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Boundedness for Hardy Type Operators on Herz Spaces with Variable Exponents

Min Wang, Lisheng Shu* and Meng Qu

School of Mathematics and Computer Science, Anhui Normal University, Wuhu 241000, China

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Abstract. In this paper, we will prove the boundedness of Hardy type operators $H_{\beta(x)}$ and $H^*_{\beta(x)}$ of variable order $\beta(x)$ on Herz spaces $K^{\alpha(\cdot)}_{p(\cdot),q}$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$, where $\alpha(\cdot)$ and $p(\cdot)$ are both variable.

Key Words: Herz spaces, Hardy type operators, variable exponent.

AMS Subject Classifications: 42B20, 42B35

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1 Introduction

Suppose that $f\in L^1_{loc}(\mathbb{R}^n)$, $0<\beta(x)<n$. The n-dimensional Hardy operator is defined by

$$Hf(x)=\frac{1}{|x|^n}\int_{|t|<|x|}f(t)dt$$

and the Hardy type operators of variable order $\beta(x)$ are defined by

$$H_{\beta(x)}f(x)=\frac{1}{|x|^{n-\beta(x)}}\int_{|t|<|x|}f(t)dt, \qquad x\in\mathbb{R}^n\setminus\{0\},$$
$$H^*_{\beta(x)}f(x)=|x|^{\beta(x)}\int_{|t|\geq|x|}\frac{f(t)}{|t|^n}dt, \qquad x\in\mathbb{R}^n\setminus\{0\},$$

when $\beta(x)=\beta$, $H_{\beta(x)}=H_\beta$ and $\beta(x)=0$, $H_{\beta(x)}=H$.

*Corresponding author. *Email addresses:* `15155369968@163.com` (M. Wang), `shulsh@mail.ahnu.edu.cn` (L. S. Shu), `qumeng@mail.ahnu.edu.cn` (M. Qu)

Christ and Grafakos [1] considered the boundedness of H on $L^p(\mathbb{R}^n)$ and obtained the best constant. The paper [2] by Fu, Liu and Lu introduced the n-dimensional fractional Hardy operators and proved the boundedness of commutators on $L^p(\mathbb{R}^n)$ and Herz spaces of homogeneous type. Hardy inequality in the generalized Lebesgue spaces was studied by Samko in [3]. Recently, Zhao, Fu and Lu [4] got endpoint estimates for n-dimensional Hardy operators and commutators.

It is well known that function spaces with variable exponents were intensively studied during the past 20 years, due to their applications to PDE with non-standard growth conditions and so on, we mention e.g., (see [5,6]). A great deal of work has been done to extend the theory of Hardy type operators on the classical Lebesgue spaces to the variable exponent case, (see [7–11]). Lukkassen, Persson, Samko and Wall [12] studied weighted Hardy type inequalities in variable exponent Morrey-type spaces. The boundedness of Hardy type operators on product of Herz-Morrey spaces with variable exponent were investigated by Zhang and Wu in [13]. Izuki (see [14,15]) first introduced the Herz spaces $K^{\alpha}_{p(\cdot),q}$ and $\dot{K}^{\alpha}_{p(\cdot),q}$ with variable exponent p but fixed $\alpha\in\mathbb{R}^n$. Recently, Almeida and Drihem [16] obtained the boundedness of a wide class of sublinear operators, which includes maximal, potential and Calderón-Zygumnd operators on Herz spaces $K^{\alpha(\cdot)}_{p(\cdot),q}$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$ where α and p are both variable. In this paper, we will study the boundedness of the n-dimensional fractional Hardy operators of variable order $\beta(x)$ on Herz spaces $K^{\alpha(\cdot)}_{p(\cdot),q}$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}$.

For brevity, C always means a positive constant independent of the main parameters and may change from one occurrence to another. $B(x,r)=\{y\in\mathbb{R}^n:|x-y|<r\}$, $B_k=\{x\in\mathbb{R}^n:|x|<2^k\}$, $R_k=B_k\backslash B_{k-1}$ and $\chi_{R_k}=\chi_k$ be the characteristic function of the set R_k for $k\in\mathbb{Z}$. $|S|$ denotes the Lebesgue measure of S. The exponent $p'(x)$ means the conjugate of $p(x)$, that is, $1/p'(x)+1/p(x)=1$. Let $p^*(x)$ be the Sobolev exponent defined by $1/p^*(x):=1/p(x)-\beta(x)/n$. We write $f\sim g$ if there exist positive constants C such that $C^{-1}g\le f\le Cg$. By $l^q,q\in(0,\infty]$, we denote the discrete Lebesgue space equipped with the usual quasinorm.

To meet the requirements in the next sections, here, the basic elements of the theory of Lebsegue spaces with variable exponent are briefly presented.

Let $p(\cdot):\Omega\to[1,\infty)$ be a measurable function. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega):=\Big\{f\text{ is measurable}:\int_{\Omega}\Big|\frac{f(x)}{\lambda}\Big|^{p(x)}dx<\infty\text{ for some constant }\lambda>0\Big\}.$$

The space $L^{p(\cdot)}_{loc}(\Omega)$ is defined by

$$L^{p(\cdot)}_{loc}(\Omega):=\Big\{f\text{ is measurable}:f\in L^{p(\cdot)}(K)\text{ for all compact subsets }K\subset\Omega\Big\}.$$

$L^{p(\cdot)}(\Omega)$ is a Banach space with the norm defined by

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf\Big\{\lambda>0: \int_\Omega \Big|\frac{f(x)}{\lambda}\Big|^{p(x)} dx \le 1\Big\}.$$

We denote

$$p^- := \operatorname{ess\,inf}_{x\in\Omega} p(x) > 1, \quad p^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x) < \infty.$$

Let $\mathcal{P}(\Omega)$ be the set of measurable function $p(\cdot)$ on Ω with value in $[1,\infty)$ such that $1 < p^-(\Omega) \le p(\cdot) \le p^+(\Omega) < \infty$.

We say a function $p(\cdot):\mathbb{R}^n \to \mathbb{R}$ is locally log-Hölder continuous, if there exists a constant C such that

$$|p(x)-p(y)| \le \frac{C}{\log(e+1/|x-y|)}$$

for all $x,y\in\mathbb{R}^n$. If

$$|p(x)-p(0)| \le \frac{C}{\log(e+1/|x|)}$$

for all $x\in\mathbb{R}^n$, then we say $p(\cdot)$ is log-Hölder continuous at the origin. If, for some $p_\infty\in\mathbb{R}$ and $C>0$, there holds

$$|p(x)-p_\infty| \le \frac{C}{\log(e+|x|)}$$

for all $x\in\mathbb{R}^n$, then we say $p(\cdot)$ is log-Hölder continuous at infinity.

By $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ and $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ we denote the class of all exponents $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ which are log-Hölder continuous at the origin and at infinity with $p_\infty := \lim_{|x|\to\infty} p(x)$. The notation $\mathcal{P}^{\log}(\mathbb{R}^n)$ is used for all those exponents $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ which are locally log-Hölder continuous and log-Hölder continuous at infinity. Obviously we have $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)\subset \mathcal{P}_0^{\log}(\mathbb{R}^n)\cap\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ and since $(p')_\infty = (p_\infty)'$ we write only p'_∞ for simplicity. Moreover, we can easily show that $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$ implies $p'(\cdot), p^*(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$. In particular, we have $(p^*)_\infty = (p_\infty)^*$, so that we write only p^*_∞ for any those quantities.

Next we define the variable exponent Herz spaces.

Definition 1.1 (see [16]). Let $0<q\le\infty, p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $\alpha(\cdot):\mathbb{R}^n\to\mathbb{R}$ with $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$. The non-homogeneous Herz space $K^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ consists of all $f\in L^{p(\cdot)}_{loc}(\mathbb{R}^n\setminus\{0\})$ such that

$$\|f\|_{K^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)} := \|f\chi_{B_0}\|_{p(\cdot)} + \Big(\sum_{k\ge1} \|2^{k\alpha(\cdot)} f\chi_k\|^q_{p(\cdot)}\Big)^{1/q} < \infty.$$

The homogeneous Herz space $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ is defined as the set of all $f\in L^{p(\cdot)}_{loc}(\mathbb{R}^n\setminus\{0\})$ such that

$$\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)}:=\Big(\sum_{k\in\mathbb{Z}}\|2^{k\alpha(\cdot)}f\chi_k\|^q_{p(\cdot)}\Big)^{1/q}<\infty,$$

with the usual modifications made when $q=\infty$.

If $\alpha(\cdot)$ and $p(\cdot)$ are constants, then $K^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)=K^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)=\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ are classical Herz spaces.

2 Preliminaries and main results

In order to prove our result, we need some conclusions as follows.

Lemma 2.1 (see [8])**.** *Let $1<\beta^-\le\beta(x)\le\beta^+<\infty$. If $\beta(\cdot)$ is log-Hölder continuous at the origin, then*

$$C^{-1}|x|^{\beta(0)}\le|x|^{\beta(x)}\le C|x|^{\beta(0)},\quad |x|<1.$$

If $\beta(\cdot)$ is log-Hölder continuous at infinity, then

$$C^{-1}|x|^{\beta_\infty}\le|x|^{\beta(x)}\le C|x|^{\beta_\infty},\quad |x|\ge1.$$

Lemma 2.2. *Let $1<\beta^-\le\beta(x)\le\beta^+<\infty$, $x\in B(0,r)\setminus B(0,r/2)$. If $\beta(\cdot)$ is log-Hölder continuous at the origin, then*

$$C^{-1}r^{\beta(0)}\le r^{\beta(x)}\le Cr^{\beta(0)},\quad 0<r<1. \tag{2.1}$$

If $\beta(\cdot)$ is log-Hölder continuous at infinity, then

$$C^{-1}r^{\beta_\infty}\le r^{\beta(x)}\le Cr^{\beta_\infty},\quad r\ge1. \tag{2.2}$$

Proof. We will prove (2.1), the argument for (2.2) is similar. It suffices to rewrite the inequality (2.1) in the form

$$C^{-1}\le r^{\beta(x)-\beta(0)}\le C.$$

It is easy to see that the above inequality is equivalent to

$$|\beta(x)-\beta(0)|\log\frac{1}{r}\le C,$$

which is valid since $\beta(\cdot)$ is log-Hölder continuous at the origin and

$$\log\frac{1}{r}<\log\Big(e+\frac{1}{|x|}\Big),$$

see also [8, 17–19]. □

Lemma 2.3 (see [20]). *If $p(\cdot)\in\mathcal{P}(\Omega)$, then for all $f\in L^{p(\cdot)}(\Omega)$ and all $g\in L^{p'(\cdot)}(\Omega)$ we have*

$$\int_\Omega |f(x)g(x)|dx\le r_p\|f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{p'(\cdot)}(\Omega)},$$

where $r_p:=1+1/p^- -1/p^+$.

For the following Lemma 2.4, see Corollary 4.5.9 in [21].

Lemma 2.4. *Let $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$, then for every ball $B=B(x,r)\subset\mathbb{R}^n$, we have*

$$\|\chi_B\|_{p(\cdot)}\sim|B|^{\frac{1}{p(x)}},\quad \text{if } r<1,\quad x\in B,$$

and

$$\|\chi_B\|_{p(\cdot)}\sim|B|^{\frac{1}{p_\infty}},\quad \text{if } r\ge 1.$$

We remark that the following Lemmas 2.5-2.9 were showed in [16].

Lemma 2.5. *Let $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$ and $r_1,r_2>0$. If $\alpha(\cdot)$ is log-Hölder continuous both at the origin and at infinity, then*

$$r_1^{\alpha(x)}\le Cr_2^{\alpha(y)}\times\Big(\frac{r_1}{r_2}\Big)^{\alpha^+},\qquad \text{if } 0<\frac{r_2}{r_1}\le\frac{1}{2}, \tag{2.3a}$$

$$r_1^{\alpha(x)}\le Cr_2^{\alpha(y)},\qquad \text{if } \frac{1}{2}<\frac{r_2}{r_1}\le 2, \tag{2.3b}$$

$$r_1^{\alpha(x)}\le Cr_2^{\alpha(y)}\times\Big(\frac{r_1}{r_2}\Big)^{\alpha^-},\qquad \text{if } \frac{r_2}{r_1}>2, \tag{2.3c}$$

for any $x\in B(0,r_1)\backslash B(0,r_1/2)$ and $y\in B(0,r_2)\backslash B(0,r_2/2)$ with the implicit constant not depending on x, y, r_1 and r_2.

Lemma 2.6. *Let $p(\cdot)\in\mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ and $R=B(0,r)\backslash B(0,r/2)$. If $|R|\ge 2^{-n}$, then*

$$\|\chi_R\|_{p(\cdot)}\sim|R|^{\frac{1}{p(x)}}\sim|R|^{\frac{1}{p_\infty}}$$

with the implicit constants independent of r and $x\in R$.

The left-hand side equivalence remains true for every $|R|>0$ if we assume, additionally, $p(\cdot)\in\mathcal{P}^{\log}_0(\mathbb{R}^n)\cap\mathcal{P}^{\log}_\infty(\mathbb{R}^n)$.

Lemma 2.7. *Let $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$, $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $q_1,q_2\in(0,\infty]$. If $q_1\le q_2$, then*

$$K^{\alpha(\cdot)}_{p(\cdot),q_1}(\mathbb{R}^n)\hookrightarrow K^{\alpha(\cdot)}_{p(\cdot),q_2}(\mathbb{R}^n)\quad \text{and}\quad \dot K^{\alpha(\cdot)}_{p(\cdot),q_1}(\mathbb{R}^n)\hookrightarrow \dot K^{\alpha(\cdot)}_{p(\cdot),q_2}(\mathbb{R}^n).$$

Lemma 2.8. *Suppose that* $0<a<1$ *and* $0<q\le\infty$. *Let* $\{\varepsilon_k\}_{k\in\mathbb{Z}}$ *be a sequence of positive real numbers, such that*

$$\|\{\varepsilon_k\}_{k\in\mathbb{Z}}\|_{l^q}=I<\infty.$$

Then the sequences $\{\delta_k:\delta_k=\sum_{j\le k}a^{k-j}\varepsilon_j\}_{k\in\mathbb{Z}}$ *and* $\{\eta_k:\eta_k=\sum_{k\le j}a^{j-k}\varepsilon_j\}_{k\in\mathbb{Z}}$ *belong to* l^q, *and*

$$\|\{\delta_k\}_{k\in\mathbb{Z}}\|_{l^q}+\|\{\eta_k\}_{k\in\mathbb{Z}}\|_{l^q}\le CI,$$

with $C>0$ *only depending* a *and* q.

Lemma 2.9. *Let* $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)$, $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ *and* $q\in(0,\infty]$. *If* $\alpha(\cdot)$ *is log-Hölder's continuous at infinity, then*

$$K^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)=K^{\alpha_\infty}_{p(\cdot),q}(\mathbb{R}^n).$$

The result in [8] is rewrited as the following Lemma.

Lemma 2.10. *Suppose that* $1<p^-\le p^+<\infty$. *Let* $p(x),\beta(x)\in\mathcal{P}^{\log}_0(\mathbb{R}^n)\cap\mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ *with* $\beta(x)<n/p^+$, *then*

$$\|H_{\beta(\cdot)}f\|_{L^{p^*(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\quad\text{and}\quad\|H^*_{\beta(\cdot)}f\|_{L^{p^*(\cdot)}(\mathbb{R}^n)}\le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Now it is the position to state our results.

Theorem 2.1. *Suppose that* $1<p^-\le p^+<\infty$, $0<q_1\le q_2<\infty$. *Let* $p(x),\beta(x)\in\mathcal{P}^{\log}_0(\mathbb{R}^n)\cap\mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ *with* $\beta(x)<n/p^+$, $\alpha(x)\in L^\infty(\mathbb{R}^n)$ *be log-Hölder's continuous both at the origin and at infinity.*

(i) If $\alpha^+<n(1-\frac{1}{p^-})$, *then* $H_{\beta(x)}$ *is bounded from* $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q_1}(\mathbb{R}^n)$ *into* $\dot{K}^{\alpha(\cdot)}_{p^*(\cdot),q_2}(\mathbb{R}^n)$.

(ii) If $\alpha^->-\frac{n}{p^+}$, *then* $H^*_{\beta(x)}$ *is bounded from* $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q_1}(\mathbb{R}^n)$ *into* $\dot{K}^{\alpha(\cdot)}_{p^*(\cdot),q_2}(\mathbb{R}^n)$.

Theorem 2.2. *Suppose that* $1<p^-\le p^+<\infty$, $0<q_1\le q_2<\infty$. *Let* $p(x),\beta(x)\in\mathcal{P}^{\log}_0(\mathbb{R}^n)\cap\mathcal{P}^{\log}_\infty(\mathbb{R}^n)$ *with* $\beta(x)<n/p^+$, $\alpha(x)\in L^\infty(\mathbb{R}^n)$ *be log-Hölder's continuous at infinity.*

(i) If $\alpha_\infty<n(1-\frac{1}{p_\infty})$, *then* $H_{\beta(x)}$ *is bounded from* $K^{\alpha(\cdot)}_{p(\cdot),q_1}(\mathbb{R}^n)$ *into* $K^{\alpha(\cdot)}_{p^*(\cdot),q_2}(\mathbb{R}^n)$.

(ii) If $\alpha_\infty>-\frac{n}{p_\infty}$, *then* $H^*_{\beta(x)}$ *is bounded from* $K^{\alpha(\cdot)}_{p(\cdot),q_1}(\mathbb{R}^n)$ *into* $K^{\alpha(\cdot)}_{p^*(\cdot),q_2}(\mathbb{R}^n)$.

Remark 2.1. In the next section, we will use the boundedness of n-dimensional fractional Hardy operators on $L^{p(\cdot)}(\mathbb{R}^n)$ in the proof of Theorem 2.2, but in the Theorem 2.1 we need not it. So, there is an interesting problem that whether we can omit the boundedness in the proof of Theorem 2.2.

3 Proof of theorems

Proof of Theorem 2.1(i). In view of Lemma 2.7, it is sufficient to consider the case $q_1=q_2=q$.

For $k\in\mathbb{Z}$ and $x\in R_k$, we split the operator into two parts and give

$$|H_{\beta(x)}f(x)|\le|H_{\beta(x)}(f\chi_{B_k})(x)|+|H_{\beta(x)}(f\chi_{\mathbb{R}^n\setminus B_k})(x)|=|H_{\beta(x)}(f\chi_{B_k})(x)|.$$

Hence, we have

$$\begin{aligned}|2^{k\alpha(x)}H_{\beta(x)}(f\chi_{B_k})(x)|&=\Big|2^{k\alpha(x)}\frac{1}{|x|^{n-\beta(x)}}\int_{|t|<|x|}(f\chi_{B_k})(t)dt\Big|\\&\le C2^{k\alpha(x)}|x|^{\beta(x)-n}\int_{B_k}|f(t)|dt.\end{aligned}$$

Lemma 2.1 gives that

$$\begin{aligned}|2^{k\alpha(x)}H_{\beta(x)}(f\chi_{B_k})(x)|&\le C2^{k\alpha(x)}|x|^{\beta^*-n}\sum_{j=-\infty}^{k}\int_{R_j}|f(t)|dt\\&\le C2^{k\alpha(x)}(2^k)^{\beta^*-n}\sum_{j=-\infty}^{k}\int_{R_j}|f(t)|dt,\end{aligned}$$

where $\beta^*=\beta(0)$, if $k<0$ and $\beta^*=\beta_\infty$, if $k\ge 0$.

By virtue of (2.3) and (2.4) in Lemma 2.5, we obtain

$$|2^{k\alpha(x)}H_{\beta(x)}(f\chi_{B_k})(x)|\le C\sum_{j=-\infty}^{k}2^{(k-j)\alpha^+}(2^k)^{\beta^*-n}\int_{R_j}2^{j\alpha(t)}|f(t)|dt.$$

After applying Lemma 2.3 to the last integral, we get

$$|2^{k\alpha(x)}H_{\beta(x)}(f\chi_{B_k})(x)|\le C\sum_{j=-\infty}^{k}2^{(k-j)\alpha^+}(2^k)^{\beta^*-n}\|2^{j\alpha(\cdot)}f\chi_j\|_{p(\cdot)}\|\chi_j\|_{p'(\cdot)}.$$

Taking $L^{p^*(\cdot)}$ norm, we have

$$\|2^{k\alpha(\cdot)}H_{\beta(\cdot)}(f\chi_{B_k})\chi_k\|_{p^*(\cdot)}\le C\sum_{j=-\infty}^{k}2^{(k-j)\alpha^+}(2^k)^{\beta^*-n}\|2^{j\alpha(\cdot)}f\chi_j\|_{p(\cdot)}\|\chi_j\|_{p'(\cdot)}\|\chi_k\|_{p^*(\cdot)}.$$

By Lemma 2.6, we obtain

$$\|\chi_j\|_{p'(\cdot)}\sim|R_j|^{\frac{1}{p'(x_j)}},\quad x_j\in R_j\quad\text{and}\quad\|\chi_k\|_{p^*(\cdot)}\sim|R_k|^{\frac{1}{p^*(x_k)}},\quad x_k\in R_k.$$

Hence, using Lemma 2.2 we have

$$\begin{aligned}
&\|2^{k\alpha(\cdot)}H_{\beta(\cdot)}(f\chi_{B_k})\chi_k\|_{p^*(\cdot)}\\
\le & C\sum_{j=-\infty}^{k}2^{(k-j)\alpha^+}(2^k)^{\beta^*-n}|R_j|^{\frac{1}{p'(x_j)}}|R_k|^{\frac{1}{p^*(x_k)}}\|2^{j\alpha(\cdot)}f\chi_j\|_{p(\cdot)}\\
\le & C\sum_{j=-\infty}^{k}2^{(k-j)(\alpha^+-n)}(2^k)^{\beta^*-\beta(x_k)}|R_j|^{-\frac{1}{p(x_j)}}|R_k|^{\frac{1}{p(x_k)}}\|2^{j\alpha(\cdot)}f\chi_j\|_{p(\cdot)}\\
\le & C\sum_{j=-\infty}^{k}2^{(k-j)(\alpha^+-n)}|R_j|^{-\frac{1}{p(x_j)}}|R_k|^{\frac{1}{p(x_k)}}\|2^{j\alpha(\cdot)}f\chi_j\|_{p(\cdot)}.
\end{aligned}$$

Now we can distinguish three cases as follows:

1. If $0\le j\le k$, by Lemma 2.6 we have
$$|R_j|^{-\frac{1}{p(x_j)}}|R_k|^{\frac{1}{p(x_k)}}\sim|R_j|^{-\frac{1}{p_\infty}}|R_k|^{\frac{1}{p_\infty}}\sim 2^{(k-j)\frac{n}{p_\infty}}\le 2^{(k-j)\frac{n}{p^-}}.$$

2. If $j<0\le k$, we obtain
$$|R_j|^{-\frac{1}{p(x_j)}}|R_k|^{\frac{1}{p(x_k)}}\le|R_j|^{-\frac{1}{p^-}}|R_k|^{\frac{1}{p^-}}\le 2^{(k-j)\frac{n}{p^-}}.$$

3. If $j\le k<0$, we get
$$|R_j|^{-\frac{n}{p(x_j)}}|R_k|^{\frac{n}{p(x_k)}}\sim\big(|R_k||R_j|^{-1}\big)^{\frac{1}{p(x_j)}}|R_k|^{\frac{1}{p(x_k)}-\frac{1}{p(x_j)}}\le 2^{(k-j)\frac{n}{p^-}}.$$

Indeed, since $|x_k|<2^k$, $|x_j|<2^j\le 2^k$ we make use of local-Hölder's continuity of $p(x)$ at the origin and have for $k<0$,

$$\begin{aligned}
\Big|\frac{1}{p(x_k)}-\frac{1}{p(x_j)}\Big|\log\frac{1}{|R_k|}&\le\Big|\frac{1}{p(x_k)}-\frac{1}{p(0)}\Big|\log\frac{1}{|R_k|}+\Big|\frac{1}{p(x_j)}-\frac{1}{p(0)}\Big|\log\frac{1}{|R_k|}\\
&\le C\frac{\log(\frac{1}{2^k})}{\log\big(e+\frac{1}{2^k}\big)}\le C
\end{aligned}$$

with $C>0$ independent of k, j, x_k, x_j.

Therefore,

$$\|2^{k\alpha(\cdot)}H_{\beta(\cdot)}(f\chi_{B_k})\chi_k\|_{p^*(\cdot)}\le C\sum_{j=-\infty}^{k}2^{(k-j)(\alpha^+-n+\frac{n}{p^-})}\|2^{j\alpha(\cdot)}f\chi_j\|_{p(\cdot)}.$$

Since $\alpha^+ - n + n/p^- < 0$, applying Lemma 2.8 we derive

$$\begin{aligned}\|H_{\beta(x)}f\|_{\dot K^{\alpha(\cdot)}_{p^*(\cdot),q}} &= \Big\{\sum_{k\in\mathbb{Z}}\|2^{k\alpha(\cdot)}H_{\beta(\cdot)}f\chi_k\|^q_{p^*(\cdot)}\Big\}^{\frac{1}{q}}\\ &\le C\Big\{\sum_{k\in\mathbb{Z}}\Big(\sum_{j=-\infty}^{k}2^{(k-j)(\alpha^+-n+\frac{n}{p^-})}\|2^{j\alpha(\cdot)}f\chi_j\|_{p(\cdot)}\Big)^q\Big\}^{\frac{1}{q}}\\ &\le C\sum_{j\in\mathbb{Z}}\|2^{j\alpha(\cdot)}f\chi_j\|_{p(\cdot)} \le C\|f\|_{\dot K^{\alpha(\cdot)}_{p(\cdot),q}},\end{aligned}$$

and hence the proof of Theorem 2.1(i) is completed. □

Proof of Theorem 2.2(ii). In the view of Lemma 2.7 and Lemma 2.9, it is sufficient to consider the case $q_1 = q_2 = q$ and we use the equivalent quasinorm $\|\cdot\|_{K^{\alpha_\infty}_{p^*(\cdot),q}}$.

Given $k\in\mathbb{N}$ and $x\in R_k$,we split the operator into two parts and give

$$|H^*_{\beta(x)}f(x)| \le |H^*_{\beta(x)}(f\chi_{B_{k-1}})(x)| + |H^*_{\beta(x)}(f\chi_{\mathbb{R}^n\setminus B_{k-1}})(x)| = |H^*_{\beta(x)}(f\chi_{\mathbb{R}^n\setminus B_{k-1}})(x)|.$$

Applying Lemma 2.1 we write

$$\begin{aligned}|H^*_{\beta(x)}(f\chi_{\mathbb{R}^n\setminus B_{k-1}})(x)| &\le |x|^{\beta(x)}\int_{\mathbb{R}^n\setminus B_{k-1}}\frac{|f(t)|}{|t|^n}dt \le C|x|^{\beta_\infty}\sum_{j\ge k}\int_{R_j}\frac{|f(t)|}{|t|^n}dt\\ &\le C\sum_{j\ge k}2^{k\beta_\infty}2^{-jn}\int_{R_j}|f(t)|dt.\end{aligned}$$

Lemma 2.3 implies that

$$\int_{R_j}|f(t)|dt \le C\|f\chi_j\|_{p(\cdot)}\|\chi_j\|_{p'(\cdot)}.$$

Taking $L^{p^*(\cdot)}$ norm, we have

$$2^{k\alpha_\infty}\|H^*_{\beta(\cdot)}f\chi_k\|_{p^*(\cdot)} \le C\sum_{j\ge k}2^{(k-j)\alpha_\infty}2^{k\beta_\infty}2^{-jn}2^{j\alpha_\infty}\|f\chi_j\|_{p(\cdot)}\|\chi_j\|_{p'(\cdot)}\|\chi_k\|_{p^*(\cdot)}.$$

Now Lemma 2.6 gives

$$\|x_j\|_{p'(\cdot)} \sim |R_j|^{\frac{1}{p'_\infty}} \sim 2^{jn(1-\frac{1}{p_\infty})} \quad \text{and} \quad \|x_k\|_{p^*(\cdot)} \sim |R_k|^{\frac{1}{p^*_\infty}} \sim 2^{kn(\frac{1}{p_\infty}-\frac{\beta_\infty}{n})}.$$

Hence,

$$2^{k\alpha_\infty}\|H^*_{\beta(\cdot)}f\chi_k\|_{p^*(\cdot)} \le C\sum_{j\ge k}2^{(k-j)(\alpha_\infty+\frac{n}{p_\infty})}2^{j\alpha_\infty}\|f\chi_j\|_{p(\cdot)}.$$

Therefore, since $\alpha_\infty+n/p_\infty>0$, we apply Lemma 2.8 to obtain

$$\begin{aligned}\Big\{\sum_{k\geq 1}2^{k\alpha_\infty q}\|H^*_{\beta(\cdot)}f\chi_k\|^q_{p^*(\cdot)}\Big\}^{1/q}\leq& C\Big\{\sum_{k\geq 1}\Big(\sum_{j\geq k}2^{(k-j)\left(\alpha_\infty+\frac{n}{p_\infty}\right)}2^{j\alpha_\infty}\|f\chi_j\|_{p(\cdot)}\Big)^q\Big\}^{1/q}\\ \leq& C\Big\{\sum_{j\geq 1}2^{j\alpha_\infty q}\|f\chi_j\|^q_{p(\cdot)}\Big\}^{1/q}\\ \leq& C\|f\|_{K^{\alpha_\infty}_{p(\cdot),q}}.\end{aligned}$$

Recalling the definition of $\|\cdot\|_{K^{\alpha_\infty}_{p^*(\cdot),q}}$, it remains to show that $\|H^*_{\beta(\cdot)}(f)\chi_{B_0}\|_{p^*(\cdot)}\leq C\|f\|_{K^{\alpha_\infty}_{p(\cdot),q}}$ to complete the proof. Since

$$|H^*_{\beta(x)}f(x)|\leq|H^*_{\beta(x)}(f\chi_{B_0})(x)|+|H^*_{\beta(x)}(f\chi_{\mathbb{R}^n\setminus B_0})(x)|.$$

Hence, by Lemma 2.10, we have

$$\|H^*_{\beta(x)}(f\chi_{B_0})\chi_{B_0}\|_{p^*(\cdot)}\leq\|H^*_{\beta(x)}(f\chi_{B_0})\|_{p^*(\cdot)}\leq C\|f\chi_{B_0}\|_{p(\cdot)}\leq C\|f\|_{K^{\alpha_\infty}_{p(\cdot),q}}.$$

For the remaining term, let $x\in B_0$, we obtain

$$\begin{aligned}|H^*_{\beta(x)}(f\chi_{\mathbb{R}^n\setminus B_0})(x)|\leq&|x|^{\beta(x)}\int_{|t|\geq|x|}\frac{|(f\chi_{\mathbb{R}^n\setminus B_0})(t)|}{|t|^n}dt\leq\int_{\mathbb{R}^n\setminus B_0}\frac{|f(t)|}{|t|^n}dt\\ =&\sum_{k\geq 1}\int_{R_k}\frac{|f(t)|}{|t|^n}dt\leq C\sum_{k\geq 1}2^{-kn}\int_{R_k}|f(t)|dt.\end{aligned}$$

Thus we get

$$|H^*_{\beta(x)}(f\chi_{\mathbb{R}^n\setminus B_0})(x)|\leq C\sum_{k\geq 1}2^{-kn}\|f\chi_k\|_{p(\cdot)}\|\chi_k\|_{p'(\cdot)},$$

by Lemma 2.3.

Since $\alpha_\infty+n/p_\infty>0$, using Lemma 2.6 we get

$$|H^*_{\beta(x)}(f\chi_{\mathbb{R}^n\setminus B_0})(x)|\leq C\sum_{k\geq 1}2^{-k(\alpha_\infty+\frac{n}{p_\infty})}2^{k\alpha_\infty}\|f\chi_k\|_{p(\cdot)}\leq C\sup_{j\in\mathbb{N}}2^{j\alpha_\infty}\|f\chi_j\|_{p(\cdot)}.$$

By Lemma 2.4, we have

$$\begin{aligned}\|\chi_{B_0}H^*_{\beta(\cdot)}(f\chi_{\mathbb{R}^n\setminus B_0})\|_{p^*(\cdot)}\leq& C\sup_{j\in\mathbb{N}}2^{j\alpha_\infty}\|f\chi_j\|_{p(\cdot)}\|\chi_{B_0}\|_{p^*(\cdot)}\\ \leq& C\sup_{j\in\mathbb{N}}2^{j\alpha_\infty}\|f\chi_j\|_{p(\cdot)}\leq C\|f\|_{K^{\alpha_\infty}_{p(\cdot),q}},\end{aligned}$$

and hence the proof of Theorem 2.2(ii) is completed. □

We omit the proof of Theorems 2.1(ii) and 2.2(i), since they are essentially similar to that of 2.1(i) and 2.2(ii), respectively.

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