let $1\leq p<\infty$, $0<\kappa<1$. For two weights u and v, a weighted Morrey space with two weights is defined by

$$L^{p,\kappa}(u,v)=\{f\in L^p_{loc}(u):\|f\|_{L^{p,\kappa}(u,v)}<\infty\},$$

where

$$\|f\|_{L^{p,\kappa}(u,v)}=\sup_Q\Big(\frac{1}{v(Q)^\kappa}\int_Q|f(x)|^pu(x)dx\Big)^{\frac{1}{p}},$$

and the supermum is taken over all balls Q in $\mathbb{R}^n$.

Definition 1.2. Let $0<\beta<1$, $1\le p<\infty$. A locally integrable function b belongs to Lipschitz space $Lip^p_\beta(\mathbb{R}^n)$, if

$$\|b\|_{Lip^p_\beta}=\sup_Q\frac{1}{|Q|^{\frac{\beta}{n}}}\Big(\frac{1}{|Q|}\int_Q|b(x)-b_Q|^p dx\Big)^{\frac{1}{p}}<\infty.$$

Let $0<\beta<1$, $1\le p<\infty$, $\omega\in A_\infty$. A locally integrable function b belongs to the weighted Lipschitz space $Lip^p_\beta(\omega)$, if

$$\|b\|_{Lip^p_\beta(\omega)}=\sup_Q\frac{1}{\omega(Q)^{\frac{\beta}{n}}}\Big(\frac{1}{\omega(Q)}\int_Q|b(x)-b_Q|^p\omega(x)^{1-p}dx\Big)^{\frac{1}{p}}<\infty.$$

When $p=1$, we denote them by $Lip_\beta(\mathbb{R}^n)$ and $Lip_\beta(\omega)$, respectively.

Definition 1.3 (see [3])**.** A weight function ω is in the Muckenhoupt class A_p with $1<p<\infty$ if there exists $C>1$ such that for any ball Q

$$\Big(\frac{1}{|Q|}\int_Q\omega(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q\omega(x)^{1-p'}dx\Big)^{p-1}\le C, \tag{1.2}$$

where $1/p+1/p'=1$ and the infimum of C satisfying the inequality (1.2) is denoted by $[\omega]_{A_p}$. We define $A_\infty=\bigcup_{1<p<\infty}A_p$.

When $p=1$, $\omega\in A_1$ if there exists $C>1$ such that for almost every x,

$$M\omega(x)\le C\omega(x) \tag{1.3}$$

and the infimum of C satisfying the inequality (1.3) is denoted by $[\omega]_{A_1}$.

Definition 1.4 (see [4])**.** A weight function ω belongs to $A_{p,q}$ for $1<p<q<\infty$ if there exists $C>1$ such that

$$\Big(\frac{1}{|Q|}\int_Q\omega(x)^q dx\Big)^{1/q}\Big(\frac{1}{|Q|}\int_Q\omega(x)^{-p'}dx\Big)^{1/p'}\le C, \tag{1.4}$$

where $1/p+1/p'=1$ and the infimum of C satisfying the inequality (1.4) is denoted by $[\omega]_{A_{p,q}}$.

When $p=1$, $\omega\in A_{1,q}$ with $1<q<\infty$ if there exists $C>1$ such that

$$\Big(\frac{1}{|Q|}\int_Q\omega(x)^q dx\Big)^{1/q}\Big(\operatorname*{esssup}_{x\in Q}\frac{1}{\omega(x)}\Big)\le C, \tag{1.5}$$

and the infimum of C satisfying the inequality (1.5) is denoted by $[\omega]_{A_{1,q}}$.

Definition 1.5. f is a locally integrable function.The Hardy-Littlewood maximal operators M and sharp maximal function $M^\sharp$ are defined by

$$M(f)(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_Q|f(y)|dy,$$
$$M^\sharp(f)(x)=\sup_{x\in Q}\frac{1}{|Q|}\int_Q|f(y)-f_Q|dy,$$

respectively, where

$$f_Q=\frac{1}{|Q|}\int_Q|f(y)|dy.$$

Following [14], for $0<\sigma<1$, we defined $\sigma-sharp$ maximal function $M^\sharp_\sigma(f)=M^\sharp(|f|^\sigma)^{1/\sigma}$. This is the transformation of $\sigma-sharp$ maximal functions by Fefferman and Stein [15]. And $M_\sigma(f)=M(|f|^\sigma)^{1/\sigma}$.

If $0<\beta<n$, $r\geq 1$, we defined the fractional maximal operator $M_{\beta,r}$

$$M_{\beta,r}(f)(x)=\sup_{x\in Q}\Big(\frac{1}{|Q|^{1-\beta r/n}}\int_Q|f(y)|^r dy\Big)^{1/r}.$$

Let ω be a weight. $M_{\beta,r,\omega}$ denotes the fractional maximal operator with respect to the measure $\omega(x)dx$ defined by

$$M_{\beta,r,\omega}(f)(x)=\sup_{x\in Q}\Big(\frac{1}{\omega(Q)^{1-\beta r/n}}\int_Q|f(y)|^r\omega(y)dy\Big)^{1/r}.$$

The supermum is taken over all balls Q and $x\in Q$.

Our main results are stated as follows.

Theorem 1.1. *Let* $b\in Lip_\beta(\mathbb{R}^n)$. *If* $0<\beta<1$, $1<p<n/\beta$, $1/s=1/p-\beta/n$, $0<\kappa<\min\{p/s,p\beta/n\}$, $\omega^s\in A_1$ *and* $\delta>(n+1)/2$, *then* $[b,T_R^\delta]$ *is bounded from* $L^{p,\kappa}(\omega^p,\omega^s)$ *to* $L^{s,\kappa s/p}(\omega^s)$.

Theorem 1.2. *Let* $b\in Lip_\beta(\omega)$, *if* $0<\beta<1$, $1<p<n/\beta$, $1/s=1/p-\beta/n$, $0<\kappa<p/s$, $\omega^{s/p}\in A_1$, $\delta>(n+1)/2$ *and*

$$r_\omega>\frac{1-\kappa}{p/s-\kappa},$$

then $[b,T_R^\delta]$ *is bounded from* $L^{p,\kappa}(\omega)$ *to* $L^{s,\kappa s/p}(\omega^{1-s},\omega)$, *where* r_ω *is the critical index of* ω *for reverse Hölder condition.*

Remark 1.1. The critical index of Bochner-Riesz operator is $\delta=(n-1)/2$. When $\delta\in((n-1)/2,(n+1)/2)$, we can not get the boundedness of these commutators from our method. Besides Theorem 1.2 is not the extend of Theorem 1.1.

Throughout this paper, C always means a positive constant independent of the main parameters involved, and may change from one occurrence to another.

2 Lemmas and the proof of Theorem 1.1

Lemma 2.1 (see [5]). *The following are true:*

(1) $A_p \subsetneqq A_q$, $1 \le p < q$.

(2) If $\omega \in A_p$, *then* $\omega^\alpha \in A_p$, $0 < \alpha < 1$.

(3) If $\omega \in RH_r$, *then* $\omega \in RH_s$, $1 < s < r$. *Moreover if* $\omega \in RH_r$, $r > 1$, *then* $\omega \in RH_{r+\varepsilon}$ *for the only one* $\varepsilon > 0$.

Lemma 2.2 (see [6]). *If* $\omega \in A_p$, $p \ge 1$, *then exists a constant* $C > 0$ *such that*

$$\omega(2Q) \le C\omega(Q),$$

for any ball Q.

More precisely, for all $\lambda > 1$ *we have*

$$\omega(\lambda Q) \le C\lambda^{np}\omega(Q),$$

where C *is a constant independent of* Q *and* λ.

Lemma 2.3 (see [6,7]). *If* $\omega \in A_p \bigcap RH_r$, $p \ge 1$ *and* $r > 1$, *then there exists* $C_1, C_2 > 0$ *such that for any ball* Q *and a measurable set* $E \subset Q$,

$$C_1\Big(\frac{|E|}{|Q|}\Big)^p \le \frac{\omega(E)}{\omega(Q)} \le C_2\Big(\frac{|E|}{|Q|}\Big)^{\frac{r-1}{r}}.$$

Lemma 2.4 (see [8]). *Assume* f *is a locally integrable function and* $\delta > (n-1)/2$, *the Bochner-Riesz maximal operator*

$$T_*^\delta(f)(x) = \sup_{R>0} |T_R^\delta(f)(x)|$$

is bounded on $L^p(\mathbb{R})$ *and of weak type* $(1,1)$.

Lemma 2.5 (see [9,10]). *(1) If* $0 < \beta < 1$, *then there exists a absolute constant* $C > 0$ *such that for* $1 \le p < \infty$,

$$\|b\|_{Lip_\beta^p} \le C\|b\|_{Lip_\beta},$$

(2) If $0 < \beta < 1$ *and* $\omega \in A_1$, *then there exists a absolute constant* $C > 0$ *such that for* $1 \le p < \infty$,

$$\|b\|_{Lip_\beta^p}(\omega) \le C\|b\|_{Lip_\beta}(\omega).$$

Lemma 2.6 (see [11]). *If* $1 < p < \infty$, $\omega \in A_p$, *then there exists a absolute constant* $C > 0$ *related to the dimension* n

$$\big\|T_*^{\frac{n-1}{2}}(f)\big\|_{L^p_\omega} \le C\|f\|_{L^p_\omega}.$$

Lemma 2.7 (see [2]). *If* $0 < \beta < n$, $1 < p < n/\beta$, $1/s = 1/p - \beta/n$, $0 < \kappa < p/s$, *and* $\omega \in A_{p,s}$, *then* $M_{\beta,1}$ *is bounded from* $L^{p,\kappa}(\omega^p, \omega^s)$ *to* $L^{s,\kappa s/p}(\omega^s)$.

Lemma 2.8 (see [12])**.** *If* $0<\beta<n$*,* $1<p<n/\beta$*,* $1/s=1/p-\beta/n$ *and* $\omega^s\in A_1$*, then for* $0<\kappa<p/s$ *and* $1<r<p$*,*

$$\|M_{\beta,r}(f)\|_{L^{s,\kappa s/p}(\omega^s)}\le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^s)}.$$

Lemma 2.9. *Let* T_R^δ *be the maximal Bochner-Riesz operator. If* $0<\beta<n$*,* $1<p<n/\beta$*,* $1/s=1/p-\beta/n$ $\omega^s\in A_1$*,* $\delta>(n-1)/2$*, and* $0<\kappa<\beta p/n$*, we have*

$$\|T_*^\delta(f)\|_{L^{p,\kappa}(\omega^p,\omega^s)}\le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^s)}.$$

Proof. Fix a ball $B=B(x,r)\subseteq\mathbb{R}^n$. We decompose $f=f_1+f_2$, where $f_1=f\chi_{2B}$. We have

$$\begin{aligned}&\sup_{R>0}\frac{1}{\omega^s(B)^{\kappa/p}}\Big(\int_B|T_R^\delta f(x)|^p\omega(x)^pdx\Big)^{\frac{1}{p}}\\ \le&\sup_{R>0}\frac{1}{\omega^s(B)^{\kappa/p}}\Big(\int_B|T_R^\delta f_1(x)|^p\omega(x)^pdx\Big)^{\frac{1}{p}}+\sup_{R>0}\frac{1}{\omega^s(B)^{\kappa/p}}\Big(\int_B|T_R^\delta f_2(x)|^p\omega(x)^pdx\Big)^{\frac{1}{p}}\\ =:&J_1+J_2.\end{aligned}$$

It is easy to estimate the term J_1. When $\delta>(n-1)/2$, using the fact

$$T_*^\delta(f)(x)\le CM(f)(x),$$

see [2]. by Lemma 2.6, we have T_*^δ is bounded on L_ω^p for $\delta\ge(n-1)/2$.

If $\omega^s\in A_1$, $1<p<s$, then $\omega^p\in A_1$, hence $\omega^p\in A_p$. Using the L_ω^p-boundedness of T_*^δ and Lemma 2.2, we obtain

$$\begin{aligned}J_1\le& C\frac{1}{\omega^s(B)^{\kappa/p}}\Big(\int_{2B}|f(x)|^p\omega(x)^pdx\Big)^{\frac{1}{p}}\le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^s)}\frac{\omega^s(2B)^{\kappa/p}}{\omega^s(B)^{\kappa/p}}\\ \le& C\|f\|_{L^{p,\kappa}(\omega^p,\omega^s)}.\end{aligned}$$

Next we estimate J_2. Observe that $x\in B$ and $y\in(2B)^c$, we get

$$\begin{aligned}|T_R^\delta f_2(x)|\le&|(f_2*\phi_{\frac{1}{R}})(x)|\le C\int_{(2B)^c}|\phi_{\frac{1}{R}}(x-y)f(y)|dy\\ \le& C\int_{(2B)^c}|R^n\phi(R(x-y))f(y)|dy.\end{aligned}$$

For $\delta\ge(n-1)/2$, using (1.1), we obtain

$$\begin{aligned}|T_R^\delta f_2(x)|\le& C\int_{(2B)^c}\Big|\frac{R^n}{(R(x-y))^n}\Big||f(y)|dy\le C\int_{(2B)^c}\frac{|f(y)|}{|y-x_0|^n}dy\\ \le& C\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)|dy\\ \le& C\sum_{j=1}^\infty\frac{1}{|2^{j+1}B|}|2^{j+1}B|[\omega^p(2^{j+1}B)]^{-\frac{1}{p}}\Big(\int_{2^{j+1}B}|f(y)|^p\omega(y)^pdy\Big)^{\frac{1}{p}}\\ \le& C\|f\|_{L^{p,\kappa}(\omega^p,\omega^s)}\sum_{j=1}^\infty\frac{\omega^s(2^{j+1}B)^{\kappa/p}}{\omega^p(2^{j+1}B)^{1/p}}.\end{aligned}$$

Then

$$J_2 \le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^s)} \sum_{j=1}^{\infty} \frac{[\omega^p(B)]^{1/p}}{[\omega^p(2^{j+1}B)]^{1/p}} \cdot \frac{[\omega^s(2^{j+1}B)]^{\kappa/p}}{[\omega^s(B)]^{\kappa/p}}.$$

Using Lemma 2.3, we get

$$C\frac{|B|}{|2^{j+1}B|} \le \frac{\omega^s(B)}{\omega^s(2^{j+1}B)}.$$

We use the fact that if $s/p>1$, and $(\omega^p)^{s/p}\in A_1$, then $\omega^p \in RH_{s/p}$. Using Lemma 2.3, we have

$$\frac{\omega^p(B)}{\omega^p(2^{j+1}B)} \le C\Big(\frac{|B|}{|2^{j+1}B|}\Big)^{1-\frac{p}{s}}.$$

Hence

$$J_2 \le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^s)} \sum_{j=1}^{\infty} (2^{jn})^{\frac{\kappa}{p}-\frac{\beta}{n}} \le C\|f\|_{L^{p,\kappa}(\omega^p,\omega^s)}.$$

We use the fact $\kappa<\beta p/n$ in the last inequality and then combining J_1 and J_2, taking the supermum over all balls $B\subseteq\mathbb{R}^n$, it completes the proof of Lemma 2.9. □

Lemma 2.10. *If* $0<\sigma<1$, $0<\beta<1$, $\delta>(n+1)/2$, *and* $b\in Lip_\beta(\mathbb{R}^n)$, *for all* $r>1$ *and* $x\in\mathbb{R}^n$, *we have*

$$M^\sharp_\sigma([b,T^\delta_R]f)(x) \le C\|b\|_{Lip_\beta}(M_{\beta,r}(T^\delta_R f)(x)+M_{\beta,r}(f)(x)+M_{\beta,1}(f)(x)).$$

Proof. Fix a ball $B=B(x_0,r_B)$, where $B(x_0,r_B)$ denotes the ball with the center x_0 and radius r_B, and decompose $f=f_1+f_2$ with $f_1=f\chi_{2B}$. We observe that

$$[b,T^\delta_R]f(x)=(b(x)-b_{2B})T^\delta_R f(x)-T^\delta_R((b-b_{2B})f)(x).$$

For $0<\sigma<1$, we have

$$\begin{aligned}
&\Big(\frac{1}{|B|}\int_B \big||[b,T^\delta_R]f(y)|^\sigma-|C|^\sigma\big|dy\Big)^{\frac{1}{\sigma}}\\
\le&\Big(\frac{1}{|B|}\int_B |[b,T^\delta_R]f(y)-C|^\sigma dy\Big)^{\frac{1}{\sigma}}\\
\le&C\Big(\frac{1}{|B|}\int_B |(b(y)-b_{2B})T^\delta_R f(y)|^\sigma dy\Big)^{\frac{1}{\sigma}}+C\Big(\frac{1}{|B|}\int_B |T^\delta_R((b(y)-b_{2B})f_1)(y)|^\sigma dy\Big)^{\frac{1}{\sigma}}\\
&\quad +C\Big(\frac{1}{|B|}\int_B |T^\delta_R((b(y)-b_{2B})f_2)(y)-C|^\sigma dy\Big)^{\frac{1}{\sigma}}\\
\triangleq& D_1+D_2+D_3.
\end{aligned}$$

For the term D_1, using Hölder inequality and Lemma 2.5, we can get

$$\begin{aligned}
D_1 \leq & C\frac{1}{|B|}\int_B |(b(y)-b_{2B})T_R^\delta f(y)|dy \\
\leq & C\frac{1}{|B|}\Big(\int_B |(b(y)-b_{2B}|^{r'}dy\Big)^{\frac{1}{r'}}\Big(\int_B |T_R^\delta f(y)|^r dy\Big)^{\frac{1}{r}} \\
\leq & C\|b\|_{Lip_\beta}\frac{1}{|B|}|B|^{\frac{\beta}{n}}|B|^{\frac{1}{r'}}\Big(\int_B |T_R^\delta f(y)|^r dy\Big)^{\frac{1}{r}} \\
\leq & C\|b\|_{Lip_\beta}\Big(\frac{1}{|B|^{1-\frac{\beta r}{n}}}\int_B |T_R^\delta f(y)|^r dy\Big)^{\frac{1}{r}} \\
\leq & C\|b\|_{Lip_\beta}M_{\beta,r}(T_R^\delta f)(x).
\end{aligned}$$

For the term D_2, using Kolmogorov inequality (see [13, pp. 12]), Hölder inequality and (1.1), we have

$$\begin{aligned}
D_2 \leq & C\frac{1}{|B|}\int_B |(b(y)-b_{2B})T_R^\delta f(y)|dy \\
\leq & C\frac{1}{|B|}\Big(\int_B |(b(y)-b_{2B}|^{r'}dy\Big)^{\frac{1}{r'}}\Big(\int_B |T_R^\delta f(y)|^r dy\Big)^{\frac{1}{r}} \\
\leq & C\|b\|_{Lip_\beta}\frac{1}{|B|}|B|^{\frac{\beta}{n}}|B|^{\frac{1}{r'}}\Big(\int_B |T_R^\delta f(y)|^r dy\Big)^{\frac{1}{r}} \\
\leq & C\|b\|_{Lip_\beta}\Big(\frac{1}{|B|^{1-\frac{\beta r}{n}}}\int_B |T_R^\delta f(y)|^r dy\Big)^{\frac{1}{r}} \\
\leq & C\|b\|_{Lip_\beta}M_{\beta,r}(T_R^\delta f)(x).
\end{aligned}$$

For the last term, we fix $C=T_R^\delta((b-b_{2B})f_2)(x_0)$. If $0<1/R<r_B$, $y\in B$, and $z\in(2B)^c$, using Hölder inequality and (1.1), we obtain

$$\begin{aligned}
D_3 \leq & C\frac{1}{|B|}\int_B |T_R^\delta((b-b_{2B})f_2)(y)-T_R^\delta((b-b_{2B})f_2)(x_0)|dy \\
\leq & C\frac{1}{|B|}\int_B \Big|\int_{\mathbb{R}^n}\phi_{\frac{1}{R}}(y-z)((b(z)-b_{2B})f_2)(z)dz-\int_{\mathbb{R}^n}\phi_{\frac{1}{R}}(x_0-z)((b(z)-b_{2B})f_2(z))dz\Big|dy \\
\leq & C\frac{1}{|B|}\int_B\int_{\mathbb{R}^n\backslash 2B} R^n|\phi(R(y-z)-\phi(R(x_0-z))||b(z)-b_{2B}||f(z)|dzdy \\
\leq & C\frac{1}{|B|}\int_B\int_{\mathbb{R}^n\backslash 2B} R^n(R|x_0-y|)\frac{1}{(1+R|x_0-z|)^{\delta+\frac{n+1}{2}}}|b(z)-b_{2B}||f(z)|dzdy \\
\leq & Cr_B^{\delta-\frac{n+1}{2}}\Big(\frac{1}{Rr_B}\Big)^{\delta-\frac{n+1}{2}}\sum_{j=1}^\infty\frac{1}{|B|}\int_B\int_{2^{j+1}B\backslash 2^jB}\frac{1}{|x_0-z|^{\delta+\frac{n+1}{2}}}|b(z)-b_{2B}||f(z)|dzdy.
\end{aligned}$$

For $\delta > (n+1)/2$, we have

$$\begin{aligned}
D_3 \leq & C\sum_{j=1}^{\infty} r_B^{\delta-\frac{n+1}{2}} \frac{1}{|B|}\int_B \int_{2^{j+1}B\setminus 2^jB} \frac{1}{|x_0-z|^{\delta+\frac{n+1}{2}}} |b(z)-b_{2B}||f(z)|dzdy \\
\leq & C\sum_{j=1}^{\infty} r_B^{\delta-\frac{n+1}{2}} \int_{2^{j+1}B\setminus 2^jB} \frac{1}{|x_0-z|^{\delta+\frac{n+1}{2}}} |b(z)-b_{2B}||f(z)|dz \\
\leq & C\sum_{j=1}^{\infty} r_B^{\delta-\frac{n+1}{2}} \frac{1}{|2^{j+1}B|}\int_{2^{j+1}B} \frac{1}{|2^j r_B|^{\delta-\frac{n-1}{2}}} |b(z)-b_{2B}||f(z)|dz \\
\leq & C\sum_{j=1}^{\infty} 2^{j(\frac{n-1}{2}-\delta)} \frac{1}{|2^{j+1}B|}\int_{2^{j+1}B} |b(z)-b_{2B}||f(z)|dz \\
\leq & C\sum_{j=1}^{\infty} 2^{j(\frac{n-1}{2}-\delta)} \frac{1}{|2^{j+1}B|}\int_{2^{j+1}B} |b(z)-b_{2^{j+1}B}||f(z)|dz \\
& +C\sum_{j=1}^{\infty} 2^{j(\frac{n-1}{2}-\delta)} |b_{2^{j+1}B}-b_{2B}| \frac{1}{|2^{j+1}B|}\int_{2^{j+1}B} |f(z)|dz \\
\triangleq & D_{31}+D_{32}.
\end{aligned}$$

To estimate the term D_{31}, using Hölder inequality,we get

$$\begin{aligned}
D_{31} \leq & C\sum_{j=1}^{\infty} 2^{j(\frac{n-1}{2}-\delta)} \frac{1}{|2^{j+1}B|}\Big(\int_{2^{j+1}B} |b(z)-b_{2^{j+1}B}|^{r'}dz\Big)^{\frac{1}{r'}}\Big(\int_{2^{j+1}B} |f(z)|^r dz\Big)^{\frac{1}{r}} \\
\leq & C\sum_{j=1}^{\infty} 2^{j(\frac{n-1}{2}-\delta)} \frac{1}{|2^{j+1}B|}\|b\|_{Lip_\beta}|2^{j+1}B|^{\frac{\beta}{n}}|2^{j+1}B|^{\frac{1}{r'}}\Big(\int_{2^{j+1}B} |f(z)|^r dz\Big)^{\frac{1}{r}} \\
\leq & C\sum_{j=1}^{\infty} 2^{j(\frac{n-1}{2}-\delta)} \|b\|_{Lip_\beta}\Big(\frac{1}{|2^{j+1}B|^{1-\frac{\beta r}{n}}}\int_{2^{j+1}B} |f(z)|^r dz\Big)^{\frac{1}{r}} \\
\leq & C\sum_{j=1}^{\infty} 2^{j(\frac{n-1}{2}-\delta)} \|b\|_{Lip_\beta} M_{\beta,r}(f)(x) \\
\leq & C|b\|_{Lip_\beta} M_{\beta,r}(f)(x).
\end{aligned}$$

For the term D_{32}, using Lemma 2.5, then $|b_{2^{j+1}B}-b_{2B}| \leq Cj\|b\|_{Lip_\beta} j|2^{j+1}|^{\beta/n}$.

Hence we have

$$\begin{aligned}
D_{32} \leq & C\sum_{j=1}^{\infty} 2^{j(\frac{n-1}{2}-\delta)} j\|b\|_{Lip_\beta} \frac{1}{|2^{j+1}B|^{1-\frac{\beta}{n}}}\int_{2^{j+1}B} |f(z)|dz \\
\leq & C\sum_{j=1}^{\infty} j2^{j(\frac{n-1}{2}-\delta)} \|b\|_{Lip_\beta} M_{\beta,1}(f)(x) \\
\leq & C\|b\|_{Lip_\beta} M_{\beta,1}(f)(x),
\end{aligned}$$

where $1/r+1/r'=1$.

Therefore we have

$$D_3\leq C\|b\|_{Lip_\beta}(M_{\beta,r}(f)(x)+M_{\beta,1}(f)(x)).$$

If $1/R\geq r_B$, fix δ_0 such that $(n+1)/2<\delta_0<\delta$. It is seen from the first condition that

$$\begin{aligned}D_3\leq& Cr_B^{\delta_0-\frac{n+1}{2}}\Big(\frac{1}{Rr_B}\Big)^{\frac{n+1}{2}-\delta_0}\sum_{j=1}^{\infty}\frac{1}{|B|}\int_B\int_{2^{j+1}B\backslash 2^jB}\frac{1}{|x_0-z|^{\delta+\frac{n+1}{2}}}|b(z)-b_{2B}||f(z)|dzdy\\ \leq& C\sum_{j=1}^{\infty}2^{j(\frac{n-1}{2}-\delta_0)}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}||f(z)|dz\\ &+C\sum_{j=1}^{\infty}2^{j(\frac{n-1}{2}-\delta_0)}|b_{2^{j+1}B}-b_{2B}|\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(z)|dz\\ \leq& C\|b\|_{Lip_\beta}(M_{\beta,r}(f)(x)+M_{\beta,1}(f)(x))\end{aligned}$$

by Hölder inequality. This completes the proof of Lemma 2.10. □

Proof of Theorem 1.1. If $0<\beta<1$ and $1<p<n/\beta$, we can take a real number r such that $1<r<p$. Using Lemma 2.10, we can get

$$\begin{aligned}&\|[b,T_R^\delta]f\|_{L^{s,\frac{\kappa s}{p}}(\omega^s)}\leq C\|M_\sigma^\sharp([b,T_R^\delta]f)\|_{L^{s,\frac{\kappa s}{p}}(\omega^s)}\\ \leq& C\|b\|_{Lip_\beta}\Big(\|M_{\beta,r}(T_R^\delta f)\|_{L^{s,\frac{\kappa s}{p}}(\omega^s)}+\|M_{\beta,r}(f)\|_{L^{s,\frac{\kappa s}{p}}(\omega^s)}+\|M_{\beta,1}(f)\|_{L^{s,\frac{\kappa s}{p}}(\omega^s)}\Big).\end{aligned}$$

We use the fact that if $\omega^s\in A_1$, then $\omega\in A_{p,s}$. For $0<\kappa<\min\{p/s,p\beta/n\}$, using Lemmas 2.7, 2.8 and 2.9, we obtain

$$\begin{aligned}&\|[b,T_R^\delta]f\|_{L^{s,\frac{\kappa s}{p}}(\omega^s)}\leq C\|b\|_{Lip_\beta}(\|T_R^\delta\|_{L^{p,\kappa}(\omega^p,\omega^s)}+\|f\|_{L^{p,\kappa}(\omega^p,\omega^s)})\\ \leq& C\|b\|_{Lip_\beta}\|f\|_{L^{p,\kappa}(\omega^p,\omega^s)}.\end{aligned}$$

This completes the proof of Theorem 1.1. □

3 Proof of Theorem 1.2

Lemma 3.1 (see [12]). *If $0<\beta<n$, $1<p<n/\beta$, $1/s=1/p-\beta/n$ and $\omega\in A_\infty$, for $0<\kappa<p/s$, we have*

$$\|M_{\beta,\omega}(f)\|_{L^{s,\frac{\kappa s}{p}}(\omega)}\leq C\|f\|_{L^{p,\kappa}(\omega)}.$$

Lemma 3.2 (see [12]). *If $0<\beta<n$, $1<p<n/\beta$, $1/s=1/p-\beta/n$, $0<\kappa<p/s$ and $\omega\in A_\infty$, then for $1<r<p$, we have*

$$\|M_{\beta,r,\omega}(f)\|_{L^{s,\frac{\kappa s}{p}}(\omega)}\leq C\|f\|_{L^{p,\kappa}(\omega)}.$$

Lemma 3.3 (see [12]). *If* $0<\beta<n$, $1<p<n/\beta$, $1/s=1/p-\beta/n$ *and* $\omega^{s/p}\in A_1$. *For* $0<\kappa<p/s$ *and* $r_\omega>(1-\kappa)/(p/s-\kappa)$, *then*

$$\|M_{\beta,1}(f)\|_{L^{s,\frac{\kappa s}{p}}(\omega)}\le C\|f\|_{L^{p,\kappa}(\omega)}.$$

Lemma 3.4. *If* $0<\sigma<1$, $\omega\in A_1$, $0<\beta<1$, $b\in Lip_\beta(\omega)$, *and* $\delta>(n+1)/2$, *then for all* $r>1$ *and* $x\in\mathbb{R}^n$, *we have*

$$\begin{aligned}M^{\sharp}_{\sigma}([b,T^\delta_R]f)(x)\le &C\|b\|_{Lip_\beta(\omega)}\big(\omega(x)M_{\beta,r,\omega}(T^\delta_R)(x)\\&+\omega(x)M_{\beta,r,\omega}(f)(x)+\omega(x)^{1+\frac{\beta}{n}}M_{\beta,1}(f)(x)\big).\end{aligned}$$

Proof. The process of proof is similar to Lemma 2.10. If $\omega\in A_1$, using Hölder inequality and Lemma 2.5, we have

$$\begin{aligned}D_1\le &C\frac{1}{|B|}\int_B|(b(y)-b_{2B})T^\delta_Rf(y)|dy\\\le &C\frac{1}{|B|}\Big(\int_B|(b(y)-b_{2B}|^{r'}\omega^{1-r'}dy\Big)^{\frac{1}{r'}}\Big(\int_B|T^\delta_Rf(y)|^r\omega(y)dy\Big)^{\frac{1}{r}}\\\le &C\|b\|_{Lip_\beta}\frac{\omega(B)}{|B|}\Big(\frac{1}{\omega(B)^{1-\frac{\beta r}{n}}}\int_B|T^\delta_Rf(y)|^r\omega(y)dy\Big)^{\frac{1}{r}}\\\le &C\|b\|_{Lip_\beta}\omega(x)M_{\beta,r,\omega}(T^\delta_Rf)(x).\end{aligned}$$

Using Kolmogorov inequality, Hölder inequality and Lemma 2.5, we have

$$D_2\le C\frac{1}{|B|}\int_{2B}|(b(y)-b_{2B})f(y)|dy\le C\|b\|_{Lip_\beta}\omega(x)M_{\beta,r,\omega}(f)(x).$$

According to the proof of Lemma 3.4, we have

$$D_3\le D_{31}+D_{32},$$

where

$$D_{31}\le C\sum_{j=1}^{\infty}2^{j(\frac{n-1}{2}-\delta)}\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|b(z)-b_{2^{j+1}B}||f(z)|dz.$$

And

$$D_{32}\le C\sum_{j=1}^{\infty}2^{j(\frac{n-1}{2}-\delta)}|b_{2^{j+1}B}-b_{2B}|\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(z)|dz.$$

We have

$$D_{31}\le C\|b\|_{Lip_\beta(\omega)}\omega(x)M_{\beta,r,\omega}(f)(x).$$

If $\omega\in A_1$, using Lemma 2.5, we can get

$$|b_{2^{j+1}B}-b_{2B}|\le C\|b\|_{Lip_\beta(\omega)}j\omega(x)\omega(2^{j+1}B)^{\frac{\beta}{n}}.$$

Hence

$$\begin{aligned}D_{32} &\le C\sum_{j=1}^{\infty}2^{j(\frac{n-1}{2}-\delta)}j\|b\|_{Lip_\beta(\omega)}\frac{\omega(x)\omega(2^{j+1}B)^{\beta/n}}{|2^{j+1}B|^{1-\beta/n}}\int_{2^{j+1}B}|f(z)|dz\\ &\le C\sum_{j=1}^{\infty}j2^{j(\frac{n-1}{2}-\delta)}\|b\|_{Lip_\beta(\omega)}M_{\beta,1}(f)(x)\omega(x)^{1+\beta/n}\\ &\le C\|b\|_{Lip_\beta(\omega)}M_{\beta,1}(f)(x)\omega(x)^{1+\beta/n}.\end{aligned}$$

Therefore we can get

$$D_3 \le C\|b\|_{Lip_\beta(\omega)}\omega(x)^{1+\beta/n}\big(M_{\beta,r,\omega}(f)(x)+M_{\beta,1}(f)(x)\big).$$

This completes the proof of Lemma 3.4. □

Proof of the Theorem 1.2. If $0<\beta<1$ and $1<p<n/\beta$, we can take a real number r such that $1<r<p$. Using Lemma 3.4, we can get

$$\begin{aligned}&\|[b,T_R^\delta]f\|_{L^{s,\frac{\kappa s}{p}}(\omega^{1-s},\omega)} \le C\|M_\sigma^\sharp([b,T_R^\delta]f)\|_{L^{s,\frac{\kappa s}{p}}(\omega^s)}\\ \le& C\|b\|_{Lip_\beta(\omega)}\Big(\|\omega(\cdot)M_{\beta,r,\omega}(T_R^\delta f)\|_{L^{s,\frac{\kappa s}{p}}(\omega^s)}+\|\omega(\cdot)M_{\beta,r,\omega}(f)\|_{L^{s,\frac{\kappa s}{p}}(\omega^s)}\\ &+\omega(\cdot)\|M_{\beta,1,\omega}(f)\|_{L^{s,\frac{\kappa s}{p}}(\omega^s)}\Big)\\ \le& C\|b\|_{Lip_\beta(\omega)}\big(\|T_R^\delta f\|_{L^{p,\kappa}(\omega^p,\omega^s)}+\|f\|_{L^{p,\kappa}(\omega^p,\omega^s)}\big)\\ \le& C\|b\|_{Lip_\beta(\omega)}\|f\|_{L^{p,\kappa}(\omega^p,\omega^s)}.\end{aligned}$$

This completes the proof of Theorem 1.2. □

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