Analysis in Theory and Applications
Anal. Theory Appl., Vol. **30**, No. 2 (2014), pp. 193-204
DOI: 10.4208/ata.2014.v30.n2.5

Necessary and Sufficient Conditions of Doubly Weighted Hardy-Littlewood-Sobolev Inequality

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Received 9 December 2013; Accepted (in revised version) 24 February 2014

Available online 30 June 2014

Abstract. Using product and convolution theorems on Lorentz spaces, we characterize the sufficient and necessary conditions which ensure the validity of the doubly weighted Hardy-Littlewood-Sobolev inequality. It should be pointed out that we consider whole ranges of p and q, i.e., $0<p\leq\infty$ and $0<q\leq\infty$.

Key Words: Hölder's inequality, Young's inequality, Hardy-Littlewood-Sobolev inequality, Lorentz space.

AMS Subject Classifications: 42B20, 42B35

1 Introduction

The Riesz potential operator

$$I_\alpha(f)(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy,$$

also called fractional integral operator, is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, provided that $1<p<q<\infty$ and $0<\alpha<n$ satisfy

$$\frac{1}{p}-\frac{\alpha}{n}=\frac{1}{q}.$$

According to the property of L^p space, we have $(L^p)^*=L^{p'}$, $1\leq p<\infty$. Thus the $L^p\to L^q$-boundedness of I_s is equivalent to Theorem 1.1.

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Theorem 1.1. *Suppose that $f\in L^{p_1}(\mathbb{R}^n)$ and $g\in L^{p_2}(\mathbb{R}^n)$. If $1<p_1,p_2<\infty$ and $0<\alpha<n$ satisfy*

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{\alpha}{n}=2,$$

then

$$\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x_1)g(x_2)}{|x_1-x_2|^\alpha}dx_1dx_2\Big|\le C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)} \tag{1.1}$$

holds.

Theorem 1.1 was obtained by Hardy and Littlewood for the case $n=1$ in [2] and by Sobolev for general n in [6]. Therefore, the inequality (1.1) is usually called the Hardy-Littlewood-Sobolev inequality in the literature. Stein and Weiss [7] considered the doubly weighted Hardy-Littlewood-Sobolev inequality and obtained Theorem 1.2 as follows.

Theorem 1.2. *If $1<p,q<\infty$ and α, β, γ satisfy the following conditions,*

$$\frac{1}{p}+\frac{1}{q}+\frac{\alpha+\beta+\gamma}{n}=2, \tag{1.2a}$$

$$\alpha+\gamma\ge 0,\quad \alpha<\frac{n}{p'},\quad \gamma<\frac{n}{q'},\quad \beta<n, \tag{1.2b}$$

$$\frac{1}{p}+\frac{1}{q}\ge 1, \tag{1.2c}$$

then

$$\Big|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^\alpha|x-y|^\beta|y|^\gamma}dxdy\Big|\le C\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)} \tag{1.3}$$

holds for $f\in L^p(\mathbb{R}^n)$ and $g\in L^q(\mathbb{R}^n)$.

Remark 1.1. In fact, both conditions (1.2a) and (1.2b) can easily imply

$$0<\beta<n. \tag{1.4}$$

It was the reason why Stein regarded (1.4) as one of the conditions in [7].

Clearly, the conditions (1.2a), (1.2b) and (1.2c) are sufficient conditions which ensure the validity of the doubly weighted Hardy-Littlewood-Sobolev inequality. In this paper, we use novel methods and ideas to investigate the sufficient and necessary conditions which makes the doubly weighted Hardy-Littlewood-Sobolev inequality hold. In particular, we apply the properties of product and convolution of two functions on Lorentz spaces to establish the doubly weighted Hardy-Littlewood-Sobolev inequality. It should be pointed out that we will consider whole ranges of p and q, i.e., $0<p\le\infty$ and $0<q\le\infty$, which cover Theorem 1.1 and Theorem 1.2, and provide us with new insightful information. The Hardy-Littlewood-Sobolev inequality is widely used in Harmonic Analysis, as well as in partial differential equation.

Now we formulate our main results as follows:

Theorem 1.3. *Let $f\in L^p(\mathbb{R}^n)$ and $g\in L^q(\mathbb{R}^n)$, $1<p,q<\infty$. Define the bilinear operator*

$$T(f,g)(x,y):=\frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{\beta}|y|^{\gamma}}. \tag{1.5}$$

The operator T is bounded from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$ if and only if the conditions (1.2a), (1.2b) *and* (1.2c) *hold simultaneously.*

We note from Theorem 1.3 that p and q are in $(1,\infty)$. To study the case of endpoints with respect to p and q, we define an operator as

$$\mathcal{T}g(x):=\frac{1}{|x|^{\alpha}}\int_{\mathbb{R}^n}\frac{g(y)}{|x-y|^{\beta}|y|^{\gamma}}dy. \tag{1.6}$$

Let $\mathcal{T}^t$ denote the transpose operator of $\mathcal{T}$. Clearly the basic properties of functional analysis imply that

$$\mathcal{T}^t f(y):=\frac{1}{|y|^{\gamma}}\int_{\mathbb{R}^n}\frac{f(x)}{|x|^{\alpha}|x-y|^{\beta}}dx. \tag{1.7}$$

For the other cases with respect to p and q, we have the following theorems.

Theorem 1.4. *Let $0<p<1$ or $0<q<1$. The operator T defined by* (1.5) *is not bounded from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$ for every real numbers α, β and γ.*

Theorem 1.5. *Let $p=1$ and $q=1$. The operator T defined by* (1.5) *is bounded from $L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$, if and only if $\alpha=\beta=\gamma=0$.*

Theorem 1.6. *Let $p=\infty$ and $1<q\le\infty$. The operator T defined by* (1.5) *is not bounded from $L^\infty(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$ for every real numbers α, β and γ.*

Theorem 1.7. *Let $1<p\le\infty$ and $q=1$. The operator T defined by* (1.5) *is bounded from $L^p(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$ if and only if*

$$\frac{1}{p}+\frac{\alpha+\beta+\gamma}{n}=1 \tag{1.8}$$

and

$$\gamma<0,\quad \alpha+\gamma>0,\quad \alpha<\frac{n}{p'}. \tag{1.9}$$

Remark 1.2. In fact, we can use a figure to indicate a domain where the operator $T_{\alpha,\lambda,\beta}$ may be bounded from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$. Theorems 1.3, 1.4, 1.5, 1.6 and 1.7 clearly imply that the domain is just the closed triangle area denoted by $\triangle EFG$ in the following Fig. 1.

We have known that when $(1/p,1/q)\in\triangle EFG$ and α, λ, β satisfy some conditions, the operator $T_{\alpha,\lambda,\beta}$ is bounded from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$. An important and interesting question is how to find the sharp bound of the operator $T_{\alpha,\lambda,\beta}$.

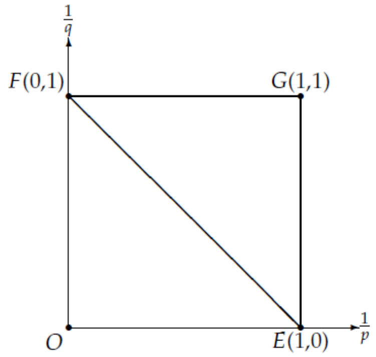

Figure 1: The closed triangle area $\triangle EFG$.

2 Some lemmas

To prove our theorems, we first provide some definitions and lemmas which will be used in the sequel. Some lemmas can be found in some books and papers, so we omit their proofs.

Definition 2.1. Let f be a measurable function defined on (X,μ).
(I) The distribution function of f is the function d_f defined on $[0,\infty)$ as follows,

$$d_f(\alpha):=\mu\big(\{x\in X:|f(x)|>\alpha\}\big).$$

(II) The decreasing rearrangement of f is the function f^* defined on $[0,\infty)$ by

$$f^*(t)=\inf\{s>0:d_f(s)\le t\}.$$

Definition 2.2. Let f be a measurable function defined on (X,μ) and $0<p<\infty$, $0<q\le\infty$. Define Lorentz space as

$$L^{p,q}(X,\mu):=\Big\{f:\|f\|_{L^{p,q}(X,\mu)}<\infty\Big\},$$

where

$$\|f\|_{L^{p,q}(X,\mu)}:=\begin{cases}\left(\int_0^\infty\left(t^{\frac{1}{p}}f^*(t)\right)^q\frac{dt}{t}\right)^{\frac{1}{q}}, & 0<q<\infty,\\ \sup_{t>0}t^{\frac{1}{p}}f^*(t), & q=\infty,\end{cases}\tag{2.1}$$

and when $p=\infty$, we only define $L^{\infty,\infty}=L^\infty$.

For the Lorentz space $L^{p,q}(X,\mu)$, the following statements are well-known.

$$L^{p,p}(X,\mu)=L^p(X,\mu),\tag{2.2a}$$
$$L^{p,\infty}(X,\mu)=\text{weak }L^p(X,\mu),\tag{2.2b}$$

and

$$L^{p,r}(X,\mu)\subset L^{p,q}(X,\mu), \tag{2.3}$$

if $0<r<q\le\infty$. In addition, if $f\in L^{p,r}(X,\mu)$, then

$$\|f\|_{L^{p,q}(X,\mu)}\le C\|f\|_{L^{p,r}(X,\mu)}. \tag{2.4}$$

As some extensions of Hölder's inequality and Young's inequality on L^p space, Richard O'Neil constructed product theorem and convolution theorem on Lorentz spaces in [5], and Richard A. Hunt also considered these questions in [3]. These results are denoted as Lemma 2.1 and Lemma 2.2 which will be useful in the next section.

Lemma 2.1. *If $f\in L^{p_0,q_0}$ and $g\in L^{p_1,q_1}$, then $fg\in L^{p,q}$, and*

$$\|fg\|_{L^{p,q}}\le C\|f\|_{L^{p_0,q_0}}\|g\|_{L^{p_1,q_1}},$$

where $1/p=1/p_0+1/p_1$ and $1/q=1/q_0+1/q_1$.

Lemma 2.2. *If G is a locally compact group with a left invariant Haar measure λ, $f\in L^{p_0,q_0}(G)$ and $g\in L^{p_1,q_1}(G)$, then $f*g\in L^{p,q}(G)$, and*

$$\|f*g\|_{L^{p,q}(G)}\le C\|f\|_{L^{p_0,q_0}(G)}\|g\|_{L^{p_1,q_1}(G)},$$

where $0<1/p=1/p_0+1/p_1-1<1$, $1<p_0,p_1<\infty$ and $1/q=1/q_0+1/q_1$.

Lemma 2.3. *The function $\chi_{|x|<1/2}(x)|x|^{-n/p}|\log|x||^{-\lambda}$ is in $L^p(\mathbb{R}^n)$ if and only if $\lambda>1/p$, so is the function $\chi_{|x|>2}(x)|x|^{-n/p}(\log|x|)^{-\lambda}$.*

Lemma 2.4 (see [1])**.** *Let $h(x)=|x|^{-d}$, $x\in\mathbb{R}^n$ and $0<d<n$. Then*

$$\hat{h}(\xi)=\pi^{d-\frac{n}{2}}\frac{\Gamma\left(\frac{n-d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}|\xi|^{d-n}.$$

Lemma 2.5 (see [1])**.** *Let $h_1(x)=|x|^{-\alpha}$, $h_2(x)=|x|^{-\beta}$ and $x\in\mathbb{R}^n$. If $\alpha<n$, $\beta<n$ and $\alpha+\beta>n$, then*

$$h_1*h_2(x)=C_{\alpha,\beta,n}|x|^{n-\alpha-\beta}, \tag{2.5}$$

where

$$C_{\alpha,\beta,n}=\pi^{\frac{n}{2}}\frac{\Gamma(\frac{n-\alpha}{2})\Gamma(\frac{n-\beta}{2})\Gamma(\frac{\alpha+\beta-n}{2})}{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})\Gamma(n-\frac{\alpha+\beta}{2})}. \tag{2.6}$$

In the paper, we always regard $C_{\alpha,\beta,n}$ as the constant defined as (2.6).

3 The proof of theorems

Proof of Theorem 1.3. We first prove the sufficiency of Theorem 1.3. Assume that the three conditions (1.2a), (1.2b) and (1.2c) all hold. Without loss of generality, we always let $f,g\geq 0$.

For the sake of clarity in writing, we first define three functions $h_1(x)=|x|^{-\alpha}$, $h_2(x)=|x|^{-\beta}$ and $h_3(x)=|x|^{-\gamma}$ for $x\in\mathbb{R}^n$. Since the condition (1.2b) holds, we have $\alpha+\gamma\geq 0$.

For the case $\alpha\geq 0$ and $\gamma\geq 0$, it easily follows that $h_1\in L^{n/\alpha,\infty}(\mathbb{R}^n)$ and $h_3\in L^{n/\gamma,\infty}(\mathbb{R}^n)$. It naturally implies from (1.4) that $h_2\in L^{n/\beta,\infty}(\mathbb{R}^n)$. Applying Lemma 2.1 to fh_1 and gh_3, we have that

$$\begin{aligned}\|fh_1\|_{L^{\frac{1}{\frac{1}{p}+\frac{\alpha}{n}},p}(\mathbb{R}^n)}&\leq C\|f\|_{L^p(\mathbb{R}^n)}\|h_1\|_{L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)},\\ \|gh_3\|_{L^{\frac{1}{\frac{1}{q}+\frac{\gamma}{n}},q}(\mathbb{R}^n)}&\leq C\|g\|_{L^q(\mathbb{R}^n)}\|h_3\|_{L^{\frac{n}{\gamma},\infty}(\mathbb{R}^n)}.\end{aligned}$$

Using Lemma 2.1 and Lemma 2.2 together with the three conditions (1.2a), (1.2b) and (1.2c), we conclude that

$$\begin{aligned}\|T(f,g)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}f(x)h_1(x)h_2(x-y)g(y)h_3(y)dxdy\\ &=\int_{\mathbb{R}^n}(gh_3)(y)(fh_1)*h_2(y)dy\\ &\leq C\big\|gh_3\big\|_{L^{\frac{1}{\frac{1}{q}+\frac{\gamma}{n}},q}(\mathbb{R}^n)}\big\|(fh_1)*h_2\big\|_{L^{\frac{1}{\frac{1}{q'}-\frac{\gamma}{n}},q'}(\mathbb{R}^n)}\\ &\leq C\|g\|_{L^q(\mathbb{R}^n)}\big\|h_3\big\|_{L^{\frac{n}{\gamma},\infty}(\mathbb{R}^n)}\big\|(fh_1)*h_2\big\|_{L^{\frac{1}{\frac{1}{q'}-\frac{\gamma}{n}},q'}(\mathbb{R}^n)}\\ &=C\|g\|_{L^q(\mathbb{R}^n)}\big\|(fh_1)*h_2\big\|_{L^{\frac{1}{\frac{1}{p}+\frac{\alpha+\beta}{n}-1},q'}(\mathbb{R}^n)}\\ &\leq C\|g\|_{L^q(\mathbb{R}^n)}\big\|fh_1\big\|_{L^{\frac{1}{\frac{1}{p}+\frac{\alpha}{n}},q'}(\mathbb{R}^n)}\big\|h_2\big\|_{L^{\frac{n}{\beta},\infty}(\mathbb{R}^n)}\\ &\leq C\|g\|_{L^q(\mathbb{R}^n)}\big\|fh_1\big\|_{L^{\frac{1}{\frac{1}{p}+\frac{\alpha}{n}},p}(\mathbb{R}^n)} &(3.1)\\ &\leq C\big\|h_1\big\|_{L^{\frac{n}{\alpha},\infty}(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}\\ &=C\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}. &(3.2)\end{aligned}$$

The inequality (3.1) results from the inequality (2.4), i.e., $\|f\|_{L^{p,r}}\leq C\|f\|_{L^{p,s}}$, when $r\geq s$.

The other case is that one of α and γ is less than zero, and another is bigger than zero, since $\alpha+\gamma\geq 0$. Without loss of generality, we assume $\gamma<0$, then it follows that

$$|y|^{-\gamma}\leq(|x|+|x-y|)^{-\gamma}\leq C\big(|x|^{-\gamma}+|x-y|^{-\gamma}\big).$$

So we have

$$\|T(f,g)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}\le C\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^{\alpha+\gamma}|x-y|^{\beta}}dxdy+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{\beta+\gamma}}dxdy\Big).$$

For the integral

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^{\alpha+\gamma}|x-y|^{\beta}}dxdy,$$

set $\alpha'=\alpha+\gamma$, $\beta'=\beta$ and $\gamma'=0$. It is easy to check that the indices p,q,α',β' and γ' satisfy the conditions (1.2a), (1.2b) and (1.2c), furthermore, $\alpha'\ge 0$ and $\gamma'\ge 0$. Thus we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^{\alpha+\gamma}|x-y|^{\beta}}dxdy\le C\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}. \tag{3.3}$$

The conditions (1.2a) and (1.2b) imply $\beta+\gamma>0$. Using the similar method as in the estimate of the inequality (3.3), we can easily obtain

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^{\alpha}|x-y|^{\beta+\gamma}}dxdy\le C\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}. \tag{3.4}$$

Combining the inequalities (3.2), (3.3) with (3.4), we have proved the sufficiency of Theorem 1.3. Next we will prove the necessity of Theorem 1.3.

We first show that the condition (1.2a) holds by dilation. Define

$$f_{p,\varepsilon}(x)=\varepsilon^{-\frac{n}{p}}f\Big(\frac{x}{\varepsilon}\Big),\quad \varepsilon>0,\quad x\in\mathbb{R}^n.$$

We clearly have

$$\|f_{p,\varepsilon}\|_{L^p(\mathbb{R}^n)}=\|f\|_{L^p(\mathbb{R}^n)}. \tag{3.5}$$

Now replacing f by $f_{p,\varepsilon}$ and g by $g_{q,\varepsilon}$, we deduce that

$$\begin{aligned}\|T(f_{p,\varepsilon},g_{q,\varepsilon})\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}=&\varepsilon^{2n-\frac{n}{p}-\frac{n}{q}-\alpha-\beta-\gamma}\|T(f,g)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}\\ \le& C\varepsilon^{2n-\frac{n}{p}-\frac{n}{q}-\alpha-\beta-\gamma}\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}\\ =&C\varepsilon^{2n-\frac{n}{p}-\frac{n}{q}-\alpha-\beta-\gamma}\|f_{p,\varepsilon}\|_{L^p(\mathbb{R}^n)}\|g_{p,\varepsilon}\|_{L^q(\mathbb{R}^n)}.\end{aligned}$$

We thus have that the norm of the operator T satisfies

$$\|T\|\le C\varepsilon^{2n-\frac{n}{p}-\frac{n}{q}-\alpha-\beta-\gamma}.$$

If $2n-n/p-n/q-\alpha-\beta-\gamma>0$, and let $\varepsilon\to 0+$, then $\|T\|=0$. If $2n-n/p-n/q-\alpha-\beta-\gamma<0$, and let $\varepsilon\to\infty$, then $\|T\|=0$. This means that if $2n-n/p-n/q-\alpha-\beta-\gamma\neq 0$, then T has to be **0** operator. Therefore, the condition (1.2a) is necessary.

Now we show that the condition (1.2b) is also necessary. To this end, we will prove that all of the conditions $\alpha+\gamma\ge 0$, $\alpha<n/p'$, $\gamma<n/q'$ and $\beta<n$ are necessary, respectively.

If $\alpha+\gamma<0$, set $f_N=\chi_{B_1(Ne_1)}$ and $g_N=\chi_{B_1((N+3)e_1)}$, where $B_r(x)$ denotes a ball with center x and radius r, $e_1=(1,0,\cdots,0)$, $N\in\mathbb{N}$. It follows that

$$\begin{aligned}\|T\|=&\sup_{\|f\|_{L^p(\mathbb{R}^n)}\neq 0,\|g\|_{L^p(\mathbb{R}^n)}\neq 0}\frac{\|T(f,g)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}}\\ \geq&\frac{\|T(f_N,g_N)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}}{\|f_N\|_{L^p(\mathbb{R}^n)}\|g_N\|_{L^q(\mathbb{R}^n)}}\geq CN^{-(\alpha+\gamma)}\to\infty,\end{aligned}\tag{3.6}$$

as $N\to\infty$. Clearly the inequality (3.6) contradicts the boundedness of T.

If $\alpha\geq n/p'$, then $n/p+\alpha+\beta\leq n$ due to (1.2a). Let $f(x)=\chi_{|x|>3}(x)|x|^{-n/p}(\log|x|)^{-1}$ and $g(y)=\chi_{1<|y|<2}(y)$. We easily have $f\in L^p(\mathbb{R}^n)$ and $g\in L^q(\mathbb{R}^n)$. However, it follows that

$$\|T(f,g)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}\geq C\int_{|x|>3}|x|^{-(\frac{n}{p}+\alpha+\beta)}(\log|x|)^{-1}dx=\infty.$$

This shows that $\alpha<n/p'$ is also necessary.

The same argument implies that $\gamma<n/q'$ is also necessary. If $\beta\geq n$, and we merely choose $f=g=\chi_{1<|\cdot|<2}$, then a simple computation implies that

$$\|T(f,g)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}=\infty.$$

This means that $\beta<n$ is also necessary.

Next it remains to prove that the condition (1.2c) is also necessary. Obviously the condition (1.2c) is equivalent to $p\leq q'$.

Note that the boundedness of T from $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$ is equivalent to that of $\mathcal{T}$ defined by (1.6) from $L^q(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$, and also is equivalent to that of $\mathcal{T}^t$ defined by (1.7) from $L^p(\mathbb{R}^n)$ to $L^{q'}(\mathbb{R}^n)$.

Assume $p>q'$. Set $f(x)=\chi_{|x|>3}(x)|x|^{-n/p}(\log|x|)^{-1/q'}$, then by Lemma 2.3, we clearly have $f\in L^p(\mathbb{R}^n)$. We conclude that

$$\begin{aligned}\|\mathcal{T}^tf\|^{q'}_{L^{q'}(\mathbb{R}^n)}=&\int_{\mathbb{R}^n}\Big(\int_{|x|>3}|x|^{-\frac{n}{p}}(\log|x|)^{-\frac{1}{q'}}|x|^{-\alpha}|x-y|^{-\beta}|y|^{-\gamma}dx\Big)^{q'}dy\\ \geq&\int_{|y|>3}\Big(|y|^{-\gamma}\int_{2|y|<|x|<4|y|}|x|^{-\frac{n}{p}-\alpha-\beta}(\log|x|)^{-\frac{1}{q'}}dx\Big)^{q'}dy\\ \geq&C\int_{|y|>3}\Big(|y|^{-\gamma}\int_{2|y|<|x|<4|y|}|x|^{-\frac{n}{p}-\alpha-\beta}(\log|y|)^{-\frac{1}{q'}}dx\Big)^{q'}dy\\ \geq&C\int_{|y|>3}\Big(|y|^{-\frac{n}{p}-\alpha-\beta-\gamma+n}(\log|y|)^{-\frac{1}{q'}}\Big)^{q'}dy\\ =&C\int_{|y|>3}|y|^{-n}(\log|y|)^{-1}dy=\infty,\end{aligned}$$

where we apply the equality

$$\frac{1}{p}+\frac{\alpha+\beta+\gamma}{n}=1+\frac{1}{q'},$$

which can be deduced from the condition (1.2a). Therefore the condition (1.2c) is also necessary.

Up to now, we have proved that (1.2a), (1.2b) and (1.2c) are necessary conditions. This finishes the proof of Theorem 1.3. □

Proof of Theorem 1.4. Without loss of generality, suppose $0<p<1$. Let

$$f(x)=\chi_{B_{\frac{1}{2}}(e_1)}(x)|x-e_1|^{-\frac{2n}{p+1}},$$

and $g=\chi_{B_{\frac{1}{2}}(-e_1)}$. Clearly we have $f\in L^p(\mathbb{R}^n)$ and $g\in L^q(\mathbb{R}^n)$ for any $0<q\le\infty$. Since f is not contained in $L^1(\mathbb{R}^n)$, we have

$$\|T(f,g)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}\ge C\|f\|_{L^1(\mathbb{R}^n)}=\infty.$$

This finishes the proof of Theorem 1.4. □

Proof of Theorem 1.5. The sufficiency of Theorem 1.5 is clear, so we only consider the necessity.

If T is bounded from $L^1(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$, then, using the same method in Theorem 1.3, we can obtain that the condition (1.2a) still holds. That is

$$\frac{1}{p}+\frac{1}{q}+\frac{\alpha+\beta+\gamma}{n}=2.$$

Since $p=q=1$, this implies

$$\alpha+\beta+\gamma=0. \tag{3.7}$$

Also using the same method in Theorem 1.3, the condition (1.2b) has to degenerate as the following condition,

$$\alpha+\gamma\ge 0,\quad \alpha\le 0,\quad \gamma\le 0, \tag{3.8}$$

which is a necessary condition of the boundedness of T for $p=q=1$. It immediately follows from (3.7) and (3.8) that $\alpha=\beta=\gamma=0$. This completes the proof of Theorem 1.5. □

Proof of Theorem 1.6. Using the same methods in Theorem 1.3, we can easily obtain that if T is bounded from $L^\infty(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$, then the conditions (1.2a) and (1.2c) must hold.

However, when $p=\infty$ and $1<q\le\infty$, $1/p+1/q=1/q<1$ contradicts the condition (1.2c), so T is not bounded from $L^\infty(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$ for every real number of α,β and γ. □

Proof of Theorem 1.7. We first prove the sufficiency of Theorem 1.7. This means that the two conditions (1.8) and (1.9) are satisfied.

If $p=\infty$, by Hölder's inequality, we obtain that

$$\begin{aligned}\|T(f,g)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^\alpha|x-y|^\beta|y|^\gamma}dxdy\\&\le\|g\|_{L^1(\mathbb{R}^n)}\big\|\mathcal{T}^t f\big\|_{L^\infty(\mathbb{R}^n)}\\&\le\|f\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}\big\|\mathcal{T}^t 1\big\|_{L^\infty(\mathbb{R}^n)}.\end{aligned}$$

Since (1.8), (1.9) hold, and $p=\infty$, a straightforward calculation implies that $\alpha<n$, $\beta<n$ and $\alpha+\beta>n$.

Now using Lemma 2.5, we deduce that

$$\mathcal{T}^t 1(y)=\int_{\mathbb{R}^n}\frac{1}{|x|^\alpha|x-y|^\beta|y|^\gamma}dx=C_{\alpha,\beta,n}|y|^{-\gamma}|y|^{n-\alpha-\beta}=C_{\alpha,\beta,n}.$$

Thus it immediately follows that

$$\|T(f,g)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}\leq C_{\alpha,\beta,n}\|f\|_{L^\infty(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}.$$

For the case of $1<p<\infty$, Hölder's inequality and the boundness of T for $p=\infty$ give the following estimate,

$$\begin{aligned}\|T(f,g)\|_{L^1(\mathbb{R}^n\times\mathbb{R}^n)}=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^\alpha|x-y|^\beta|y|^\gamma}dxdy\\ \leq&\|g\|_{L^1(\mathbb{R}^n)}\|\mathcal{T}^t f\|_{L^\infty(\mathbb{R}^n)}\\ \leq&\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^1(\mathbb{R}^n)}\Big\|\Big(\int_{\mathbb{R}^n}\frac{1}{|x|^{p'\alpha}|x-\cdot|^{p'\beta}|\cdot|^{p'\gamma}}dx\Big)^{\frac{1}{p'}}\Big\|_{L^\infty(\mathbb{R}^n)}.\end{aligned}$$

Since (1.8) and (1.9) hold, we easily have $p'\alpha<n$, $p'\beta<n$ and $p'\alpha+p'\beta=n-p'\gamma>n$. Using the same method with the case $p=\infty$, we obtain that T is bounded from $L^p(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$.

Next we will prove that (1.8) and (1.9) are necessary conditions.

Using the almost similar methods in Theorem 1.3, we can easily prove that (1.8) is a necessary condition.

We next prove that (1.9) is necessary. In fact, $\alpha+\gamma\geq 0$ and $\alpha<n/p'$ can easily be obtained just by using the similar way in Theorem 1.3, so we omit it. The condition $\gamma\leq 0$ holds as the same reason of the proof of Theorem 1.4.

Now we must eliminate the cases $\gamma=0$ and $\alpha+\gamma=0$, respectively.

Set

$$f(x)=\chi_{|x|>2}(x)|x|^{-\frac{n}{p}}(\log|x|)^{-1}.$$

Lemma 2.3 implies $f\in L^p(\mathbb{R}^n)$.

Let $|y|<1/2$ and $\gamma=0$. It follows that

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{f(x)}{|x|^\alpha|x-y|^\beta}dx=&\int_{|x|>2}\frac{|x|^{-\frac{n}{p}}(\log|x|)^{-1}}{|x|^\alpha|x-y|^\beta}dx\\ \geq&C\int_{|x|>2}|x|^{-\frac{n}{p}-\alpha-\beta}(\log|x|)^{-1}dx\\ =&C\int_{|x|>2}|x|^{-n}(\log|x|)^{-1}dx=\infty.\end{aligned}\tag{3.9}$$

We thus have that

$$\int_{|y|<\frac{1}{2}}\int_{\mathbb{R}^n}\frac{f(x)}{|x|^\alpha|x-y|^\beta}dxdy=\infty.$$

This shows that if $\gamma=0$, then T is not bounded from $L^p(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$. This is equivalent to $\gamma<0$.

When $p=\infty$, since (1.8) holds and we obviously have $\beta<n$, we obtain $\alpha+\gamma>0$. For the case $1<p<\infty$, if $\alpha+\gamma=0$, since (1.8) holds, we have $\beta=n/p'$.

Let

$$f(x)=\chi_{|x-e_1|<\frac{1}{2}}(x)|x-e_1|^{-\frac{n}{p}}\Big|\log|x-e_1|\Big|^{-\lambda},$$

where $1/p<\lambda<1$ and $e_1=(1,0,\cdots,0)$. It implies from Lemma 2.3 that $f\in L^p(\mathbb{R}^n)$.

Since T is bounded from $L^p(\mathbb{R}^n)\times L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n\times\mathbb{R}^n)$, $\mathcal{T}^t$ is bounded from $L^p(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

Let $|y-e_1|<1/4$. We have

$$\begin{aligned}\mathcal{T}^tf(y)&\geq C\int_{|x-e_1|<\frac{1}{2}}|x-y|^{-\frac{n}{p'}}|x-e_1|^{-\frac{n}{p}}|\log|x-e_1||^{-\lambda}dx\\&\geq C\int_{2|y-e_1|<|x-e_1|<\frac{1}{2}}|x-y|^{-\frac{n}{p'}}|x-e_1|^{-\frac{n}{p}}|\log|x-e_1||^{-\lambda}dx\\&\geq C\int_{2|y-e_1|<|x-e_1|<\frac{1}{2}}|x-e_1|^{-\frac{n}{p'}}|x-e_1|^{-\frac{n}{p}}|\log(2|y-e_1|)|^{-\lambda}dx\\&\geq C|\log(2|y-e_1|)|^{-\lambda}\int_{2|y-e_1|<|x-e_1|<\frac{1}{2}}|x-e_1|^{-n}dx\\&\geq C\Big|\log(2|y-e_1|)|^{-\lambda}|\log(4|y-e_1|)\Big|.\end{aligned}$$

We can easily deduce that

$$\big|\log(2|y-e_1|)\big|^{-\lambda}\big|\log(4|y-e_1|)\big|\to\infty,$$

as $|y-e_1|\to 0$.

Consequently, we obtain that $\mathcal{T}^t$ is not bounded from $L^p(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. Thus we have shown that (1.9) is necessary. This finishes the proof of Theorem 1.7. $\square$

Acknowledgements

The author D. Y. Yan is supported in part by National Natural Foundation of China (Grant Nos. 11071250 and 11271162)

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