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On Some Class of n-Normed Generalized Difference Sequences Related to ℓ_p-Space

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Abstract. In this paper, we introduce the class of n-normed generalized difference sequences related to ℓ_p-space. Some properties of this sequence space like solidness, symmetricity, convergence-free etc. are studied. We obtain some inclusion relations involving this sequence space.

Key Words: Generalized difference operator, n-norm, n-Banach space, symmetricity, solidness, convergence free, completeness.

AMS Subject Classifications: 40A05, 40A25, 40A30, 40C05

1 Introduction

The notion of n-normed space was studied at the initial stage by Gahler [7], Misiak [10], Gunawan [8] and many others from different aspects.

Let $n\in N$ and X be a real vector space. A real valued function on X^n satisfying the following $\|\cdot,\cdots\|$ four properties:

1. $\|(z_1,z_2,\cdots,z_n)\|_n=0$ if and only if $z_1,z_2,\cdots,z_n$ are linearly dependent;
2. $\|(z_1,z_2,\cdots,z_n)\|_n$ is invariant under permutation;
3. $\|(z_1,z_2,\cdots,z_{n-1},\alpha z_n)\|_n=\|\alpha\|\|(z_1,z_2,\cdots,z_n)\|_n$, for all $\alpha\in R$;
4. $\|(z_1,z_2,\cdots,z_{n-1},x+y)\|_n\le\|(z_1,z_2,\cdots,z_{n-1},x)\|_n+\|(z_1,z_2,\cdots,z_{n-1},y)\|_n$;

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is called an n-norm on X and the pair $(X,\|,\cdot,\cdot\|_n)$ is called an n-normed space.

Kizmaz [9] studied the notion of difference sequence spaces at the initial stage. Kizmaz [9] investigated the difference sequence spaces $\ell_\infty(\Delta)$,$c(\Delta)$ and $c_0(\Delta)$ of crisp sets. The notion is defined as follows:

$$Z(\Delta)=\{x=(x_k):(\Delta x_k)\in Z\},$$

for $Z=\ell_\infty$,c and c_0, where $\Delta x=(\Delta x_k)=(x_k-x_{k+1})$, for all $k\in N$. The above spaces are Banach spaces, normed by

$$\|x\|_\Delta=\|x_1\|+\sup_k\|\Delta x_k\|.$$

The idea of Kizmaz [9] was applied to introduce different types of difference sequence spaces and study their different properties by Tripathy (see [13, 14]), Tripathy, Altin and Et [15], Tripathy and Baruah (see [16, 17]), Tripathy abd Borgohain [18], Tripathy and Chandra [19], Tripathy, Choudhary and Sarma [20], Tripathy and Dutta [21], Tripathy and Esi (see [22, 23]), Tripathy, Esi and Tripathy [24], Tripathy and Mahanta [25] and many others.

Tripathy and Esi [22] introduced the new type of difference sequence spaces, for fixed $m\in N$,

$$Z(\Delta_m)=\{x=(x_k):(\Delta_m x_k)\in Z\},$$

for $Z=\ell_\infty$,c and c_0, where $\Delta_m x=(\Delta_m x_k)=(x_k-x_{k+m})$, for all $k\in N$.

This generalizes the notion of difference sequence spaces studied by Kizmaz [9]. The above spaces are Banach spaces, normed by

$$\|x\|_{\Delta_m}=\sum_{r=1}^{m}\|x_r\|+\sup_k\|\Delta_m x_k\|.$$

Tripathy, Esi and Tripathy [24] further generalized this notion and introduced the following notion. For $m\geq 1$ and $n\geq 1$,

$$Z(\Delta_m^n)=\{x=(x_k):(\Delta_m^n x_k)\in Z\},$$

for $Z=\ell_\infty$,c and c_0.

This generalized difference has the following binomial representation,

$$\Delta_m^n x_k=\sum_{r=0}^{n}(-1)^r\binom{n}{r}x_{k+rm}. \tag{1.1}$$

Sargent [12] introduced the crisp set sequence space $m(\phi)$ and studied some properties of this space. Later on it was studied from the sequence space point of view and some matrix classes were characterized with one member as $m(\phi)$ by Rath and Tripathy [11]. Afterwards the notion was further investigated by Esi [5], Tripathy and Borgohain [18], Tripathy and Sen [27] and others. In this article we introduce the class of sequences $(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)$ with respect to n-norm.

2 Definitions and background

A sequence (x_k) in an n-normed space is said to be convergent to $x\in X$ if,

$$\lim_{k\to\infty}\|(z_1,z_2,\cdots,z_{n-1},x_k-x)\|_n=0,\quad \text{for all } z_1,z_2,\cdots,z_{n-1}\in X.$$

A sequence (x_k) in an n-normed space is called Cauchy (with respect to n-norm) if,

$$\lim_{k,j\to\infty}\|(z_1,z_2,\cdots,z_{n-1},x_k-x_j)\|_n=0,\quad \text{for all } z_1,z_2,\cdots,z_{n-1}\in X.$$

If every Cauchy sequence in X converges to an $x\in X$, then X is said to be complete (with respect to the n-norm). A complete n-normed space is called n-Banach space.

An n-normed sequence space E is said to be solid if $(y_k)\in E$, whenever $(x_k)\in E$ and $\|(z_1,z_2,\cdots,z_{n-1},y_n)\|_n\le\|(z_1,z_2,\cdots,z_{n-1},x_n)\|_n$ for all $k\in N$.

Let $x=(x_k)$ be a sequence, then $S(x)$ denotes the set of all permutations of the elements of (x_k) i.e., $S(x)=\{(x_{\pi(n)}):\pi \text{ is a permutation of N}\}$. A sequence space E is said to be symmetric if $S(x)\in E$ for all $x\in E$.

A sequence space E is said to be convergence free if $(y_k)\in E$ whenever $(x_k)\in E$ with $y_k=0$ whenever $x_k=0$.

A sequence space E is said to be monotone if E contains the canonical pre-images of all its step spaces.

Lemma 2.1. *A sequence space E is solid implies that E is monotone.*

Let $\wp_s$ be the class of all subsets of N those do not contain more than s number of elements. Throughout $\{\phi_n\}$ is a non-decreasing sequence of positive real numbers such that $n\phi_{n+1}\le(n+1)\phi_n$, for all $n\in N$. The space $m(\phi)$ introduced by Sargent [12] is defined by

$$m(\phi)=\Big\{(x_k)\in X:\|x\|_{m(\phi)}=\sup_{s\ge1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\|x_k\|<\infty\Big\}.$$

In this article we introduce the following sequence space:

$$\begin{aligned}&(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)\\&=\Big\{(x_k)\in X:\|(x_k)\|_{n,m(\phi,\Delta_p^q)}=\sup_{s\ge1,\sigma\in\wp}\frac{1}{\phi_s}\sum_{k\in\sigma}\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q x_k)\|_n<\infty\Big\}.\end{aligned}$$

3 Main results

Theorem 3.1. *$(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)$ is an n-Banach space with respect to the norm defined by*

$$\|x\|_{n,m(\phi,\Delta_p^q)}=\sum_{r=1}^{pq}\|(z_1,z_2,\cdots,z_{n-1},x_r)\|_n+\sup_{s\ge1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q x_k)\|_n,$$

for all $z_1,z_2,\cdots,z_{n-1}\in X$ where X is an n-Banach space.

Proof. Let $(x^{(i)})$ be a Cauchy sequence in $(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)$ such that, $x^{(i)}=(x_k^{(i)})_{i=1}^{\infty}$. Then there exists a positive integer $n_0=n_0(\varepsilon)$ such that,

$$\|x^{(i)}-x^{(j)}\|_{n,m(\phi,\Delta_p^q)}<\varepsilon, \quad \text{for all } i,j\geq n_0,$$

which implies

$$\sum_{r=1}^{pq}\|(z_1,z_2,\cdots,z_{n-1},x_r^{(i)}-x_r^{(j)})\|_n+\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q(x_k^{(i)}-x_k^{(j)}))\|_n<\varepsilon, \tag{3.1}$$

for all $i,j\geq n_0$. Which implies that,

$$\begin{aligned}&\sum_{r=1}^{pq}\|(z_1,z_2,\cdots,z_{n-1},x_r^{(i)}-x_r^{(j)})\|_n<\varepsilon, \quad \text{for all } i,j\geq n_0,\\ \Rightarrow&\|(z_1,z_2,\cdots,z_{n-1},x_r^{(i)}-x_r^{(j)})\|_n<\varepsilon, \quad \text{for all } i,j\geq n_0, \quad r=1,2,3,\cdots,pq.\end{aligned}$$

i.e., $(x_r^{(i)})$ is a Cauchy sequence in X. Since X is an n-Banach space, so it is convergent in X, for $r=1,2,3,\cdots,\cdots,pq$.

$$\lim_{i\to\infty}x_r^{(i)}=x_r, \quad r=1,2,3,\cdots,pq. \tag{3.2}$$

Also,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q(x_k^{(i)}-x_k^{(j)}))\|_n<\varepsilon, \quad \text{for all } i,j\geq n_0. \tag{3.3}$$

On taking $s=1$ and σ varying over $\wp_s$, we get,

$$\begin{aligned}&\sum_{k\in\sigma}\big\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q(x_k^{(i)}-x_k^{(j)}))\big\|_n<\varepsilon, \quad \text{for all } i,j\geq n_0.\\ \Rightarrow&\sum_{k\in\sigma}\big\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q(x_k^{(i)}-x_k^{(j)}))\big\|_n<\varepsilon, \quad \text{for all } i,j\geq n_0.\end{aligned}$$

i.e., $(\Delta_p^q(x_k^{(i)}))$ is a Cauchy sequence in X. Since X is an n-Banach space, so it is convergent in X.

Let

$$\lim_{i\to\infty}\Delta_p^q(x_k^{(i)})=y_k$$

in X, for each $k\in N$. We have to prove that,

$$\lim_{i\to\infty}x^{(i)}=x \quad \text{and} \quad x\in(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n).$$

For $k=1$, we get from (1.1) and (3.3),

$$\lim_{i\to\infty}x_{pq+1}^{(i)}=x_{pq+1}, \quad \text{for } p\geq 1, \quad q\geq 1.$$

Hence we get that,

$$\lim_{i\to\infty} x_k^{(i)} = x_k, \quad \text{for each } k\in N.$$

Also,

$$\lim_{i\to\infty} \Delta_p^q x_k^{(i)} = \Delta_p^q x_k, \quad \lim_{i\to\infty} x_k^{(i)} = x_k, \tag{3.4}$$

i.e., for each $k\in N$.

Since $(x^{(i)})$ is a Cauchy sequence in $(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)$, so we get, for each $k\in N$, there exists a positive integer $n_0(\epsilon)$ such that, (from (3.4)),

$$\sup_{s\geq 1,\sigma\in\wp_s} \frac{1}{\phi_s}\sum_{k\in\sigma}\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q(x_k^{(i)}-x_k^{(j)}))\|_n<\varepsilon, \quad \text{for all } i,j\geq n_0.$$

Taking as $j\to\infty$, we have, from (3.1) and (3.4),

$$\sum_{r=1}^{pq}\|(z_1,z_2,\cdots,z_{n-1},x_r^{(i)}-x_r)\|_n+\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q(x_k^{(i)}-x_k))\|_n$$
$$<\varepsilon, \quad \text{for all } i\geq n_0.$$

Which implies that,

$$\|x^{(i)}-x\|_{n,m(\phi,\Delta_p^q)}<\varepsilon, \quad \text{for all } i\geq n_0.$$

i.e., $\lim_{i\to\infty} x^{(i)} = x$.

Now we have for all $i\geq n_0$,

$$\|x\|_{n,m(\phi,\Delta_p^q)}\leq\|x+x^{(i)}-x^{(i)}\|_{n,m(\phi,\Delta_p^q)}\leq\|x-x^{(i)}\|_{n,m(\phi,\Delta_p^q)}+\|x^{(i)}\|_{n,m(\phi,\Delta_p^q)}<\varepsilon+M,$$

i.e., $\|x^{(i)}\|_{n,m(\phi,\Delta_p^q)}<\infty$.

Hence, $x\in(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)$. Hence $(m(\phi,\Delta_p^q),\|\cdots,\|_n)$ is an n-Banach space. □

Theorem 3.2. *The class of sequences $(m(\phi,\Delta_p^q),\|\cdots,\|_n)$ is not solid.*

Proof. The proof follows from the following example:

Example 3.1. Let $n=3$, $p=3$, $q=2$ and $x_k=2k$, for all $k\in N$. Then, $\Delta_3^2 x_k=0$, for all $k\in N$. Let $\phi_s=s$, for all $s\in N$. Then we have,

$$\begin{aligned}\|(z_1,z_2,\Delta_3^2 x_k)\|_3 =& \text{volume of the parallelopiped with the vertices } (0,0,0),(z_1,0,0),(0,z_2,0),\\ &(0,0,\Delta_3^2x_k),(z_1,z_2,0),(0,z_2,\Delta_3^2x_k),(z_1,0,\Delta_3^2x_k),(z_1,z_2,\Delta_3^2x_k)\\ =& z_1\times z_2\times\Delta_3^2x_k<\infty, \quad \text{for all } z_1,z_2\in X.\end{aligned}$$

Which implies that,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{s}\sum_{k\in\sigma}\|(z_1,z_2,\Delta_3^2x_k)\|_3<\infty.$$

i.e., $(x_k)\in(m(\phi,\Delta_3^2),\|\cdots,\cdot\|_3)$.

Consider the sequence (α_k) of scalars defined by,

$$\alpha_k=\begin{cases}1, & \text{for } k \text{ even},\\ 0, & \text{otherwise},\end{cases}\qquad \alpha_k x_k=\begin{cases}2k, & \text{for } k \text{ even},\\ 0, & \text{otherwise}.\end{cases}$$

Then, $\Delta_3^2\alpha_k x_k=(-16,20,-24,\cdots,)$, for all $k\in N$. We get,

$$\begin{aligned}\|(z_1,z_2,\Delta_3^2\alpha_k x_k)\|_3=&\text{volume of the parallelopiped with the vertices } (0,0,0),(z_1,0,0),(0,z_2,0),\\ &(0,0,\Delta_3^2\alpha_k x_k),(z_1,z_2,0),(0,z_2,\Delta_3^2\alpha_k x_k),(z_1,0,\Delta_3^2\alpha_k x_k),(z_1,z_2,\Delta_3^2\alpha_k x_k)\\ =&z_1\times z_2\times\Delta_3^2\alpha_k x_k=\infty,\quad \text{for all } z_1,z_2\in X,\end{aligned}$$

which implies that,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{s}\sum_{k\in\sigma}\|(z_1,z_2,\Delta_3^2\alpha_k x_k)\|_3=\infty.$$

i.e., $(\alpha_k x_k)\notin(m(\phi,\Delta_3^2),\|\cdots,\cdot\|_3)$. Which shows that $(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)$ is not solid. □

Theorem 3.3. *The class of sequences* $(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)$ *is not symmetric.*

Proof. The proof follows from the following example:

Example 3.2. Let $n=3$, $p=1$, $q=1$ and $x_k=k$, for all $k\in N$. Then, $\Delta x_k=-1$. Let $\phi_s=s$, for all $s\in N$. We have,

$$\begin{aligned}\|(z_1,z_2,\Delta x_k)\|_3=&\text{volume of the parallelopiped with the vertices } (0,0,0),(z_1,0,0),(0,z_2,0),\\ &(0,0,\Delta x_k),(z_1,z_2,0),(0,z_2,\Delta x_k),(z_1,0,\Delta x_k),(z_1,z_2,\Delta x_k)\\ =&z_1\times z_2\times\Delta x_k<\infty,\quad \text{for all } z_1,z_2\in X.\end{aligned}$$

Which implies that,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{s}\sum_{k\in\sigma}\|(z_1,z_2,\Delta x_k)\|_3<\infty.$$

i.e., $(x_k)\in(m(\phi,\Delta),\|\cdots,\cdot\|_3)$.

Let (y_k) be a rearrangement of (x_k) such that,

$$(y_k)=(x_1,x_2,x_4,x_3,x_9,x_5,x_{16},x_6,x_{25},x_7,x_{36},\cdots),$$

from which we get that,

$$\Delta y_k\approx k-(k-1)^2\approx k^2,\quad \text{for all } k\in N.$$

Which implies that, $\|(z_1,z_2,\Delta y_k)\|_3=\infty$, i.e.,

$$\sup_{s\geq 1,\sigma\in\wp_s}\frac{1}{s}\sum_{k\in\sigma}\|(z_1,z_2,\Delta y_k)\|_3=\infty.$$

Hence, $(y_k)\notin(m(\phi,\Delta),\|\cdots,\cdot\|_3)$. Thus, $(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)$ is not symmetric. □

Theorem 3.4. *The class of sequences* $(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)$ *is not convergence free.*

Proof. The proof follows from the following example:

Example 3.3. Let $p=4$, $q=1$, $n=2$, and $x_k=k$. Then, $\Delta_4 x_k=-4$, for all $k\in N$. Let $\phi_s=s$, for all $s\in N$.

$$\begin{aligned}\|(z,\Delta_4 x_k)\|_2 &= \text{area of the triangle with the vertices } (0,0),(\Delta_4 x_k,0),(0,z)\\ &=\frac{1}{2}(\text{area of the parallelopiped with the vertices } (0,0),(\Delta_4 x_k,0),(0,z),(\Delta_4 x_k,z))\\ &=\frac{1}{2}\times\Delta_4 x_k\times z<\infty,\quad \text{for all } \ z\in X,\end{aligned}$$

which implies that,

$$\sup_{s\geq1,\sigma\in\wp_s}\frac{1}{s}\sum_{k\in\sigma}\|(z,\Delta_4 x_k)\|_2<\infty,$$

which shows that, $(x_k)\in(m(\phi,\Delta_4),\|\cdots,\cdot\|_2)$.

Let us take another sequence (y_k) such that, $y_k=k^2$. Then, $\Delta_4 y_k=-8k-16$, for all $k\in N$. Which shows that, $\|(z,\Delta_4 y_k)\|_2=\infty$, which implies that,

$$\sup_{s\geq1,\sigma\in\wp_s}\frac{1}{s}\sum_{k\in\sigma}\|(z,\Delta_4 y_k)\|_2=\infty.$$

i.e., $(y_k)\notin(m(\phi,\Delta_4),\|\cdots,\cdot\|_2)$.

Hence, $(m(\phi,\Delta_p^q),\|\cdots,\|_n)$ is not convergence free. □

Theorem 3.5. $(m(\phi,\Delta_p^q),\|\cdots,\|_n)\subseteq(m(\psi,\Delta_p^q),\|\cdots,\cdot\|_n)$ *if and only if,*

$$\sup_{s\geq1}\frac{\phi_s}{\psi_s}<\infty,\quad \textit{for}\ \ 0<p<\infty.$$

Proof. First, suppose that,

$$\sup_{s\geq1}\frac{\phi_s}{\psi_s}=K<\infty.$$

We have, $\phi_s\leq K\Psi_s$. Now, if $(x_k)\in(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)$, we get,

$$\begin{aligned}&\sup_{s\geq1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q x_k)\|_n<\infty\\ \Rightarrow&\sup_{s\geq1,\sigma\in\wp_s}\frac{1}{K\psi_s}\sum_{k\in\sigma}\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q x_k)\|_n<\infty.\end{aligned}$$

i.e., $(x_k)\in(m(\psi,\Delta_p^q),\|\cdots,\cdot\|_n)$.

Hence, $(m(\phi,\Delta_p^q),\|\cdots,\|_n)\subseteq(m(\psi,\Delta_p^q),\|\cdots,\cdot\|_n)$. Conversely, suppose that,

$$(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)\subseteq(m(\psi,\Delta_p^q),\|\cdots,\cdot\|_n).$$

We have to prove that,

$$\sup_{s\ge1}\frac{\phi_s}{\psi_s}=\sup_{s\ge1}(\eta_s)<\infty.$$

Suppose that, $\sup_{s\ge1}(\eta_s)=\infty$. Then there exists a subsequence (η_{s_i}) of (η_s) such that,

$$\lim_{i\to\infty}(\eta_{s_i})=\infty.$$

For $(x_k)\in(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)$, we have,

$$\sup_{s\ge1,\sigma\in\wp_s}\frac{1}{\psi_s}\sum_{k\in\sigma}\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q x_k)\|_n\ge\sup_{s\ge1,\sigma\in\wp_s}\frac{\eta_{s_i}}{\phi_{s_i}}\sum_{k\in\sigma}\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q x_k)\|_n=\infty,$$

i.e.,

$$\sup_{s\ge1,\sigma\in\wp_s}\frac{1}{\psi_s}\sum_{k\in\sigma}\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q x_k)\|_n=\infty,$$

which implies that,

$$(x_k)\in(m(\psi,\Delta_p^q),\|\cdots,\cdot\|_n),$$

a contradiction. This completes the proof. □

Corollary 3.1. $(m(\phi,\Delta_p^q),\|\cdots,\|_n)=(m(\psi,\Delta_p^q),\|\cdots,\cdot\|_n)$ if and only if,

$$\sup_{s\ge1}(\eta_s)<\infty\quad\text{and}\quad\sup_{s\ge1}(\eta_s^{-1})<\infty,$$

where $\eta_s=\phi_s/\psi_s$.

Theorem 3.6. $(\ell_p(\Delta_p^q),\|\cdots,\cdot\|_n)\subseteq(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)\subseteq(\ell_\infty(\Delta_p^q),\|\cdots,\cdot\|_n)$.

Proof. By taking $\phi_s=1$, for all $s\in N$, $(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)=(\ell_p(\Delta_p^q),\|\cdots,\cdot\|_n)$. So, the first inclusion is proved.

Next suppose that, $(x_k)\in(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n)$. This implies that,

$$\sup_{s\ge1,\sigma\in\wp_s}\frac{1}{\phi_s}\sum_{k\in\sigma}\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q x_k)\|_n=K<\infty.$$

For $s=1$, $\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q x_k)\|_n\le K\phi_1$, for $k\in\sigma$. Which implies that,

$$\sup_{k\ge1}\|(z_1,z_2,\cdots,z_{n-1},\Delta_p^q x_k)\|_n<\infty,$$

thus we have, $(x_k)\in(\ell_\infty(\Delta_p^q),\|\cdots,\cdot\|_n)$. This completes the proof. □

Putting $\psi_n = 1$, for all $n \in N$, in Corollary 3.1, we get

Proposition 3.1. $(m(\phi,\Delta_p^q),\|\cdots,\cdot\|_n) = (\ell_p(\Delta_p^q),\|\cdots,\cdot\|_n)$, if and only if,

$$\sup_{s\geq 1}(\phi_s) < \infty \quad \text{and} \quad \sup_{s\geq 1}(\phi_s^{-1}) < \infty.$$

Corollary 3.2. $(m(\phi,\Delta_p^q),\|\cdots,\|_n) = (\ell_p(\Delta_p^q),\|\cdots,\cdot\|_n)$, if

$$\lim_{s\to\infty}\frac{\phi_s}{s} > 0.$$

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