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A Remark on Pál Type Interpolation on Non-Uniformly Distributed Nodes on the Unit Circle

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Abstract. In this paper we study the problem of explicit representation and convergence of Pál type $(0;1)$ interpolation and its converse, with some additional conditions, on the non-uniformly distributed nodes on the unit circle obtained by projecting the interlaced zeros of $P_n(x)$ and $P_n'(x)$ on the unit circle. The motivation to this problem can be traced to the recent studies on the regularity of Birkhoff interpolation and Pál type interpolations on non-uniformly distributed zeros on the unit circle.

Key Words: Pál type interpolation, non-uniformly distributed set of points on unit circle, Legendre polynomials.

AMS Subject Classifications: 41A05, 41A10, 41A25

1 introduction

Let $\{x_{k,n}\}_{k=1}^{n}$ and $\{y_{k,n-1}\}_{k=1}^{n-1}$ be the zeros of $P_n(x)$ and $P_n'(x)$ respectively in $[-1,1]$, where $P_n(x)$ is the nth Legendre polynomial, which are interlaced such that

$$-1=x_{0,n}<x_{1,n}<y_{1,n-1}<\cdots<x_{n-1,n}<y_{n-1,n-1}<x_{n,n}<x_{n+1,n}=1. \tag{1.1}$$

We project these points on the unit circle by the inverse of the transformation

$$x=\frac{1}{2}(z+z^{-1}).$$

Let $\{z_{k,2n+2}\}_{k=1}^{2n}$ and $\{w_{k,2n-2}\}_{k=1}^{2n-2}$ be the transformations of $\{x_{k,n}\}_{k=1}^{n}$ and $\{y_{k,n-1}\}_{k=1}^{n-1}$ respectively on the unit circle and $z_{0,2n+2}=-1$, $z_{2n+1,2n+2}=1$. The set of points

$$\Lambda=\{z_{k,2n+2}\}_{k=0}^{2n+1}\cup\{w_{k,2n-2}\}_{k=1}^{2n-2}, \tag{1.2}$$

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thus obtained on the unit circle is non-uniformly distributed.

A Pál type $(0;1)$ interpolation on Λ means the determination of a polynomial R_n of minimum possible degree when the function values are prescribed on $\{z_{k,2n+2}\}_{k=0}^{2n+1}$ and the first derivatives are prescribed on $\{w_{k,2n-2}\}_{k=1}^{2n-2}$, i.e.,

$$R_n(z_{k,2n+2})=\alpha_{k,2n+2}, \qquad k=0,1,2\cdots,2n,2n+1, \tag{1.3a}$$

$$R_n'(w_{k,2n-2})=\beta_{k,2n-2}, \qquad k=1,2\cdots,2n-2, \tag{1.3b}$$

where $\{\alpha_{k,2n+2}\}_{k=0}^{2n+1}$ and $\{\beta_{k,2n-2}\}_{k=1}^{2n-2}$ are arbitrary given complex numbers. In this paper, it has been shown that Pál type $(0;1)$ interpolation on Λ is regular i.e., there exists a unique polynomial of degree $\le 4n-1$ satisfying the conditions (1.3a) and (1.3b). The explicit representation of the interpolatory polynomial has been obtained and a convergence theorem has also been proved for the same. If $\{z_{k,2n+2}\}_{k=0}^{2n+1}$ and $\{w_{k,2n-2}\}_{k=1}^{2n-2}$ are interchanged in (1.3a) and (1.3b) respectively (the converse of the above problem) then it has been shown that the Pál type $(0;1)$ interpolation is regular together with some additional interpolatory conditions. The explicit representation of the corresponding interpolatory polynomial and its convergence has also been dealt with. Further it has been shown, by numerical calculations, that the maximum absolute error in (i) the Lagrange interpolation for the simple function $f(z)=\exp(z)$, $z\in\mathbb{C}$ and (ii) the barycentric [2] form of Lagrange interpolation for the simple function $f(z)=1/(1+z^2)$, $z\in\mathbb{C}$ is least when the interpolation points are chosen to be the points of the set Λ given by (1.2) in comparison to that obtained when the points of interpolation are (i) projected zeros of the Chebyshev polynomial $T_N(x)$, rescaled so that the first and last zeros coincide, respectively, with -1 and 1, to the unit circle and (ii) the N^{th} roots of unity.

The study of interpolation processes on the unit circle was initiated in 1960 by O. Kiš [12], when he considered the $(0,2)$ and $(0,1,\cdots,r-2,r)$ interpolations, for any integer $r\geq 2$ on nth roots of unity. Since then several mathematicians have taken up the study of various interpolatory problems on the unit circle. Considerable literature has come up on the subject of lacunary and Pál type interpolations on the roots of unity [13]. Recently, the regularity of Birkhoff interpolation and Pál type interpolation on non-uniformly distributed nodes on the unit circle has been attracting much attention, which motivated us to study the above problem.

R. Brück [4] has studied the convergence of Lagrange interpolation of a function on the nodes $z_{kn}^{\alpha}=T_\alpha(w_{kn})$, $k=1(1)2n$, where T_α is a Möbius transform of a unit disk into itself and $w_{kn}=\exp(2\pi ik/(2n+1))$, $n\geq 0$. In [3] Bokhari et al. and in [10] de Bruin et al. have obtained the regularity of certain interpolation problems on the zeros of $(z-\xi)Q(z)$, where $Q(z)$ is a polynomial, whose all zeros lie on the unit circle. They have also determined the range of the values of ξ in the complex plane. In [11], Dikshit has shown the regularity of certain Pál type interpolation problems involving Möbius transforms of the zeros of z^n+1 and z^n-1 along with an additional point ξ or two additional points ξ and 1. For more references of recent works in this direction, we refer to [3–11,14].

In another paper, S. Xie [17] considered among others regularity of $(0,1,\cdots,r-2,r)$ and

$(0,1,\cdots,r-2,r)^*$ interpolations on a special set of nodes on the unit circle obtained by projecting vertically the zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$, $-1<\alpha,\beta\le 1/2$, where $P_n^{(\alpha,\beta)}(x)$ stands for the nth Jacobi polynomial. In [18], S. Xie dealt with the problem of explicit representation and convergence of $(0,1,3)^*$-interpolation in the case when $\alpha=\beta=0$. In another paper [15] the author of this paper, on the suggestion of Prof. A. Sharma, considered an analogous problem on the nodes obtained by projecting the zeros of $(1-x^2)P_n^{(\lambda)}(x)$, where $P_n^{(\lambda)}(x)$ is the nth Ultraspherical polynomial with $-1/2<\lambda\le 1/2$, on the unit circle.

In Section 2, we give some preliminaries. Explicit representation and quantitative estimates of the fundamental polynomials leading to the convergence problem will be dealt with in Sections 4 and 5 respectively.

2 Preliminaries

Let $\{z_{k,2n+2}\}_{k=1}^{2n}$ and $\{w_{k,2n-2}\}_{k=1}^{2n-2}$, given in (1.2), be the zeros of the polynomials $W_{2n}(z)$ and $H_{2n-2}(z)$ respectively. Then we can write

$$W_{2n}(z)=\prod_{k=1}^{2n}(z-z_{k,2n+2})=K_nP_n\Big(\frac{1+z^2}{2z}\Big)z^n, \tag{2.1}$$

and

$$H_{2n-2}(z)=\prod_{k=1}^{2n-2}(z-w_{k,2n-2})=K_n^*P_n'\Big(\frac{1+z^2}{2z}\Big)z^{n-1}, \tag{2.2}$$

where K_n, K_n^* are the inverses of the coefficients of heighest degree term in the expansion of $P_n(x)$ and $P_n'(x)$ respectively. The differential equation satisfied by $P_n(x)$ is

$$(1-x^2)P_n{''}(x)-2xP_n'(x)+n(n+1)P_n(x)=0. \tag{2.3}$$

By (2.1) and (2.2), we have

$$W_{2n}'(z)=K_nz^{n-2}\Big[\frac{1}{2}P_n'\Big(\frac{1+z^2}{2z}\Big)(z^2-1)+nzP_n\Big(\frac{1+z^2}{2z}\Big)\Big], \tag{2.4}$$

and

$$H_{2n-2}'(z)=K_n^*z^{n-3}\Big[(n-1)P_n'\Big(\frac{1+z^2}{2z}\Big)z+\frac{(z^2-1)}{2}P_n''\Big(\frac{1+z^2}{2z}\Big)\Big]. \tag{2.5}$$

Let

$$L_{k,2n}(z)=\frac{W_{2n}(z)}{(z-z_{k,2n+2})W_{2n}'(z_{k,2n+2})},\quad k=1,2,\cdots,2n, \tag{2.6}$$

and

$$l_{k,2n-2}(z)=\frac{H_{2n-2}(z)}{(z-w_{k,2n-2})H_{2n-2}'(w_{k,2n-2})},\quad k=1,2,\cdots,2n-2^{\dagger}. \tag{2.7}$$

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†We will use "k" in the place of k, n in the subscript.

For $-1\le x\le 1$ by (see [19, (7.21.1), (7.3.8)]) and (see [1, pp. 207]), we have

$$|P_n(x)|\le 1, \tag{2.8a}$$

$$(1-x^2)^{\frac{1}{4}}|P_n(x)|\le\sqrt{\frac{2}{n\pi}}, \tag{2.8b}$$

$$|P_n'(x)|\le\frac{n(n+1)}{2}, \tag{2.8c}$$

$$(1-x^2)^{\frac{3}{4}}|P_n'(x)|\le\sqrt{2n}. \tag{2.8d}$$

Let $x_k=\cos\theta_k$, $k=1,2,\cdots,n$, be the zeros of the nth Legendre polynomial $P_n(x)$, then (see [1, pp. 207]), we have

$$(1-x_k^2)\ge k^2n^{-2}, \qquad k=1,2,\cdots,\left[\frac{n}{2}\right], \tag{2.9a}$$

$$(1-x_k^2)\ge (n-k+1)^2n^{-2}, \qquad k=\left[\frac{n}{2}\right]+1,\cdots,n, \tag{2.9b}$$

and (see [19, (8.9.2)])

$$|P_n'(x_k)|\ge ck^{-\frac{3}{2}}n^2, \qquad k=1,2,\cdots,\left[\frac{n}{2}\right], \tag{2.10a}$$

$$|P_n'(x_k)|\ge c(n-k+1)^{-\frac{3}{2}}n^2, \qquad k=\left[\frac{n}{2}\right]+1,\cdots,n. \tag{2.10b}$$

Let the fundamental functions of the Lagrange interpolation on the zeros of Legendre polynomial be given by

$$l_k^*(x)=\frac{P_n(x)}{(x-x_k)P_n'(x_k)}, \quad k=1,2\cdots,n, \tag{2.11}$$

and those on the zeros of $\pi_n(x)=(1-x^2)P_{n-1}'(x)$ are given as

$$l_k^{**}(x)=\frac{\pi_n(x)}{(x-y_k)\pi_n'(y_k)}, \quad k=1,2\cdots,n-1. \tag{2.12}$$

Further for these fundamental functions, from [19] and [1], we have

$$\sum_{k=1}^{n}|l_k^*(x)|=c\log n \tag{2.13}$$

and

$$(l_j^{**})^2(x)\le\sum_{k=1}^{n}(l_k^{**}(x))^2\le 1, \quad 1\le j\le n. \tag{2.14}$$

If y_k's are the zeros of $P_{n-1}'(x)$, then by [1]

$$|P_n(y_k)|>\frac{1}{\sqrt{8\pi k}}. \tag{2.15}$$

Suppose that

$$J_{1j}(z)=\int_0^z t^{n+j-1}H_{2n-2}(t)dt, \quad j=0,1, \tag{2.16}$$

which satisfies

$$J_{1j}(-1)=(-1)^{n+j}J_{1j}(1). \tag{2.17}$$

3 Regularity and explicit forms

3.1 Pál Type $(0;1)$-Interpolation

Let Λ be an interlacing set of non-uniformly distributed nodes on the unit circle given by (1.2), then we have

Theorem 3.1. *The Pál type interpolation satisfying the conditions* (1.3a) *and* (1.3b) *is regular on* Λ.

Proof. For regularity it is sufficient to show that if $\alpha_k=0$, $k=0,1,\cdots,2n+1$ and $\beta_k=0$, $k=1,\cdots,2n-2$ then $R_{4n-1}(z)\equiv 0$. Let

$$R_{4n-1}(z)=(z^2-1)W_{2n}(z)q_{2n-3}(z), \tag{3.1}$$

where $q_{2n-3}(z)$ is a polynomial of degree $\le 2n-3$. Clearly $R_{4n-1}(z)$ is a polynomial of degree $\le 4n-1$ and $R_{4n-1}(z_k)=0$ for $k=0,1,\cdots,2n,2n+1$, due to (2.1). For $R'_{4n-1}(w_k)=0, k=1,2,\cdots,2n-2$, we must have

$$q'_{2n-3}(w_k)+\Big[\frac{W'_{2n}(w_k)}{W_{2n}(w_k)}+\frac{2w_k}{w_k^2-1}\Big]q_{2n-3}(w_k)=0, \tag{3.2}$$

which owing to (2.1) and (2.4) implies that for $k=1,2,\cdots,2n-2$,

$$q'_{2n-3}(w_k)+\Big(\frac{n}{w_k}+\frac{1}{w_k+1}+\frac{1}{w_k-1}\Big)q_{2n-3}(w_k)=0. \tag{3.3}$$

Hence $q_{2n-3}(z)$ satisfies the first order linear differential equation

$$z(z^2-1)q'_{2n-3}(z)+\{n(z^2-1)+z(z+1)+z(z-1)\}q_{2n-3}(z)=(Az+B)H_{2n-2}(z), \tag{3.4}$$

where A, B are constants. On solving this differential equation we get

$$q_{2n-3}(z)=\frac{1}{z^n(z^2-1)}\int_0^z t^{n-1}(At+B)H_{2n-2}(t)dt=\frac{1}{z^n(z^2-1)}[AJ_{11}(z)+BJ_{10}(z)],$$

where $J_{1j}(z)$, $j=0,1$ are given by (2.16). Since $q_{2n-3}(z)$ is a polynomial of degree $\le 2n-3$, hence we must have

$$AJ_{11}(1)+BJ_{10}(1)=0, \quad AJ_{11}(-1)+BJ_{10}(-1)=0,$$

which due to (2.17), on solving give $A=B=0$. Hence $q_{2n-3}(z)=0$ and hence $R_{4n-1}(z)\equiv 0$, which proves the Theorem. $\square$

The interpolatory polynomial $R_{4n-1}(z)$ satisfying the conditions (1.3a) and (1.3b) has the form

$$R_{4n-1}(z)=\sum_{k=0}^{2n+1}\alpha_k A_k(z)+\sum_{k=1}^{2n-2}\beta_k B_k(z), \tag{3.5}$$

where $\{A_k(z)\}_{k=0}^{2n+1}$ and $\{B_k(z)\}_{k=1}^{2n-2}$ are fundamental polynomials each of degree $\le 4n-1$ satisfying the conditions: For $k=0,1,\cdots,2n,2n+1$,

$$\begin{cases} A_k(z_j)=\delta_{jk}, & j=0,1,\cdots,2n+1, \\ A_k'(w_j)=0, & j=1,\cdots,2n-2, \end{cases} \tag{3.6}$$

and for $k=1,2,\cdots,2n-2$,

$$\begin{cases} B_k(z_j)=0, & j=0,1,\cdots,2n+1, \\ B_k'(w_j)=\delta_{jk}, & j=1,\cdots,2n-2. \end{cases} \tag{3.7}$$

The explicit forms of the fundamental polynomials are given in the following Theorems.

Theorem 3.2. *Let $J_{1j}(z)$, $j=0,1$ be given by* (2.16). *The fundamental polynomials $B_k(z)$, $k=1,2,\cdots,2n-2$, have the form*

$$B_k(z)=\frac{z^{-n}W_{2n}(z)}{W_{2n}(w_k)}\big[S_k(z)+b_{0k}J_{10}(z)+b_{1k}J_{11}(z)\big], \tag{3.8}$$

where

$$S_k(z)=\int_0^z t^n l_k(t)dt, \tag{3.9a}$$

$$b_{0k}=-\frac{S_k(1)+(-1)^n S_k(-1)}{2J_{10}(1)}, \tag{3.9b}$$

and

$$b_{1k}=-\frac{S_k(1)-(-1)^n S_k(-1)}{2J_{11}(1)}. \tag{3.10}$$

Proof. Obviously, for $k=1,2,\cdots,2n-2$, $B_k(z)$ given by (3.8) is a polynomial of degree $\le(4n-1)$ and $B_k(z_j)=0$, $j=1,2,\cdots,2n$ owing to (2.1). Also for $z_0=-1$ and $z_{2n+1}=1$, $B_k(z_0)=0$ and $B_k(z_{2n+1})=0$, if

$$S_k(1)+b_{0k}J_{10}(1)+b_{1k}J_{11}(1)=0,\quad S_k(-1)+(-1)^n b_{0k}J_{10}(1)+(-1)^{n+1}b_{1k}J_{11}(1)=0, \tag{3.11}$$

owing (2.17). On solving the above equations we get b_{0k} and b_{1k} as given in (3.9b) and (3.10) respectively. Since, for $j=1,\cdots,2n-2$,

$$(z^{-n}W_{2n}(z))'_{w_j}=w_j^{-n-1}\big[w_jW_{2n}'(w_j)-nW_{2n}(w_j)\big]. \tag{3.12}$$

On substituting the value of $W'_{2n}(w_j)$ from (2.4) in the above equation and using (2.1), we have that for $j=1,\cdots,2n-2$,

$$(z^{-n}W_{2n}(z))'_{w_j}=w_j^{-n-1}\Big[nK_nw_j^nP_n\Big(\frac{1+w_j^2}{2w_j}\Big)-nW_{2n}(w_j)\Big]=0. \tag{3.13}$$

Hence for $j=1,\cdots,2n-2$, we have due to (3.13)

$$B'_k(w_j)=\frac{(w_j^{-n}W_{2n}(w_j))}{W_{2n}(w_k)}\big[S'_k(w_j)+b_{0k}J'_{10}(w_j)+b_{1k}J'_{11}(w_j)\big], \tag{3.14}$$

which owing to (2.2), (2.16) and (3.9a) reduces to

$$B'_k(w_j)=\frac{l_k(w_j)W_{2n}(w_j)}{W_{2n}(w_k)}=\delta_{jk}. \tag{3.15}$$

Thus $B_k(z)$, $k=1,2,\cdots,2n-2$, given by (3.8) is a uniquely determined polynomial of degree $\leq 4n-1$ satisfying the conditions (3.7). □

Theorem 3.3. *Let $J_{1j}(z)$, $j=0,1$ be given by* (2.16)*. The fundamental polynomials $A_k(z)$, $k=0,2n+1$, have the form*

$$A_k(z)=\frac{z^{-n}W_{2n}(z)}{2W_{2n}(z_k)}[a_{0k}J_{10}(z)+a_{1k}J_{11}(z)], \tag{3.16}$$

where

$$a_{00}=a_{0,2n+1}=\frac{1}{J_{10}(1)}, \tag{3.17a}$$

$$a_{10}=-a_{1,2n+1}=-\frac{1}{J_{11}(1)}. \tag{3.17b}$$

For $k=1,2,\cdots,2n$,

$$\begin{aligned}A_k(z)=&\frac{(z^2-1)H_{2n-2}(z)L_k(z)}{(z_k^2-1)H_{2n-2}(z_k)}\\&+\frac{z^{-n}W_{2n}(z)}{(z_k^2-1)H_{2n-2}(z_k)W'_{2n}(z_k)}\big[J_k(z)+a^*_{0k}J_{10}(z)+a^*_{1k}J_{11}(z)\big],\end{aligned} \tag{3.18}$$

where

$$J_k(z)=-\int_0^z t^{n-1}(t^2-1)\frac{tH'_{2n-2}(t)+c_kH_{2n-2}(t)}{t-z_k}dt, \tag{3.19a}$$

$$a^*_{0k}=-\frac{J_k(1)+(-1)^nJ_k(-1)}{2J_{10}(1)}, \tag{3.19b}$$

$$a^*_{1k}=-\frac{J_k(1)-(-1)^nJ_k(-1)}{2J_{11}(1)}, \tag{3.19c}$$

$$c_k=-\frac{z_kH'_{2n-2}(z_k)}{H_{2n-2}(z_k)}. \tag{3.19d}$$

Proof. Obviously, $A_k(z)$ for $k=0,2n+1$ given by (3.16) is a polynomial of degree $\le(4n-1)$ and $A_k(z_j)=0$, $j=1,2,\cdots,2n$ owing to (2.1). Also $A_k(z_j)=\delta_{jk}$, $j=0,2n+1$, provided

$$a_{00}J_{10}(1)+a_{10}J_{11}(1)=0, \quad a_{00}J_{10}(1)-a_{10}J_{11}(1)=2, \tag{3.20}$$

and

$$a_{0,2n+1}J_{10}(1)+a_{1,2n+1}J_{11}(1)=2, \quad a_{0,2n+1}J_{10}(1)-a_{1,2n+1}J_{11}(1)=0, \tag{3.21}$$

owing to (2.17). On solving the above equations we get a_{0k} and a_{1k}, $k=0,2n+1$ as given in (3.17a) and (3.17b) respectively. Lastly, for $j=1,\cdots,2n-2$, we have

$$A_k'(w_j)=\frac{w_j^{-n}W_{2n}(w_j)}{W_{2n}(w_k)}\left[a_{0k}J_{10}'(w_j)+a_{1k}J_{11}'(w_j)\right]=0, \tag{3.22}$$

owing to (2.16) and (3.13). Thus $A_k(z)$, $k=0,2n+1$ given by (3.16) is a uniquely determined polynomial of degree $\le 4n-1$ satisfying the conditions (3.6).

As $z_kH_{2n-2}'(z_k)+c_kH_{2n-2}(z_k)=0$ for $k=1,2,\cdots,2n$ due to $\{c_k\}_{k=1}^{2n}$ given by (3.19d) thus $\{J_k(z)\}_{k=1}^{2n}$ given by (3.19a) is a polynomial of degree $\le 3n-1$. Hence $\{A_k(z)\}_{k=1}^{2n}$ given by (3.18) is a polynomial of degree $\le(4n-1)$. Obviously, $A_k(z_j)=\delta_{jk}$, $j=1,2,\cdots,2n$ owing to (2.6). Also for $z_0=-1$ and $z_{2n+1}=1$, $A_k(z_0)=0$ and $A_k(z_{2n+1})=0$, if

$$J_k(1)+a_{0k}^*J_{10}(1)+a_{1k}^*J_{11}(1)=0, \quad J_k(-1)+(-1)^n a_{0k}^*J_{10}(1)+(-1)^{n+1}a_{1k}^*J_{11}(1)=0, \tag{3.23}$$

owing to (2.17). On solving the above equations we get a_{0k}^* and a_{1k}^* as given in (3.19b) and (3.19c) respectively. Lastly, for $j=1,\cdots,2n-2$, we have due to (3.13), (2.2), (2.16) and (3.19a)

$$\begin{aligned}A_k'(w_j)&=\frac{(w_j^2-1)H_{2n-2}'(w_j)L_k(w_j)}{(z_k^2-1)H_{2n-2}(z_k)}+\frac{w_j^{-n}W_{2n}(w_j)}{(z_k^2-1)H_{2n-2}(z_k)W'(z_k)}J_k'(w_j)\\&=\frac{(w_j^2-1)H_{2n-2}'(w_j)L_k(w_j)}{(z_k^2-1)H_{2n-2}(z_k)}-\frac{(w_j^2-1)H_{2n-2}'(w_j)W_{2n}(w_j)}{(z_k^2-1)H_{2n-2}(z_k)(w_j-z_k)W'(z_k)}=0.\end{aligned}$$

Thus $A_k(z)$, $k=1,2,\cdots,2n$, given by (3.18) is a uniquely determined polynomial of degree $\le 4n-1$ satisfying the conditions (3.6). □

3.2 (0;1) Pál type interpolation with interchanged nodes

In this section we consider the converse of the problem dealt in last section. Precisely, we have

Theorem 3.4. *The Pál type interpolation $Q_{4n+1}(z)$ satisfying the conditions*

$$\begin{aligned}&Q_{4n+1}(w_k)=\sigma_k, && k=0,1,\cdots,2n-2,2n+1, && (3.24a)\\&Q_{4n+1}'(z_k)=\rho_k, && k=0,1,\cdots,2n,2n+1, && (3.24b)\end{aligned}$$

is regular if $\{w_k\}_{k=0,k\neq 2n-1,2n}^{2n+1}$ *and* $\{z_k\}_{k=0}^{2n+1}$ *are zeros of* $(z^2-1)H_{2n-2}(z)$ *and* $(z^2-1)W_{2n}(z)$ *respectively, where* $w_0=-1$, $w_{2n+1}=1$, $W_{2n}(z)$ *and* $H_{2n-2}(z)$ *are given by* (2.1) *and* (2.2) *respectively and* $\{\sigma_k\}_{k=0,k\neq 2n-1,2n}^{2n+1}$, $\{\rho_k\}_{k=0}^{2n+1}$ *are arbitrary given complex constants.*

Proof. The proof of the theorem is quite similar to that of Theorem 3.1. □

The interpolatory polynomial $Q_{4n+1}(z)$ satisfying the conditions (3.24a) and (3.24b) has the form

$$Q_{4n+1}(z)=\sum_{k=0,k\neq 2n-1,2n}^{2n+1}\sigma_k A_k^*(z)+\sum_{k=0}^{2n+1}\rho_k B_k^*(z), \tag{3.25}$$

where $\{A_k^*(z)\}_{k=0,k\neq 2n-1,2n}^{2n+1}$ and $\{B_k^*(z)\}_{k=0}^{2n+1}$ are fundamental polynomial each of degree $\leq 4n+1$ satisfying the conditions: For $k=0,1,\cdots,2n+1$, $k\neq 2n-1,2n$

$$\begin{cases} A_k^*(w_j)=\delta_{jk}, & j=0,1,\cdots,2n+1, \quad j\neq 2n-1,2n, \\ (A_k^*)'(z_j)=0, & j=0,1,\cdots,2n,2n+1, \end{cases} \tag{3.26}$$

and for $k=0,1,2,\cdots,2n,2n+1$,

$$\begin{cases} B_k^*(w_j)=0, & j=0,1,\cdots,2n+1, \quad j\neq 2n-1,2n, \\ (B_k^*)'(z_j)=\delta_{jk}, & j=0,1,\cdots,2n,2n+1. \end{cases} \tag{3.27}$$

The explicit forms of the fundamental polynomials are given in the following Theorems.

Theorem 3.5. *The fundamental polynomials* $B_k^*(z)$, $k=1,2\cdots,2n$, *have the form*

$$B_k^*(z)=\frac{z^{-n-1}(z^2-1)^2H_{2n-2}(z)}{H_{2n-2}(z_k)}\int_0^z t^n\frac{tL_k(t)+\{a(t-1)^2(t+1)+b(t-1)^2+c(t-1)+d\}W_{2n}(t)}{(t^2-1)^2}dt,$$

where $L_k(z)$ *are given by* (2.6) *and* a,b,c,d *are given by*

$$\begin{aligned} &L_k(1)+dW_{2n}(1)=0,\\ &L_k(1)+L_k'(1)+dW_{2n}'(1)+cW_{2n}(1)=0,\\ &-L_k(-1)+(4b-2c+d)W_{2n}(-1)=0,\\ &L_k(-1)+L_k'(-1)+(4b-2c+d)W_{2n}'(-1)+(4a-4b+c)W_{2n}(-1)=0, \end{aligned}$$

and for $k=0,2n+1$,

$$\begin{aligned} B_k^*(z)=&\frac{(z^2-1)(z-z_k)H_{2n-2}(z)W_{2n}(z)}{4W_{2n}(z_k)H_{2n-2}(z_k)}\\ &-\frac{z^{-n-1}(z^2-1)^2H_{2n-2}(z)}{4W_{2n}(z_k)H_{2n-2}(z_k)}\int_0^z t^n\frac{tW_{2n}'(t)W_{2n}(z_k)-z_kW_{2n}'(z_k)W_{2n}(t)}{t-z_k}dt. \end{aligned}$$

Theorem 3.6. *The fundamental polynomials $A_k^*(z)$, $k=1,2\cdots,2n-2$ have the form*

$$A_k^*(z)=\frac{(z^2-1)^2W_{2n}(z)l_k(z)}{(w_k^2-1)^2W_{2n}(w_k)} -\frac{z^{-n-1}(z^2-1)^2H_{2n-2}(z)}{(w_k^2-1)^2W_{2n}^2(w_k)H'_{2n-2}(w_k)}\int_0^z t^n\frac{tW'_{2n}(t)W_{2n}(w_k)-w_kW'_{2n}(w_k)W_{2n}(t)}{t-w_k}dt,$$

where $l_k(z)$, are given by (2.7) *and for $k=0,2n+1$,*

$$A_k^*(z)=\frac{(z-z_k)^2H_{2n-2}(z)W_{2n}(z)}{4W_{2n}(z_k)H_{2n-2}(z_k)} -\frac{z^{-n-1}(z^2-1)^2H_{2n-2}(z)}{4W_{2n}(z_k)H_{2n-2}(z_k)}\int_0^z t^n\frac{tW'_{2n}(t)W_{2n}(z_k)+\{a_{1k}^*+a_{2k}^*(t-z_k)W_{2n}(t)}{(t-z_k)^2}dt,$$

where

$$a_{1k}^*=-\frac{z_kW'_{2n}(z_k)}{W_{2n}(z_k)},$$

and

$$a_{2k}^*=-\frac{z_kW''_{2n}(z_k)+W'_{2n}(z_k)-a_{1k}^*W'_{2n}(z_k)}{W_{2n}(z_k)}.$$

The proof of the Theorems 3.5 and 3.6 follow on the same lines as the proof of Theorem 3.2. We omit details.

3.3 Convergence theorems

In this section we state our main Theorems which deal with the convergence of the interpolatory polynomials $\{R_{4n-1}(z)\}$ and $\{Q_{4n+1}(z)\}$ defined by (3.5) and (3.25) respectively.

Theorem 3.7. *Let $f(z)$ be continuous in a region $|z|\le 1$ and analytic in $|z|<1$, then the sequence of interpolatory polynomials $\{R_{4n-1}(z)\}$ defined by* (3.5) *with*

$$\alpha_k=f(z_k),\quad \beta_k=\mathcal{O}\Big(n\omega\Big(f,\frac{1}{n}\Big)\Big), \tag{3.28}$$

satisfy the relation

$$|R_{4n-1}(z)-f(z)|\le\mathcal{O}\Big(n^2\omega\Big(f,\frac{1}{n}\Big)\Big), \tag{3.29}$$

where $\omega(f,\delta)$ is the modulus of continuity of f.

Theorem 3.8. *Let $f(z)$ be continuous in a region $|z|\le 1$ and analytic in $|z|<1$, then the sequence of interpolatory polynomials $\{Q_{4n+1}(z)\}$ defined by* (3.25) *with*

$$\alpha_k=f(z_k),\quad \beta_k=\mathcal{O}\Big(n\omega\Big(f,\frac{1}{n}\Big)\Big), \tag{3.30}$$

satisfy the relation

$$|Q_{4n+1}(z)-f(z)| \leq \mathcal{O}\Big(n^2\omega\Big(f,\frac{1}{n}\Big)\Big), \tag{3.31}$$

where $\omega(f,\delta)$ *is the modulus of continuity of* f.

We will prove only Theorem 3.7, as the proof of Theorem 3.8 is quite similar to that of Theorem 3.7, so we omit details. To prove Theorem 3.7, we shall need the estimates of the fundamental polynomials which are given in the next section.

4 Estimation of the fundamental polynomials: $(0;1)$-case

Lemma 4.1. *For* $|z|\leq 1$, $J_{1j}(z)$, $j=0,1$, *given by* (2.16) *satisfy*

$$|J_{1j}(1)| > \frac{K_n^* n(n+1)}{2(2n+j)}, \tag{4.1}$$

and

$$|J_{1j}(z)| \leq \frac{K_n^* n(n+1)}{2(2n+j)}. \tag{4.2}$$

Proof. By (2.16), we have

$$\begin{aligned}|J_{1j}(1)| &= \Big|K_n^*\int_0^1 t^{2n+j-1}P_n'\Big(\frac{t^2+1}{2t}\Big)dt\Big| = \Big|K_n^*\int_1^\infty t^{-2n-j-1}P_n'\Big(\frac{t^2+1}{2t}\Big)dt\Big| \\ &> \Big|K_n^*P_n'(1)\int_1^\infty t^{-2n-j-1}dt\Big| = \frac{K_n^* n(n+1)}{2(2n+j)},\end{aligned}$$

owing to (2.3). Also by (2.8c) and for $|z|\leq 1$, we have

$$\begin{aligned}|J_{1j}(z)| &\leq K_n^*\Big|\int_0^1 t^{2n+j-1}P_n'\Big(\frac{t^2+1}{2t}\Big)dt\Big| \leq \frac{K_n^* n(n+1)}{2}\Big|\int_0^1 t^{2n+j-1}dt\Big| \\ &\leq \frac{K_n^* n(n+1)}{2(2n+j)},\end{aligned}$$

which proves the Lemma. ☐

Lemma 4.2. *Let* $B_k(z)$, $k=1,2\cdots,2n-2$ *be given as in Theorem 3.2 then for* $|z|\leq 1$, *we have*

$$\sum_{k=1}^{2n-2}|B_k(z)| \leq c_1 n, \tag{4.3}$$

where c_1 *does not depend on* n *and* x.

Proof. By (3.8), we have

$$|B_k(z)| \le \Big|\frac{P_n(x)}{P_n(y_k)}\Big|\big[|S_k(z)|+|b_{0k}J_{01}(z)|+|b_{1k}J_{11}(z)|\big] \equiv I_1+I_2+I_3, \tag{4.4}$$

where

$$|S_k(z)| \le \Big|\int_0^1 t^n l_k(t)dt\Big| \le \frac{|l_k(1)|}{n}. \tag{4.5}$$

By using (2.2), (2.3) after transformation and (2.5) in (2.7) then for $|z|\le 1$, we have

$$|l_k(z)| = \frac{|P_n{}'(x)|}{|(z-w_k)||(w_k^2-1)P_n''(y_k)|} = \frac{|(w_k^2-1)||P_n{}'(x)|}{4n(n+1)|(z-w_k)||P_n(y_k)|}.$$

Let $z=x+i\sqrt{1-x^2}$, then for $0\le \arg z\le \pi$,

$$|(1-z^2)| = 2\big|(1-x^2)^{\frac{1}{2}}\big|, \tag{4.6a}$$

$$|(z-w_k)| = \sqrt{2}|(1-xy_k)-(1-x^2)^{\frac{1}{2}}(1-y_k^2)^{\frac{1}{2}}|^{\frac{1}{2}}, \tag{4.6b}$$

and

$$|(1-x^2)^{\frac{1}{2}}(1-y_k^2)^{\frac{1}{2}}| \le |(1-xy_k)| \quad \text{for } |x|\le 1, \quad |y_k|\le 1. \tag{4.7}$$

Hence,

$$\begin{aligned} |l_k(z)| &= \frac{|(1-y_k^2)^{\frac{1}{2}}\big[(1-xy_k)+(1-x^2)^{\frac{1}{2}}(1-y_k^2)^{\frac{1}{2}}\big]^{\frac{1}{2}}||P_n{}'(x)|}{2\sqrt{2}n(n+1)|(x-y_k)||P_n(y_k)|} \\ &\le \frac{|(1-y_k^2)^{\frac{1}{2}}(1-xy_k)^{\frac{1}{2}}||P_n{}'(x)|}{2n(n+1)|(x-y_k)||P_n(y_k)|}. \end{aligned} \tag{4.8}$$

If $|x-y_k|\ge (1-y_k^2)/4$, we have

$$|1-xy_k| \le |1-y_k^2|+|y_k^2-xy_k| \le 5|x-y_k|, \tag{4.9}$$

then by (2.8c) and (2.15), it follows that

$$|l_k(z)| \le \frac{\sqrt{5}|P_n{}'(x)|}{n(n+1)|P_n(y_k)|} = \mathcal{O}(\sqrt{k}). \tag{4.10}$$

Thus, in this case, we have due to (2.8a), (2.15) and (4.10)

$$I_1 \le \frac{|P_n(x)|}{(n+1)|P_n(y_k)|}|l_k(1)| = \mathcal{O}\Big(\frac{k}{n}\Big). \tag{4.11}$$

Also, since by (3.9b), (3.10) and (4.5) for $j=0,1$

$$|b_{jk}J_{1j}(z)| \le \frac{|S_k(1)|+|S_k(-1)|}{2} \le \frac{|l_k(1)|}{n}, \tag{4.12}$$

from which it follows that

$$I_2 = \Big|\frac{b_{0k}P_n(x)J_{10}(z)}{P_n(y_k)}\Big| \le I_1, \tag{4.13}$$

and

$$I_3 = \Big|\frac{b_{1k}P_n(x)J_{11}(z)}{P_n(y_k)}\Big| \le I_1. \tag{4.14}$$

By substituting the values of (4.11), (4.13) and (4.14) in (4.4), (4.3) follows in this case.

If $|x-y_k| \le (1-y_k^2)/4$, we have

$$|1-xy_k| \le |1-y_k^2| + |y_k^2 - xy_k| \le \frac{5|1-y_k^2|}{4}, \tag{4.15a}$$

$$|1-x^2| \ge |1-y_k^2| - |x^2-y_k^2| \ge \frac{|1-y_k^2|}{2}, \tag{4.15b}$$

then by (4.8), we have

$$|l_k(z)| \le \frac{\sqrt{5}|P_n{}'(x)|}{4|(x-y_k)P_n''(y_k)|} \le \frac{\sqrt{5}|(1-x^2)P_n{}'(x)|}{2|(x-y_k)(1-y_k^2)P_n''(y_k)|} = \frac{\sqrt{5}|l_k^{**}(x)|}{2},$$

where $l_k^{**}(x)$ are given by (2.12). Thus, in this case

$$I_1 \le \frac{\sqrt{5}|P_n(x)|}{2n|P_n(y_k)|}|l_k^{**}(x)|, \tag{4.16}$$

which together with (2.14), (4.13), (4.14) and (4.4) imply (4.3) in this case as well. ☐

Lemma 4.3. *Let $A_k(z)$, $k=0,1,2\cdots,2n+1$ be given as in Theorem 3.3 then for $|z|\le 1$, we have*

$$\sum_{k=0}^{2n+1} |A_k(z)| \le cn^2, \tag{4.17}$$

where c does not depend on n and x.

Proof. Since $|P_n(1)| = |P_n(-1)| = 1$ thus for $k=0,2n+1$, it follows that

$$|a_{0k}J_{10}(z)| = \frac{|J_{10}(z)|}{2|J_{10}(1)|}, \quad |a_{1k}J_{11}(z)| = \frac{|J_{11}(z)|}{2|J_{11}(1)|},$$

which implies that for $k=0,2n+1$

$$|A_k(z)| \le |K_nP_n(x)|\big[|a_{0k}J_{10}(z)| + |a_{1k}J_{11}(z)|\big] = \mathcal{O}(1). \tag{4.18}$$

For $k=1,2,\cdots,2n$,

$$\begin{aligned} |A_k(z)| \le & \Big|\frac{(1-x^2)^{\frac{1}{2}}P_n'(x)L_k(z)}{(1-x_k^2)^{\frac{1}{2}}P_n'(x_k)}\Big| + \Big|\frac{P_n(x)}{(1-x_k^2)P_n'(x_k)^2}\Big|\big[|J_k(z)| + |a_{0k}J_{10}(z)| + |a_{1k}J_{11}(z)|\big] \\ \equiv & I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.19}$$

As in the last lemma for $z=x+i\sqrt{1-x^2}$ together with $0\le \arg z\le\pi$ and $k=1,2,\cdots,n$,

$$|(z-z_k)|=\sqrt{2}|(1-xx_k)-(1-x^2)^{\frac{1}{2}}(1-x_k^2)^{\frac{1}{2}}|^{\frac{1}{2}}, \tag{4.20}$$

and

$$|(1-x^2)^{\frac{1}{2}}(1-x_k^2)^{\frac{1}{2}}|\le|(1-xx_k)| \quad \text{for } \ |x|\le 1,\ |x_k|\le 1. \tag{4.21}$$

Hence,

$$\begin{aligned}
I_1 &\le \Big|\frac{(1-x^2)^{\frac{1}{2}}P_n(x)P_n'(x)}{(1-x_k^2)(z-z_k)P_n'(x_k)^2}\Big| \\
&\le \frac{|(1-x^2)^{\frac{1}{2}}\left[(1-xx_k)+(1-x^2)^{\frac{1}{2}}(1-x_k^2)^{\frac{1}{2}}\right]^{\frac{1}{2}}P_n'(x)||P_n(x)|}{|\sqrt{2}(1-x_k^2)(x-x_k)|P_n'(x_k)^2} \\
&\le \frac{|(1-x^2)^{\frac{1}{2}}(1-xx_k)^{\frac{1}{2}}P_n'(x)||P_n(x)|}{|(1-x_k^2)(x-x_k)|P_n'(x_k)^2}.
\end{aligned}$$

When $|x-x_k|\ge(1-x_k^2)/4$, we have

$$|(1-x^2)|\le|1-x_k^2|+|x^2-x_k^2|\le 6|x-x_k|, \tag{4.22a}$$

$$|1-xx_k|\le|1-x_k^2|+|x_k^2-xx_k|\le 5|x-x_k|, \tag{4.22b}$$

which by (2.8a) and (2.9)-(2.10b) implies

$$I_1\le\Big|\frac{\sqrt{30}P_n(x)P_n'(x)}{(1-x_k^2)P_n'(x_k)^2}\Big|\le\frac{\sqrt{30}(n+1)k}{2c^2n}. \tag{4.23}$$

Also, since

$$\begin{aligned}
|J_k(z)| &\le \sup_{|t|\le 1}\Big|(t^2-1)\frac{tH_{2n-2}'(t)+c_kH_{2n-2}(t)}{t-z_k}\Big|\Big|\int_0^1 t^{n-1}dt\Big| \\
&\le \frac{1}{n}\sup_{|t|\le 1}\Big|(t^2-1)\frac{tH_{2n-2}'(t)+c_kH_{2n-2}(t)}{t-z_k}\Big| \\
&\le \frac{1}{n}\sup_{|t|\le 1}\Big[\Big|\frac{t(t^2-1)H_{2n-2}'(t)}{t-z_k}\Big|+\Big|\frac{z_kH_{2n-2}'(z_k)}{H_{2n-2}(z_k)}\frac{(t^2-1)H_{2n-2}(t)}{t-z_k}\Big|\Big] \\
&\equiv \Big(\frac{1}{n}\Big)\sup_{|t|\le 1}[|I_{21}|+|I_{22}|].
\end{aligned} \tag{4.24}$$

Now I_{21}, can be written as,

$$\begin{aligned}
I_{21} &= \frac{K_n^*t^{n-2}(t^2-1)[(t^2-1)P_n''(x)+2(n-1)tP_n'(x)]}{2(t-z_k)} \\
&= \frac{K_n^*t^{n-2}[\{(n-1)(t^2-1)-2(t^2+1)\}tP_n'(x)+2n(n+1)t^2P_n(x)]}{(t-z_k)} \\
&\equiv I_{211}+I_{212}+I_{213},
\end{aligned} \tag{4.25}$$

where,

$$
\begin{aligned}
|I_{211}| &= \Big|\frac{K_n^*(n-1)(t^2-1)P_n'(x)}{(t-z_k)}\Big| \\
&\le \frac{K_n^*(n-1)|2(1-x^2)^{\frac{1}{2}}[(1-xx_k)+(1-x^2)^{\frac{1}{2}}(1-x_k^2)^{\frac{1}{2}}]^{\frac{1}{2}}P_n'(x)|}{|\sqrt{2}(x-x_k)|} \\
&\le \frac{2K_n^*(n-1)|(1-x^2)^{\frac{1}{2}}(1-xx_k)^{\frac{1}{2}}P_n'(x)|}{|(x-x_k)|} \\
&\le 2\sqrt{30}(n-1)K_n^*|P_n'(x)|.
\end{aligned}
\tag{4.26}
$$

Similarly,

$$
|I_{212}| = \Big|\frac{2K_n^* t^{n-1}(t^2+1)P_n'(x)}{(t-z_k)}\Big| \le \frac{4K_n^*|(1-xx_k)^{\frac{1}{2}}P_n'(x)|}{|(x-x_k)|} \le \frac{8\sqrt{5}K_n^*|P_n'(x)|}{|\sqrt{(1-x_k^2)}|},
\tag{4.27}
$$

and

$$
\begin{aligned}
|I_{213}| &= \Big|\frac{2K_n^* n(n+1)P_n(x)}{(t-z_k)}\Big| \le \frac{2K_n^* n(n+1)|(1-xx_k)^{\frac{1}{2}}P_n(x)|}{|(x-x_k)|} \\
&\le \frac{4\sqrt{5}n(n+1)K_n^*|P_n(x)|}{|\sqrt{1-x_k^2}|}.
\end{aligned}
\tag{4.28}
$$

Hence, by (4.26), (4.27) and (4.28), we have

$$
|I_{21}| \le 2\sqrt{5}K_n^*\Big[\sqrt{6}(n-1)|P_n'(x)| + \frac{4|P_n'(x)|+2n(n+1)|P_n(x)|}{|\sqrt{(1-x_k^2)}|}\Big].
\tag{4.29}
$$

Again,

$$
\begin{aligned}
|I_{22}| &= \Big|\frac{(t^2-1)H_{2n-2}(t)}{(t-z_k)}\Big|\Big|\frac{[2(n-1)z_kP_n'(x_k)+(z_k^2-1)P_n''(x_k)]}{2z_kP_n'(x_k)}\Big| \\
&\le \frac{2K_n^*|(1-x^2)^{\frac{1}{2}}(1-xx_k)^{\frac{1}{2}}P_n'(x)|}{|(x-x_k)|}\Big[(n-1)+\frac{2}{|(z_k^2-1)|}\Big] \\
&\le 2\sqrt{30}K_n^*\Big[(n-1)+\frac{1}{|\sqrt{(1-x_k^2)}|}\Big]|P_n'(x)|.
\end{aligned}
\tag{4.30}
$$

Therefore, by (4.24), (4.29) and (4.30), we have

$$
\begin{aligned}
I_2 &\le \Big|\frac{2\sqrt{5}P_n(x)}{n(1-x_k^2)P_n'(x_k)^2}\Big|\Big[\Big\{2\sqrt{6}(n-1)+\frac{\sqrt{6}+4}{|\sqrt{(1-x_k^2)}|}\Big\}|P_n'(x)|+\frac{2n(n+1)|P_n(x)|}{|\sqrt{(1-x_k^2)}|}\Big] \\
&\le \Big[\frac{c_2(n^2-1)k}{n^2}+\frac{c_3(n+1)}{n}\Big],
\end{aligned}
\tag{4.31}
$$

where c_2, c_3 are constants independent of n and x. Also,

$$I_3=\Big|\frac{P_n(x)}{K_n^*(1-x_k^2)P_n'(x_k)^2}\Big||a_{0k}J_{10}(z)|\le c_5 I_2, \tag{4.32}$$

and

$$I_4=\Big|\frac{P_n(x)}{K_n^*(1-x_k^2)P_n'(x_k)^2}\Big||a_{1k}J_{11}(z)|\le c_6 I_2. \tag{4.33}$$

Hence by using (4.23), (4.31)-(4.33) in (4.19), (4.17) follows in this case.

Now let $|x-x_k|\le(1-x_k^2)/4$, then

$$|(1-x^2)|\le|1-x_k^2|+|x^2-x_k^2|\le\frac{3}{2}|1-x_k^2|, \tag{4.34a}$$

$$|1-xx_k|\le|1-x_k^2|+|x_k^2-xx_k|\le\frac{5}{4}|1-x_k^2|, \tag{4.34b}$$

$$|(1-x^2)|\ge|1-x_k^2|-|x^2-x_k^2|\ge\frac{1}{2}|1-x_k^2|. \tag{4.34c}$$

In this case, (4.23), has the form

$$I_1\le\frac{|(1-x_k^2)P_n'(x)P_n(x)|}{|(x-x_k)|P_n'(x_k)^2}\le c_7 l_k^*(x), \tag{4.35}$$

where c_7 is a constant and $l_k^*(x)$ is given by (2.11). Similarly, in this case, by using (4.34a) and (4.34b), (4.26), (4.27), (4.28) and (4.30) reduce to

$$|I_{211}|\le\frac{2K_n^*(n-1)|(1-x^2)^{\frac12}(1-xx_k)^{\frac12}P_n'(x)|}{|(x-x_k)|}\le\frac{\sqrt{30}K_n^*(n-1)|(1-x_k^2)P_n'(x)|}{2|(x-x_k)|}, \tag{4.36a}$$

$$|I_{212}|\le\frac{4K_n^*|(1-xx_k)^{\frac12}P_n'(x)|}{|(x-x_k)|}\le\frac{2\sqrt{5}K_n^*|\sqrt{(1-x_k^2)}P_n'(x)|}{|(x-x_k)|}, \tag{4.36b}$$

$$|I_{213}|\le\frac{2K_n^*n(n+1)|(1-xx_k)^{\frac12}P_n(x)|}{|(x-x_k)|}\le\frac{\sqrt{5}n(n+1)K_n^*|\sqrt{1-x_k^2}P_n(x)|}{|(x-x_k)|}, \tag{4.36c}$$

and

$$\begin{aligned}|I_{22}|&\le\frac{2K_n^*|(1-x^2)^{\frac12}(1-xx_k)^{\frac12}P_n'(x)|}{|(x-x_k)|}\Big[(n-1)+\frac{2}{|(z_k^2-1)|}\Big]\\&\le\frac{\sqrt{30}K_n^*|(1-x_k^2)P_n'(x)|}{2|(x-x_k)|}\Big[(n-1)+\frac{1}{|\sqrt{(1-x_k^2)}|}\Big].\end{aligned} \tag{4.37}$$

Thus by using (4.34c), (4.36a)-(4.37), (4.31) in this case, has the form

$$I_2 \leq \Big| \frac{2^{1/4}\sqrt{5} l_k^*(x)}{n(1-x_k^2)^{3/4} P_n'(x_k)} \Big| \Big[\sqrt{2}(1-x^2)^{3/4} |P_n'(x)| \Big\{ \frac{(4+\sqrt{6})}{2\sqrt{(1-x_k^2)}} + \sqrt{6}(n-1) \Big\} ++ n(n+1)(1-x^2)^{1/4} |P_n(x)| \Big],$$

which by (2.8b), (2.8d) (2.9) and (2.10a) gives

$$I_2 \leq \Big[\frac{c_8}{k} + \frac{c_9(n-1)}{n} + \frac{c_{10}(n+1)}{n} \Big] |l_k^*(x)|, \tag{4.38}$$

where c_8,c_9,c_{10} are constants. Hence by virtue of (2.13), (4.35), (4.38), (4.32), (4.33) and (4.19), (4.17) follows in this case as well. □

To prove our Theorem 3.7 we shall need the following Lemma [12, 13]:

Lemma 4.4. *Let $f(z)$ be continuous in the region $|z| \leq 1$ and analytic in $|z| < 1$. Let $\omega(f,1/n)$ be modulus of continuity of $f(e^{i\theta})$, $0 \leq \theta \leq 2\pi$. Then there exists a polynomial $F_{4n-1}(z)$ of degree $\leq 4n-1$ such that*

$$|f(z) - F_{4n-1}(z)| \leq c\omega\Big(f, \frac{1}{n}\Big), \tag{4.39}$$

and

$$|F_{4n-1}^{(m)}(z)| \leq c n^m \omega\Big(f, \frac{1}{n}\Big). \tag{4.40}$$

5 Proof of Theorem 3.7

Proof. The polynomial $F_{4n-1}(z)$ satisfying the Lemma 4.4 can be represented as:

$$F_{4n-1}(z) = \sum_{k=0}^{2n+1} F_{4n-1}(z_k) A_k(z) + \sum_{k=1}^{2n} F_{4n-1}'(w_k) B_k(z), \tag{5.1}$$

then,

$$\begin{aligned} |R_{4n-1}(z) - f(z)| &\leq |R_{4n-1}(z) - F_{4n-1}(z)| + |F_{4n-1}(z) - f(z)| \\ &\leq \sum_{k=0}^{2n+1} |F_{4n-1}(z_k) - f(z_k)| |A_k(z)| + \sum_{k=1}^{2n} \big[|F_{4n-1}'(w_k)| \\ &\quad + |\beta_k| \big] |B_k(z)| + |F_{4n-1}(z) - f(z)|. \end{aligned} \tag{5.2}$$

By lemmas 4.2, 4.3 and 4.4 the Theorem follows. □

6 Numerical example

As a numerical example of the effectiveness of the interpolation in the points Λ given by (1.2) we computed the maximum (uniform) error in

1. the Lagrange interpolation for the simple function $f(z)=\exp(z)$, $z\in\mathbb{C}$.
2. the Barycentric [2] form of Lagrange interpolation for the simple function $f(z)=1/(1+25z^2)$, $z\in\mathbb{C}$.

For comparison purposes, Table 1 and Table 2 list the maximum error over the unit circle in the above two cases when the interpolation points are chosen to be

1. projected zeros of the Chebyshev polynomial $T_N(x)$, rescaled so that the first and last zeros coincide, respectively, with -1 and 1, to the unit circle.
2. the set of points Λ given by (1.2).
3. the N^{th} roots of unity.

We see from Tables 1 and 2 that in both the examples considered here the maximum absolute error of interpolation is least for the listed values of N if the interpolation points are chosen to be the points from the set Λ in comparison to the projected Chebyshev points or the roots of unity.

Table 1: Maximum error of Lagrange Interpolation in N points for $f(z)=\exp(z)$, $z\in\mathbb{C}$.

N	**Chebyshev**	Λ	**Roots of unity**
2	0.00181271	0.00144706	0.00150764
4	3.09957×10^{-11}	1.74393×10^{-11}	3.1805×10^{-8}
6	8.66746×10^{-16}	1.65861×10^{-16}	1.81413×10^{-13}
7	5.38654×10^{-16}	7.88428×10^{-17}	5.40216×10^{-16}
9	1.62699×10^{-16}	6.92666×10^{-17}	3.87692×10^{-16}
11	1.59809×10^{-16}	5.70193×10^{-17}	2.94879×10^{-16}
13	3.87991×10^{-17}	3.52045×10^{-17}	2.88607×10^{-16}
15	3.2634×10^{-17}	2.2593×10^{-17}	2.53211×10^{-16}
19	2.7823×10^{-17}	1.8971×10^{-17}	2.13673×10^{-16}
25	2.0368×10^{-17}	1.6523×10^{-17}	1.97628×10^{-16}
27	1.8781×10^{-17}	1.3218×10^{-17}	1.45321×10^{-16}
31	1.4343×10^{-17}	1.1564×10^{-17}	1.13345×10^{-16}
35	1.1293×10^{-17}	1.0238×10^{-17}	8.8843×10^{-17}

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Table 2: Maximum error of Barycentric form of Lagrange Interpolation in N points for $f(z)=1/(1+25z^2)$, $z\in\mathcal{C}$.

N	**Chebyshev**	Λ	**Roots of unity**
2	0.0309206	0.0255228	0.0315358
3	0.0124176	0.0116602	0.019307
5	0.00196875	0.00186229	0.0024008
8	0.000122967	0.000116439	0.00015853
11	7.73215×10^{-6}	7.46413×10^{-6}	1.03497×10^{-5}
14	4.90258×10^{-7}	4.74106×10^{-7}	6.68694×10^{-7}
17	3.12666×10^{-8}	3.04738×10^{-8}	4.31352×10^{-8}
20	8.38225×10^{-10}	8.22371×10^{-10}	2.49053×10^{-9}
23	4.6312×10^{-11}	4.58926×10^{-11}	1.64524×10^{-10}
25	8.11901×10^{-12}	7.96671×10^{-12}	2.6684×10^{-11}
28	5.80264×10^{-13}	5.69955×10^{-13}	1.76953×10^{-12}
30	9.93097×10^{-14}	9.76493×10^{-14}	2.88169×10^{-14}
34	2.87588×10^{-15}	2.82716×10^{-15}	7.62225×10^{-15}
38	8.48963×10^{-17}	8.24678×10^{-17}	1.98575×10^{-16}

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