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A Local Property of Hausdorff Centered Measure of Self-Similar Sets

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Abstract. We analyze the local behavior of the Hausdorff centered measure for self-similar sets. If E is a self-similar set satisfying the open set condition, then

$$C^s(E\cap B(x,r))\leq(2r)^s$$

for all $x\in E$ and $r>0$, where C^s denotes the s-dimensional Hausdorff centered measure. The above inequality is used to obtain the upper bound of the Hausdorff centered measure. As the applications of above inequality, We obtained the upper bound of the Hausdorff centered measure for some self-similar sets with Hausdorff dimension equal to 1, and prove that the upper bound reach the exact Hausdorff centered measure.

Key Words: Hausdorff centered measure, Hausdorff measure, self-similar sets.

AMS Subject Classifications: 28A78, 28A80

1 Introduction

Measure and dimension are basic concepts in fractals. Computing and estimating the dimension and measure of the fractal sets are active problems in recent years. The basic and important example of fractal measures are the Hausdorff measure H^s. Let $s\geq 0$, $\delta>0$, $E\subset R^n$, the s-dimensional Hausdorff measure of E is defined by

$$H^s(E)=\lim_{\delta\to 0}H^s_\delta(E)=\sup_{\delta>0}H^s_\delta(E), \tag{1.1}$$

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where $H^s_\delta(E)=\inf\{\sum_{i>0}|U_i|^s:E\subset\bigcup_{i>0}U_i,\ 0<|U_i|\le\delta\}$, with the infimum taken over all δ-covering of E by countable sets $\{U_i\}_{i>0}$.

The Hausdorff dimension of E is defined as

$$dim_H(E)=\sup\{s\ge 0:H^s(E)=\infty\}=\inf\{s\ge 0:H^s(E)=0\}.$$

Another two kinds of measures, the packing measure and the Hausdorff centered measure, have been used by comparison with the Hausdorff measure to study the irregularity of sets in R^n. The packing measure P^s where introduced by Tricot [13] and Taylor and Tricot [12]. In [10], Saint Raymond and Tricot, introduced the Hausdorff centered measure C^s.

We first recall the definition of the Hausdorff centered measure for the use of the paper.

Let $E\subset R^n$, $s\ge 0$, $\delta>0$, the s-dimensional Hausdorff centered measure of E is defined as

$$C^s(E)=\sup\{C^s_0(F):F\subset E\}, \tag{1.2}$$

where $C^s_0(F)=\lim_{\delta\to 0}C^s_\delta(F)$, and $C^s_\delta(F)=\inf\{\sum_{i>0}(2r_i)^s\}$, the infimum taken over all δ-covering by countable ball collection $\{B(x_i,r_i)\}_{i>0}$ with center $x_i\in F$. We call $C^s_0(F)$ the s-dimensional Hausdorff centered pre-measure of F. Notice that $C^s_0(\cdot)$ is not a measure since it is not monotone. Thus we define a measure $C^s(\cdot)$ by (1.2). The Hausdorff centered dimension of E is deduced by

$$dim_C(E)=\sup\{s\ge 0:C^s(E)=\infty\}=\inf\{s\ge 0:C^s(E)=0\}.$$

The Hausdorff measure and the packing measure have been extensively studied in the later years. In self-similar setting, there are some results in computation and theory. For example, Zhou [14] obtained the following inequality for self-similar sets

$$H^s(E\cap U)\le|U|^s, \tag{1.3}$$

where E is a self-similar set satisfies the open set condition, U is a set intersects with E. The similar results for the packing measure are also obtained in [5]. Moreover, There are some computable results for the Hausdorff and packing measures (see for example, [1–4]).

Much less is known about the Hausdorff centered measure, although it has some computable results recently (see [7–9, 15, 16]). As the Hausdorff centered measure differs by a constant factor from the Hausdorff measure, it may be used in the computation of the Hausdorff dimension.

In [6], Marta Liorente and Manuel Moran showed that if E is a self-simllar set satisfying the open set condition, then $C^s_0(E)=C^s(E)$, making unnecessary the step (1.2) in its definition above.

In this paper, we should be desirable to extend the inequality (1.3) for the Hausdorff measure to the Hausdorff centered measure. By the results obtained in [6], we have

$$C^s(E\cap B(x,r))\leq (2r)^s, \tag{1.4}$$

for any $x\in E$ and $r>0$, where E is a self-similar set satisfies the open set condition, and $B(x,r)$ is a ball with center $x\in E$.

The early researches of computation of the exact measure values for self-similar sets mainly concerned with the one with the dimension is no more than 1. The common difficulties for these measures is that it is hard to get the exact measure value when the set with dimension greater than 1. So far, there is no any result of exact value of the Hausdorff measure for nontrivial self-similar sets with dimension greater than 1, but there exist an exception to the packing measure. In [2], Jia and Zhou etc. obtained the exact packing measure for a self-similar with dimension $dim_H(E)=\log_3 4>1$. It is natural asked that how about the results for the Hausdorff centered measure? From the results have been known now, there is no relative results for Hausdorff centered measure, even through the set has dimension $dim_H(E)=1$.

As the application of the inequality (1.4) above, we obtained the upper bound of the Hausdorff centered measure for some self-similar sets with their dimensions 1. By the relations between the Hausdorff measure and the Hausdorff centered measure, we proved that the upper bound reached the exact Hausdorff center measure.

This paper is organized as follows. In Section 2 we review the needed facts and notations, and give the main results of the paper. In Section 3, we construct some nontrivial self-similar sets with dimension equal to 1, which the exact Hausdorff centered measure can be determined.

2 General facts and main theorem

We give some basic properties of above measure and dimension, these properties can be found in [3, 4, 10, 11].

Lemma 2.1. *Let $s\geq 0$, then*

(i) $H^s(\cdot)$, $P^s(\cdot)$ and $C^s(\cdot)$ are Borel regular measures.

(ii) For any set $A\subset R^n$, $H^s(A)\leq C^s(A)\leq P^s(A)$, and $2^{-s}C^s(A)\leq H^s(A)\leq C^s(A)$.

(iii) For any set $A\subset R^n$, $dim_H(A)=dim_C(A)\leq dim_P(A)$, where $dim_H(A)$, $dim_C(A)$ and $dim_P(A)$, denote the Hausdorff dimension, the Hausdorff centered dimension and the packing dimension of A, respectively.

Lemma 2.2 (see [4]). *Let $s\geq 0$, E be any subset of R^n and Π be any subspace of R^n. Then*

$$H^s(proj_\Pi E)\leq H^s(E), \tag{2.1}$$

where $proj_\Pi$ denotes the orthogonal projection from R^n onto Π.

The above Lemma and Lemma 2.1 imply that

Corollary 2.1. We have

$$H^s(proj_{\Pi}E) \le H^s(E) \le C^s(E). \tag{2.2}$$

From the computability point of view, most peoples pay attention to the self-similar setting. Recall that the definition of the self-similar sets.

Let $\Phi=\{f_1,f_2,\cdots,f_m\}$ be an iterated function system(IFS), where f_i is a contracting similitude of R^n. The self-similar set E generated by Φ is the unique non-empty compact set which satisfies

$$E=\bigcup_{i=1}^{m} f_i(E).$$

We assume that the following conditions:

(i) Φ satisfies the open condition, if there exists a bound open set $U\subset R^n$ such that $\bigcup_{i=1}^m f_i(U)\subset U$ and $f_i(U)\cap f_j(U)=\emptyset$ for $i\neq j$, $i,j=1,2,\cdots,m$.

(ii) Φ satisfies the strong separation condition, if $f_i(E)\cap f_j(E)=\emptyset$ for $i\neq j$, $i,j=1,2,\cdots,m$.

If Φ satisfies the strong separation condition, Φ also satisfies the open set condition. Under the open set condition, we have the following two concludes.

(i) $dim_H(E)=dim_C(E)=dim_p(E)=s$, s is determined by $\sum_{i=1}^m \lambda_i^s=1$, where λ_i is the contraction ratio of the similarity f_i, $i=1,2,\cdots,m$.

(ii) $0<H^s(E)\le C^s(E)\le P^s(E)<\infty$, i.e., E is a s-set.

Marta Llorente and Manuel Moran proved the following result recently [6].

Lemma 2.3 (see [6])**.** *Let E be the self-similar set satisfying the open set condition, $s=dim_H(E)$, and A either a close or open subset of E, Then*

$$C^s(A)=C_0^s(A). \tag{2.3}$$

It is obvious that E is the close subset of itself, then $C^s(E)=C_0^s(E)$. Lemma 2.3 indicated that the extra step (1.2) in the definition of the Hausdorff centered measure is not needed. By the use of Lemma 2.3, we obtained the following result.

Theorem 2.1. *Let E be the self-similar set determined by the IFS $\Phi=\{f_1,f_2,\cdots,f_m\}$, and Φ satisfying the open set condition, $s=dim_H(E)$. Then for any ball $B(x,r)$ with center $x\in E$, we have*

$$C^s(E\cap B(x,r))\le (2r)^s. \tag{2.4}$$

Proof. We first prove that for $\forall\delta>0$, $C^s_\delta(E)=C^s_0(E)$. By the definition of the Hausdorff centered measure, $C^s_0(E)\geq C^s_\delta(E)$ is obvious, the reminder is to prove the inequality $C^s_0(E)\leq C^s_\delta(E)$.

For $\forall\varepsilon>0$, there exists a cenered ball δ-covering $\{B(x_p,r_p)\}_{p>0}$ of E, such that

$$\sum_{p>0}(2r_p)^s<C^s_\delta(E)+\varepsilon. \tag{2.5}$$

Then for any positive integer $k>0$, the collection of balls

$$\Big\{\bigcup_{(i_1i_2\cdots i_k)} f_{i_1}\circ f_{i_2}\circ\cdots\circ f_{i_k}(B(x_p,r_p))\Big\}_{p>0}$$

is a centered ball δ_k-covering of E, where

$$\delta_k=\max_{(i_1i_2\cdots i_k)}\{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}\delta\},\quad i_j\in\{1,2,\cdots,m\},\quad j=1,2,\cdots,k,$$

and λ_i is the contraction ratio of the similarity f_i. Thus

$$\begin{aligned}C^s_{\delta_k}(E)&\leq\sum_{p>0}\sum_{(i_1i_2\cdots i_k)}|f_{i_1}\circ f_{i_2}\circ\cdots\circ f_{i_k}(B(x_p,r_p))|^s=\sum_{p>0}\sum_{(i_1i_2\cdots i_k)}\lambda^s_{i_1}\lambda^s_{i_2}\cdots\lambda^s_{i_k}(2r_p)^s\\&=\sum_{p>0}(2r_p)^s\sum_{(i_1i_2\cdots i_k)}\lambda^s_{i_1}\lambda^s_{i_2}\cdots\lambda^s_{i_k}=\sum_{p>0}(2r_p)^s.\end{aligned}$$

By (2.5), we have

$$C^s_{\delta_k}(E)<C^s_\delta(E)+\varepsilon.$$

Let $k\to\infty$ in the above inequality, we have

$$C^s_0(E)\leq C^s_\delta(E)+\varepsilon.$$

Thus $C^s_0(E)\leq C^s_\delta(E)$.

Let $B(x,r)$ be the ball with center $x\in E$, then for $\forall\varepsilon>0$, there exists a centered ball δ-covering $\{B(x'_p,r'_p)\}_{p>0}$ of $E-B(x,r)$, such that

$$C^s_0(E-E\cap B(x,r))+\varepsilon\geq C^s_\delta(E-E\cap B(x,r))+\varepsilon>\sum_{p>0}(2r'_p)^s. \tag{2.6}$$

Then the collection of balls $\{B(x,r)\}\cup\{B(x'_p,r'_p)\}_{p>0}$ is a centered ball δ'-covering of E, where $\delta'=\max\{\delta,2r\}$. By the relation $C^s_0(E)=C^s_\delta(E)$ and Lemma 2.3, we have

$$\begin{aligned}&C^s(E-E\cap B(x,r))+C^s(E\cap B(x,r))=C^s(E)=C^s_0(E)=C^s_{\delta'}(E)\\&\leq(2r)^s+\sum_{p>0}(2r'_p)^s<(2r)^s+C^s_0(E-E\cap B(x,r))+\varepsilon.\end{aligned}$$

That is

$$C^s(E\cap B(x,r))+C^s(E-E\cap B(x,r))<(2r)^s+C_0^s(E-E\cap B(x,r))+\varepsilon.$$

Note that

$$C^s(E-E\cap B(x,r))\geq C_0^s(E-E\cap B(x,r)),$$

the above inequalities indicate that

$$C^s(E\cap B(x,r))<(2r)^s+\varepsilon.$$

Then

$$C^s(E\cap B(x,r))\leq(2r)^s. \tag{2.7}$$

This completes the proof of Theorem 2.1. □

3 Self-similar sets on the plane

In this section, we construct some examples on the plane such that the sets with dimension equal to 1, and the exact Hausdorff centered measure of these sets can be determined. For the self-similar sets construct on the line, the related results can be found in [7–9,15]. The self-similar sets given below provided useful examples for further researches in this fields.

Let $\Phi=\{f_1,f_2,\cdots,f_m\}$ be an IFS defined on the plane, and E be the self-similar set generated by Φ. Without loss of generality we assume that the IFS Φ iterate on E_0, where E_0 be the unit square in R^2. We establish an orthogonal coordinate system as follows. Take the origin to be a vertex of E_0. Then $E_0=[0,1]\times[0,1]$, the self-similar set E is regarded as the attractor of IFS Φ iterated on E_0, the map f_i in Φ satisfies the following conditions:

(i) $m\geq 2$, the contraction ratio λ_i of f_i satisfies $0<\lambda_i<1$ and $\sum_{i=1}^m\lambda_i=1$.

(ii) $f_1(x)=\lambda_1 x, f_m(x)=\lambda_m x+(1-\lambda_m,1-\lambda_m)$.

(iii) For $2\leq j\leq m-1$, $f_j(x)=\lambda_j x+(x_j,2(\lambda_1+\lambda_2+\cdots+\lambda_{j-1})-x_j)$.

(iv) For $2\leq j\leq m-1$, the domain of each x_j satisfies $0\leq x_j\leq 2(\lambda_1+\lambda_2+\cdots+\lambda_{j-1})$ when $\lambda_1+\lambda_2+\cdots+\lambda_j+\lambda_{j+1}/2<1/2$, and $1-(\lambda_j+\lambda_{j+1}+\cdots+\lambda_m)\leq x_j\leq 1-\lambda_m$ when $\lambda_1+\lambda_2+\cdots+\lambda_j+\lambda_{j+1}/2\geq 1/2$.

(v) The point $(1/2,1/2)$ is the fixed point of some f_j for $1\leq j\leq m$, i.e., $f_j(1/2,1/2)=(1/2,1/2)$.

The conditions (ii) and (iii) ensure that E satisfies the open set condition, and (i) implies that $dim_H(E)=s=1$. We give some concrete examples for different parameters of IFS Φ.

Example 3.1. Let $m=4$, the contraction ratios satisfy $\lambda_1=\lambda_2=\lambda_3=\lambda_4=1/4$, and

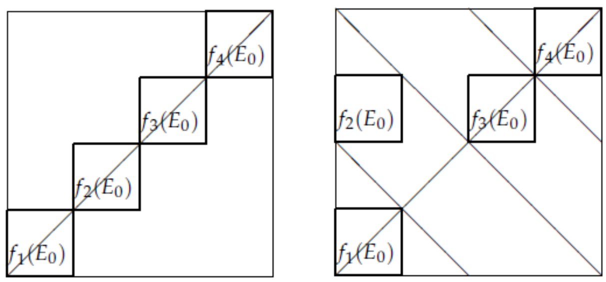
$$f_1(x)=\frac{1}{4}x, \qquad f_2(x)=\frac{1}{4}x+\Big(x_2,\frac{1}{2}-x_2\Big),$$
$$f_3(x)=\frac{1}{4}x+(x_3,1-x_3), \qquad f_4(x)=\frac{1}{4}x+\Big(\frac{3}{4},\frac{3}{4}\Big),$$

where $0\le x_2\le 1/2$, $1/4\le x_3\le 3/4$. Fig. 1(a) corresponding to the case when $x_2=1/4$, $x_3=1/2$. Fig. 1(b) corresponding to the case when $x_2=0$, $x_3=1/2$.

Example 3.2. Let $m=5$, the contraction ratios satisfy $\lambda_1=1/5$, $\lambda_2=1/4$, $\lambda_3=1/10$, $\lambda_4=1/4$, $\lambda_5=1/5$, and

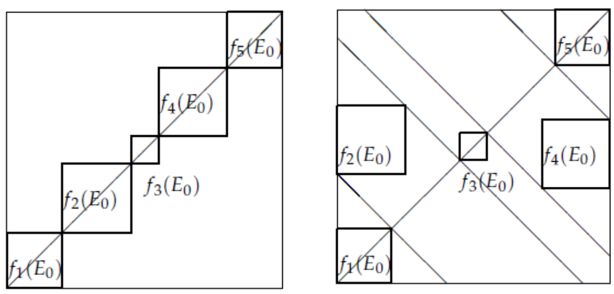
$$f_1(x)=\frac{1}{5}x, \qquad f_2(x)=\frac{1}{4}x+\Big(x_2,\frac{2}{5}-x_2\Big), \qquad f_3(x)=\frac{1}{10}x+\Big(x_3,\frac{9}{10}-x_3\Big),$$
$$f_4(x)=\frac{1}{4}x+\Big(x_4,\frac{11}{10}-x_4\Big), \qquad f_5(x)=\frac{1}{5}x+\Big(\frac{3}{4},\frac{3}{4}\Big),$$

where $0\le x_2\le 2/5$, $0\le x_3\le 9/10$, $7/20\le x_4\le 3/4$. Fig. 2(a) corresponding to the case when $x_2=1/4$, $x_3=9/20$, $x_4=11/20$. Fig. 2(b) corresponding to the case when $x_2=0$, $x_3=9/20$, $x_4=3/4$.



(a) (b)

Figure 1:



(a) (b)

Figure 2:

We have the following result about the Hausdorff centered measure of E.

Theorem 3.1. *Let E be the self-similar set generated by IFS Φ satisfying the above conditions (i)-(v). Then*

$$C^s(E)=H^s(E)=\sqrt{2},$$

where $s=dim_H(E)=1$.

Proof. By the condition (v), the point $(1/2,1/2)\in E$, we take $x_0=(1/2,1/2)$, $r_0=\sqrt{2}/2$, Theorem 2.1 indicates that

$$C^s(E\cap B(x_0,r_0))\le(2r_0)^s.$$

As $C^s(E\cap B(x_0,r_0))=C^s(E)$, then $C^s(E)\le\sqrt{2}$.

On the other hand, the construction of E imply that $proj_\Pi(E)=\{(x,y)|y=x,\ 0\le x\le 1\}$, where $\Pi=\{(x,y)|y=x\}$ is the main diagonal line of E_0. Combing this fact with (2.2) in Corollary 2.1, then

$$C^s(E)\ge H^s(E)\ge H^s(proj_\Pi E)=L^1(proj_\Pi E)=\sqrt{2},$$

where $L^1(\cdot)$ denotes the 1-dimensional Lebesgue measure. Thus

$$C^s(E)=H^s(E)=\sqrt{2}.$$

This completes the proof of Theorem 3.1. □

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