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Approximation of Generalized Bernstein Operators

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Abstract. This paper is devoted to study direct and converse approximation theorems of the generalized Bernstein operators $C_n(f,s_n,x)$ via so-called unified modulus $\omega^2_{\varphi^\lambda}(f,t)$, $0\le\lambda\le 1$. We obtain main results as follows

$$\omega^2_{\varphi^\lambda}(f,t)=O(t^\alpha)\Longleftrightarrow|C_n(f,s_n,x)-f(x)|=\mathcal{O}\big((n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^\alpha\big),$$

where $\delta_n^2(x)=\max\{\varphi^2(x),1/n\}$ and $0<\alpha<2$.

Key Words: Bernstein type operator, Ditzian-Totik modulus, direct and converse approximation theorem.

AMS Subject Classifications: 41A25, 41A27, 41A36

1 Introduction

Let $C(I)$ be the class of all continuous functions defined on $I=[0,1]$. A generalized Bernstein operator first introduced in [1] is defined by

$$C_n(f,s_n,x)=\frac{1}{s_n}\sum_{i=0}^{n}\sum_{j=0}^{s_n-1}f\Big(\frac{i+j}{n+s_n-1}\Big)p_{n,i}(x),\quad x\in I, \tag{1.1}$$

where

$$p_{n,i}(x)=\binom{n}{i}x^i(1-x)^{n-i},\quad f(x)\in C(I),$$

and $\{s_n\}$ is a sequence of natural numbers.

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Obviously, if $s_n=1$ $(n=1,2,\cdots)$, then $C_n(f,s_n,x)$ degenerates into the well-known Bernstein operators

$$B_nf(x)=\sum_{k=0}^{n}f\Big(\frac{k}{n}\Big)b_{nk}(x),\quad b_{nk}(x)=\binom{n}{k}x^k(1-x)^{n-k}, \tag{1.2}$$

for a given $f(x)$ on I.

For Bernstein operators, Ditzian has established the following direct theorem of approximation in [2]

$$|B_nf(x)-f(x)|\le C\omega^2_{\varphi^\lambda}\Big(f,\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\Big), \tag{1.3}$$

where $\omega^2_{\varphi^\lambda}(f,t)$ is the unified modulus which will be defined in the next section.

When $\lambda=0$, (1.3) degenerates

$$|B_nf(x)-f(x)|\le C\omega^2\Big(f,\frac{\varphi(x)}{\sqrt{n}}\Big),$$

which is a pointwise approximation result; and when $\lambda=1$, (1.3) degenerates

$$|B_nf(x)-f(x)|\le C\omega^2_{\varphi}\Big(f,\frac{1}{\sqrt{n}}\Big),$$

which is a global approximation result. Since (1.3) incorporates the pointwise and global approximation theorems of Bernstein operators, it is a very interesting estimate. Later in 1998, an inverse theorem of approximation for Bernstein operators in the following form was present in [3].

$$|B_nf(x)-f(x)|=\mathcal{O}\big((n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))^{\alpha}\big)\Longleftrightarrow\omega^2_{\varphi^\lambda}(f,t)=\mathcal{O}(t^\alpha). \tag{1.4}$$

In this paper, we will establish the same result as (1.4) for the operators $C_n(f,s_n,x)$ defined in (1.1), but it must be restricted the sequence $\{s_n\}$ to be bounded.

2 Preliminary

We start with notation. Let $\delta^2_n(x)=\max\{\varphi^2(x),1/n\}$, $\varphi^2(x)=x(1-x)$, $\|f\|=\sup_{x\in I}|f(x)|$, and denoting

$$\begin{aligned}&\overrightarrow{\Delta}^2_hf(x)=f(x+2h)-2f(x+h)+f(x),\\&\omega^2_{\varphi^\lambda}(f,t)=\sup_{0<h\le t}\|\Delta^2_{h\varphi^\lambda}f(x)\|,\end{aligned}$$

$$\Delta^2_{h\varphi^\lambda}f(x)=f(x+h\varphi^\lambda(x))-2f(x)+f(x-h\varphi^\lambda(x)),$$
$$K^2_{\varphi^\lambda}(f,t^2)=\inf_{g\in D^2_\lambda}\{\|f-g\|+t^2\|\varphi^{2\lambda}g''\|\},$$
$$\bar{K}^2_{\varphi^\lambda}(f,t^2)=\inf_{g\in D^2_\lambda}\{\|f-g\|+t^2\|\varphi^{2\lambda}g''\|+t^{\frac{4}{2-\lambda}}\|g''\|\},$$

where
$$D^2_\lambda=\{f\in C_B(I)|\ f'\in A.C._{loc};\ \|\varphi^{2\lambda}f''\|<\infty\},$$
$C_B(I)$ is the class of bounded continuous functions on I and $A.C._{loc}$ is the space of local absolute continuous functions. In [4] it can be found that
$$\omega^2_{\varphi^\lambda}(f,t)\sim K^2_{\varphi^\lambda}(f,t^2)\sim\bar{K}^2_{\varphi^\lambda}(f,t^2).$$

Next we introduce some lemmas that are necessary in this paper.

Lemma 2.1 (see [5]). *For $f\in C(I)$, $0<\alpha<2/(2-\lambda)$, $0\le\lambda\le1$, we have*
$$\omega^2_{\varphi^\lambda}(f,t)=\mathcal{O}(t^\alpha)\Longrightarrow\omega_1(f,t)=\mathcal{O}(t^{\alpha(1-\frac{\lambda}{2})}),\tag{2.1}$$
where
$$\omega_1(f,t)=\sup_{0<h\le t}\big\{|f(x+h)-f(x)|:x,x+h\in I\big\}.$$

Lemma 2.2 (see [3]). *Let $0\le\beta\le2$, $x\in(0,1)$, then for $0<t<1/4$, $t\le x\le1-t$, there exists a constant C such that*
$$\iint_{-\frac{t}{2}}^{\frac{t}{2}}\varphi^{-\beta}(x+u_1+u_2)du_1du_2\le Ct^2\varphi^{-\beta}(x).\tag{2.2}$$

Lemma 2.3. *For $f\in C(I)$, $0\le\lambda\le1$, there exists a constant C such that*
$$|\varphi^{2\lambda}(x)C''_n(f,s_n,x)|\le Cn\varphi^{-2(1-\lambda)}(x)\|f\|.\tag{2.3}$$

Proof. By directly calculating, we have
$$C''_n(f,s_n,x)=\frac{n(n-1)}{s_n}\sum_{i=0}^{n-2}\sum_{j=0}^{s_n-1}\Big[f\Big(\frac{i+j+2}{n+s_n-1}\Big)-2f\Big(\frac{i+j+1}{n+s_n-1}\Big)+f\Big(\frac{i+j}{n+s_n-1}\Big)\Big]p_{n-2,i}(x)\tag{2.4}$$
$$=\frac{n^2}{\varphi^4(x)s_n}\sum_{i=0}^{n}\sum_{j=0}^{s_n-1}f\Big(\frac{i+j}{n+s_n-1}\Big)\Big[\Big(\frac{i}{n}-x\Big)^2-\Big(\frac{i}{n}-x\Big)\Big(\frac{1-2x}{n}\Big)-\frac{x(1-x)}{n}\Big]p_{n,i}(x).\tag{2.5}$$

Therefore,
(i) if $x\in[0,1/n]\bigcup[1-1/n,1]$, then $\varphi^2(x)\le1/n$, by (2.4) we may get
$$|\varphi^{2\lambda}(x)C''_n(f,s_n,x)|\le Cn^2\varphi^{2\lambda}(x)\|f\|\le Cn\varphi^{2\lambda}(x)\varphi^{-2}(x)\|f\|=Cn\varphi^{-2(1-\lambda)}(x)\|f\|.$$

(ii) if $x\in(1/n,1-1/n)$, then by (2.5) we can reach

$$\begin{aligned}|\varphi^{2\lambda}(x)C_n''(f,s_n,x)|&\le n^2\varphi^{2\lambda-4}(x)\|f\|\cdot\sum_{i=0}^{n}\Big[\Big|\frac{i}{n}-x\Big|^2+\Big|\frac{i}{n}-x\Big|\Big|\frac{1-2x}{n}\Big|+\Big|\frac{x(1-x)}{n}\Big|\Big]p_{n,i}(x)\\&=:I_1+I_2+I_3,\end{aligned}$$

$$I_1=n^2\varphi^{2\lambda-4}(x)\|f\|\sum_{i=0}^{n}\Big|\frac{i}{n}-x\Big|^2p_{n,i}(x)=n^2\varphi^{2\lambda-4}(x)\|f\|\frac{\varphi^2(x)}{n}=n\|f\|\varphi^{2\lambda-2}(x),$$

$$\begin{aligned}I_2&=n^2\varphi^{2\lambda-4}(x)\|f\|\sum_{i=0}^{n}\Big|\frac{i}{n}-x\Big|\Big|\frac{1-2x}{n}\Big|p_{n,i}(x)\le n\varphi^{2\lambda-4}(x)\|f\||1-2x|\Big\{\sum_{i=0}^{n}(\frac{i}{n}-x)^2p_{n,i}(x)\Big\}^{\frac12}\\&=n\|f\|\varphi^{2\lambda-2}(x)\frac{|1-2x|}{\varphi(x)\sqrt{n}}.\end{aligned}$$

Noting that, If $n\ge3$, $x\in(1/n,1/2)$, then

$$\frac{|1-2x|}{\varphi(x)\sqrt{n}}\le\frac{1-2x}{\sqrt{1-x}}\le\sqrt{2}.$$

If $n\ge3$, $x\in(1/2,1-1/n)$, then

$$\frac{|1-2x|}{\varphi(x)\sqrt{n}}\le\frac{2x-1}{\sqrt{x}}\le\sqrt{2}.$$

Hence there is a constant C such that

$$I_2\le Cn\|f\|\varphi^{2\lambda-2}(x).$$

As for I_3, it is easy to get that

$$I_3=n^2\varphi^{2\lambda-4}(x)\|f\|\sum_{i=0}^{n}\Big|\frac{x(1-x)}{n}\Big|p_{n,i}(x)=n\|f\|\varphi^{2\lambda-2}(x).$$

That completes the proof. □

Lemma 2.4. *For $f''\in C[0,1]$, $0\le\lambda\le1$,there exists a constant C such that*

$$|\varphi^{2\lambda}(x)C_n''(f,s_n,x)|\le C\|\varphi^{2\lambda}f''\|.$$

Proof. By (2.2) and (2.4), we have

$$|\varphi^{2\lambda}(x)C_n''(f,s_n,x)|=\Big|\varphi^{2\lambda}(x)\frac{n(n-1)}{s_n}\sum_{i=0}^{n-2}\sum_{j=0}^{s_n-1}\overrightarrow{\Delta}^2_{\frac{1}{n+s_n-1}}f\Big(\frac{i+j}{n+s_n-1}\Big)p_{n-2,i}(x)\Big|$$

$$
\begin{aligned}
&=\Big|\varphi^{2\lambda}(x)\frac{n(n-1)}{s_n}\sum_{i=0}^{n-2s_n-1}\sum_{j=0}\iint_{-\frac{1}{2(n+s_n-1)}}^{\frac{1}{2(n+s_n-1)}}f''\Big(\frac{i+j}{n+s_n-1}+u+v\Big)dudvp_{n-2,i}(x)\Big|\\
&\le\|\varphi^{2\lambda}f''\|\Big|\varphi^{2\lambda}(x)\frac{n(n-1)}{s_n}\sum_{i=0}^{n-2s_n-1}\sum_{j=0}\iint_{-\frac{1}{2(n+s_n-1)}}^{\frac{1}{2(n+s_n-1)}}\varphi^{-2\lambda}\Big(\frac{i+j+1}{n+s_n-1}+u+v\Big)dudv\cdot p_{n-2,i}(x)\Big|\\
&\le\|\varphi^{2\lambda}f''\|\varphi^{2\lambda}(x)\Big|\frac{n(n-1)}{s_n}\sum_{i=0}^{n-2s_n-1}\sum_{j=0}\frac{1}{(n+s_n-1)^2}\varphi^{-2\lambda}\Big(\frac{i+j+1}{n+s_n-1}\Big)p_{n-2,i}(x)\Big|\\
&=:\|\varphi^{2\lambda}f''\|\varphi^{2\lambda}(x)k,
\end{aligned}
$$

and

$$
\begin{aligned}
K&=\frac{n(n-1)}{s_n}\frac{1}{(n+s_n-1)^2}\Big|\sum_{i=0}^{n-2s_n-1}\sum_{j=0}\Big[\frac{1}{\frac{i+j+1}{n+s_n-1}\big(1-\frac{i+j+1}{n+s_n-1}\big)}\Big]^{\lambda}p_{n-2,i}(x)\Big|\\
&\le\frac{n(n-1)}{s_n}\frac{1}{(n+s_n-1)^2}\Big|\sum_{i=0}^{n-2s_n-1}\sum_{j=0}\Big[\frac{(n+s_n-1)^2}{(i+1)(n-i-1)}\Big]^{\lambda}p_{n-2,i}(x)\Big|\\
&=\frac{n(n-1)}{(n+s_n-1)^2}\frac{(n+s_n-1)^{2\lambda}}{n^{2\lambda}}\Big|\sum_{i=0}^{n-2}\Big[\frac{1}{\frac{i+1}{n}\Big(1-\frac{i+1}{n}\Big)}\Big]^{\lambda}p_{n-2,i}(x)\Big|\\
&=\frac{n(n-1)}{(n+s_n-1)^2}\frac{(n+s_n-1)^{2\lambda}}{n^{2\lambda}}\Big|\sum_{i=0}^{n-2}\varphi^{-2\lambda}\Big(\frac{i+1}{n}\Big)p_{n-2,i}(x)\Big|\\
&\le C.
\end{aligned}
$$

Thus we obtain that

$$|\varphi^{2\lambda}(x)C_n''(f,s_n,x)|\le C\|\varphi^{2\lambda}f''\|\varphi^{2\lambda}(x)\le C\|\varphi^{2\lambda}f''\|.$$

That proves this lemma. ☐

Lemma 2.5. *For $x\in(0,1)$, $t\in[0,1]$, we have*

$$\int_x^t\frac{|t-u|}{\delta_n^{2\lambda}(u)}du\le C\frac{|t-x|^2}{\delta_n^{2\lambda}(x)}.$$

Proof. This Lemma results from Lemma 2.4 [6] and $\delta_n^2(x)\sim\varphi^2(x)+1/n$. ☐

3 Main results

Theorem 3.1. *For $f(x)\in C[0,1]$, $0\le\lambda\le1$, there exists a constant C such that*

$$|C_n(f,s_n,x)-f(x)|\le C\Big(\omega^2_{\varphi^\lambda}\Big(f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)\Big)+\omega_1\Big(f,\frac{1}{n}\Big).$$

Proof. From [1], we can get without difficulty

$$C_n(1,s_n,x)=1,\qquad C_n(t-x,s_n,x)=\frac{s_n-1}{n+s_n-1}\Big(\frac{1}{2}-x\Big),$$
$$C_n((t-x)^2,s_n,x)=\frac{n-(s_n-1)^2}{(n+s_n-1)^2}\varphi^2(x)+\frac{(s_n-1)(2s_n-1)}{6(n+s_n-1)^2}.$$

Let

$$L_n(f,s_n,x)=f\Big(x+\frac{s_n-1}{n+s_n-1}\Big(\frac{1}{2}-x\Big)\Big)-f(x), \tag{3.1a}$$
$$A_n(f,s_n,x)=C_n(f,s_n,x)-L_n(f,s_n,x),\quad \|A_n\|\le 3, \tag{3.1b}$$
$$L_n(f,s_n,x)\le C\omega_1\Big(f,\frac{1}{n}\Big), \tag{3.1c}$$
$$A_n(t-x,s_n,x)=C_n(t-x,s_n,x)-L_n(t-x,s_n,x)=0, \tag{3.1d}$$
$$\begin{aligned}A_n((t-x)^2,s_n,x)&=C_n((t-x)^2,s_n,x)-L_n((t-x)^2,s_n,x)\\&=\frac{n-(s_n-1)^2}{(n+s_n-1)^2}\varphi^2(x)+\frac{(s_n-1)(2s_n-1)}{6(n+s_n-1)^2}-\Big[\frac{s_n-1}{n+s_n-1}\Big(\frac{1}{2}-x\Big)\Big]^2\\&=\frac{1}{(n+s_n-1)^2}\Big[n\varphi^2(x)+\frac{s_n^2-1}{12}\Big]\\&\le\frac{1}{n^2}[n\varphi^2(x)+C]\le\frac{C}{n}\delta_n^2(x).\end{aligned} \tag{3.1e}$$

Since

$$\omega^2_{\varphi^\lambda}(f,t)\sim\bar{K}^2_{\varphi^\lambda}(f,t^2)$$

in the definition of $\bar{K}^2_{\varphi^\lambda}$, we can choose $g\in D^2_\lambda$ such that

$$\|f-g\|+\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^2\|\varphi^{2\lambda}g''\|+\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{\frac{4}{2-\lambda}}\|g''\|\le C\omega^2_{\varphi^\lambda}\Big(f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big). \tag{3.2}$$

Using this g, we get

$$|A_n(f,s_n,x)-f(x)|\le C\|f-g\|+|A_n(g,s_n,x)-g(x)|. \tag{3.3}$$

In view of (3.1d), Lemma 2.5 and

$$g(t)=g(x)+g'(x)(t-x)+\int_x^t g''(u)(t-u)du,$$

it is followed that

$$\begin{aligned}|A_n(g,s_n,x)-g(x)|=&\Big|A_n\Big(\int_x^t g''(u)\Big)(t-u)du,x)\Big|\le\|\delta_n^{2\lambda}g''\|\Big|A_n\Big(\int_x^t\frac{t-u}{\delta_n^{2\lambda}(u)}du,x\Big)\Big|\\ \le&C\|\delta_n^{2\lambda}g''\|\Big|A_n\Big(\frac{(t-x)^2}{\delta_n^{2\lambda}(x)},x\Big)\Big|\le C\|\delta_n^{2\lambda}g''\|\frac{\delta_n^{2-2\lambda}(x)}{n}\\ \le&C\|\varphi^{2\lambda}g''\|\frac{\delta_n^{2-2\lambda}(x)}{n}+C\|g''\|\frac{\delta_n^{2-2\lambda}(x)}{n^{1+\lambda}}\\ \le&C\|\varphi^{2\lambda}g''\|\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^2+C\|g''\|\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{\frac{4}{2-\lambda}},\end{aligned}\tag{3.4}$$

the reason of the last inequality in (3.4) is that

$$\begin{aligned}\Big(\frac{\delta_n^{2-2\lambda}(x)}{n^{1+\lambda}}\Big)^{1-\frac{\lambda}{2}}\le& C\Big(\frac{\varphi(x)n^{-\frac{1+\lambda}{2}}}{\varphi^{\lambda}(x)}+\frac{1}{n}\Big)^{2-\lambda}=C\Big(\frac{\varphi(x)n^{-\frac{1}{2}}}{\varphi^{\lambda}(x)}+\frac{1}{n^{1-\frac{\lambda}{2}}}\Big)^2\Big(\frac{\varphi(x)n^{-\frac{1-\lambda}{2}}}{\varphi^{\lambda}(x)}+1\Big)^{-\lambda}\\ \le& C\Big(\frac{\varphi(x)n^{-\frac{1}{2}}}{\varphi^{\lambda}(x)}+\frac{1}{n^{1-\frac{\lambda}{2}}}\Big)^2,\end{aligned}$$

that is

$$\frac{\delta_n^{2-2\lambda}(x)}{n^{1+\lambda}}\le\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{\frac{4}{2-\lambda}}.$$

Hence, with (3.1c), (3.2), (3.3) and (3.4), we complete the proof of Theorem 3.1. □

Theorem 3.2. *For* $0<\alpha<2$, $0\le\lambda\le1$, *we have*

$$\omega^2_{\varphi^\lambda}(f,t)=\mathcal{O}(t^\alpha)\Longrightarrow|C_n(f,s_n,x)-f(x)|=\mathcal{O}\Big(\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^\alpha\Big).\tag{3.5}$$

Proof. By Lemma 2.1 and $\omega^2_{\varphi^\lambda}(f,t)=\mathcal{O}(t^\alpha)$, we get

$$\omega_1\Big(f,\frac{1}{n}\Big)\le C\Big(\frac{1}{n}\Big)^{\alpha(1-\frac{\lambda}{2})}\le C(n^{\frac{-1}{2}}n^{\frac{\lambda-1}{2}})^\alpha\le C(n^{\frac{-1}{2}}\delta_n^{1-\lambda})^\alpha,$$

again by Theorem 3.1, (3.5) follows. □

Theorem 3.3. *For* $0<\alpha<2$, $0\le\lambda\le1$, $f(x)\in C[0,1]$, *we have*

$$|C_n(f,s_n,x)-f(x)|=\mathcal{O}\Big(\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^\alpha\Big)\Longrightarrow\omega^2_{\varphi^\lambda}(f,t)=\mathcal{O}(t^\alpha).$$

Proof. We know that $\varphi^{\alpha(1-\lambda)}(x)$ is convex on I, so for $t\varphi^\lambda(x)<x$, we have

$$\delta_n^{\alpha(1-\lambda)}(x+t\varphi^\lambda(x))+\delta_n^{\alpha(1-\lambda)}(x-t\varphi^\lambda(x))\le2\delta_n^{\alpha(1-\lambda)}(x).$$

Since

$$|C_n(f,s_n,x)-f(x)|=\mathcal{O}\Big(\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{\alpha}\Big),$$

for each n $(n>2)$, it holds that

$$\begin{aligned}|\Delta^2_{t\varphi^\lambda}f(x)|&\le|\Delta^2_{t\varphi^\lambda}(C_n(f,s_n,x)-f(x))|+|\Delta^2_{t\varphi^\lambda}C_n(f,s_n,x)|\\ &\le C\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{\alpha}+|\Delta^2_{t\varphi^\lambda}C_n(f,s_n,x)|.\end{aligned}$$

Choose g as in (3.2), by Lemma 2.2, 2.3 and 2.4, we have

$$\begin{aligned}|\Delta^2_{t\varphi^\lambda}C_n(f,s_n,x)|&=\Big|\iint_{-\frac{t\varphi^\lambda(x)}{2}}^{\frac{t\varphi^\lambda(x)}{2}}C''_n(f,s_n,x+u_1+u_2)du_1du_2\Big|\\ &\le\Big|\iint_{-\frac{t\varphi^\lambda(x)}{2}}^{\frac{t\varphi^\lambda(x)}{2}}C''_n(f-g,s_n,x+u_1+u_2)du_1du_2\Big|\\ &\quad+\Big|\iint_{-\frac{t\varphi^\lambda(x)}{2}}^{\frac{t\varphi^\lambda(x)}{2}}C''_n(g,s_n,x+u_1+u_2)du_1du_2\Big|\\ &\le Cn\|f-g\|\Big|\iint_{-\frac{t\varphi^\lambda(x)}{2}}^{\frac{t\varphi^\lambda(x)}{2}}\varphi^{-2}(x+u_1+u_2)du_1du_2\Big|\\ &\quad+C\|\varphi^{2\lambda}g''\|\Big|\iint_{-\frac{t\varphi^\lambda(x)}{2}}^{\frac{t\varphi^\lambda(x)}{2}}\varphi^{-2\lambda}(x+u_1+u_2)du_1du_2\Big|\\ &\le Cn\|f-g\|t^2\varphi^{2(\lambda-1)}(x)+Ct^2\|\varphi^{2\lambda}g''\|\\ &\le C\|f-g\|t^2[n^{\frac{-1}{2}}\delta_n^{1-\lambda}(x)]^{-2}+Ct^2\|\varphi^{2\lambda}g''\|\\ &\le Ct^2[n^{\frac{-1}{2}}\delta_n^{1-\lambda}(x)]^{-2}\omega^2_{\varphi^\lambda}\Big(f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big).\end{aligned}$$

So now we come to

$$|\Delta^2_{t\varphi^\lambda}f(x)|\le C\Big(\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big)^{\alpha}+Ct^2[n^{\frac{-1}{2}}\delta_n^{1-\lambda}(x)]^{-2}\omega^2_{\varphi^\lambda}\Big(f,\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\Big).$$

For each fixed h $(0<h<1/16)$ and each $x\ge t$, we can choose n such that

$$\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}}\le h<2\frac{\delta_n^{1-\lambda}(x)}{\sqrt{n}},$$

then it follows that

$$|\Delta^2_{t\varphi^\lambda}f(x)|\le C\Big(h^{\alpha}+\Big(\frac{t}{h}\Big)^2\omega^2_{\varphi^\lambda}(f,h)\Big),$$

that is

$$\omega_{\varphi^\lambda}^2(f,t) \le C\left(h^\alpha + \left(\frac{t}{h}\right)^2 \omega_{\varphi^\lambda}^2(f,h)\right).$$

Finally by Berens-Lorentz Lemma, we can complete the proof of Theorem 3.3.

□

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