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Convergence of the q Analogue of Szász-Beta-Stancu Operators

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Abstract. In the present paper, we propose the q analogue of Szász-Beta-Stancu operators. By estimate the moments, we establish direct results in terms of the modulus of smoothness. Investigate the rate of point-wise convergence and weighted approximation properties of the q operators. Voronovskaja type theorem is also obtained. Our results generalize and supplement some convergence results of the q-Szász-Beta operators, thus they improve the existing results.

Key Words: q-Szász-Beta operators, q-analogues, modulus of smoothness, stancu, weighted approximation.

AMS Subject Classifications: 41A25, 41A35, 41A36

1 Introduction

For $f\in C_\gamma[0,\infty)$, a new type of Szász-Beta operator studied by Gupta and Noor in [1] is defined as

$$S_n(f;x)=\int_0^\infty W_n(x,t)f(t)=\sum_{v=1}^{\infty}s_{n,k}(x)\int_0^\infty f(t)b_{n,k}(t)dt+s_{n,0}(x)f(0), \tag{1.1}$$

where $W_n(x,t)=\sum_{k=1}^{\infty}s_{n,k}(x)b_{n,k}(t)+s_{n,0}(x)\delta(t)$, $\delta(t)$ being Dirac delta-function and

$$s_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!},\quad b_{n,k}(t)=\frac{1}{B(n+1,k)}\frac{t^{k-1}}{(1+t)^{n+k+1}},$$

are respectively Szász and Beta basis functions. In [1] Gupta and Noor studied some approximation properties for the operators defined in (1.1) and obtained the rate of point-wise convergence, a Voronovskaja type asymptotic formula and an error estimate in simultaneous approximation.

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In 1987, Lupaş introduced a q-analogue of the Bernstein operator and investigated its approximating and shape preserving properties. Ten years later, Phillips [2] proposed another generalization of the classical Bernstein polynomials based on q-integers. He obtained the rate of convergence and Voronovskaya-type asymptotic formula for these new Bernstein operators. An extension to q-calculus of Szász-Mirakyan operators was given by Aral [7] and established a Voronovskaja theorem related to q-derivatives for these operators.

In recent years, the application of q calculus is the most interesting areas of research in the approximation theory. Several authors have proposed the q analogues of different linear positive operators and studied their approximation behaviors. Gupta [5] introduced a q-analogue of usual Bernstein-Durrmeyer operators and established the rate of convergence of these operators. Gupta [6] proposed a generalization of the Baskakov operators based on q integers and estimated the rate of convergence in the weighted norm and some shape preserving properties.

Very recently in [9], Gupta introduced the q-analogue of Szász-Beta operators defined as

$$S_{n,q}(f(t);x)=\sum_{k=1}^{\infty}q^{\frac{3k^2-3k}{2}}s^q_{n,k}(x)\int_0^{\infty/A}p^q_{n,k}(t)f(qt)d_qt+E_q(-[n]_qx)f(0), \tag{1.2}$$

where

$$s^q_{n,k}(x)=\frac{([n]_qx)^k}{[k]_q!}E_q(-[n]_qq^kx),\quad p^q_{n,k}(t)=\frac{1}{B_q(n+1,k)}\frac{t^{k-1}}{(1+t)_q^{n+k+1}}. \tag{1.3}$$

While for $q=1$, these operators coincide with the Szász-Beta operators defined by (1.1).

First, we give some basic definitions and notations of q-calculus. All of the results can be found in [10, 11]. Throughout the present paper, we consider q as a real number such that $0<q<1$. For $n\in\mathbb{N}$. The q integer and q factorial are respectively defined as

$$[n]_q=\frac{1-q^n}{1-q},\qquad [n]_q!=\begin{cases}[n]_q[n-1]_q\cdots[1]_q, & n\geq 1,\\ 1, & n=0.\end{cases}$$

The q-binomial coefficients are given by

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!},\quad 0\leq k\leq n.$$

The q-Jackson integrals and the q-improper integrals are defined as (see [12])

$$\int_0^a f(x)d_qx=(1-q)a\sum_{n=0}^{\infty}f(aq^n)q^n,\quad a>0,$$

and

$$\int_0^{\infty/A}f(x)d_qx=(1-q)\sum_{n=-\infty}^{\infty}f\Big(\frac{q^n}{A}\Big)\frac{q^n}{A},\quad A>0, \tag{1.4}$$

provided the sum converge absolutely.

For $t>0$, the q-Gamma integral (see [12]) is defined by

$$\Gamma_q(t)=\int_0^{\frac{1}{1-q}} x^{t-1}E_q(-qx)d_qx, \tag{1.5}$$

where

$$E_q(x)=\sum_{n=0}^{\infty} q^{n(n-1)/2}\frac{x^n}{[n]_q!}.$$

Also

$$\Gamma_q(t+1)=[t]_q\Gamma_q(t),\quad \Gamma_q(1)=1.$$

We denote

$$(1+x)_q^n=\begin{cases}(1+x)(1+qx)\cdots(1+q^{n-1}), & n=1,2,\cdots,\\ 1, & n=0.\end{cases}$$

The q-Beta integral is defined by

$$B_q(t,s)=K(A,t)\int_0^{\infty/A}\frac{x^{t-1}}{(1+x)_q^{n+k}}d_qx, \tag{1.6}$$

where

$$K(x,t)=\frac{1}{x+1}x^t\Big(1+\frac{1}{x}\Big)_q^t(1+x)_q^{1-t}.$$

It was observed in that $K(x,t)$ is a q-constant i.e., $K(qx,t)=K(x,t)$. In particular for any positive integer n, one has

$$K(x,n)=q^{\frac{n(n-1)}{2}},\quad K(x,0)=1\quad \text{and}\quad B_q(t,s)=\frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}. \tag{1.7}$$

For details on q-Beta function, we refer the readers to [19]. Inspired by the Stancu type generalization of q-Baskakov operators. For $0\le\alpha\le\beta$ we introduce the q-Szász-Beta-Stancu operators defined as

$$\begin{aligned}&S_{n,q}^{\alpha,\beta}(f(t);x)\\ =&\sum_{k=1}^{\infty}q^{\frac{3k^2-3k}{2}}s_{n,k}^q(x)\int_0^{\infty/A}p_{n,k}^q(t)f\Big(\frac{q[n]_qt+\alpha}{[n]_q+\beta}\Big)d_qt+E_q(-[n]_qx)f\Big(\frac{\alpha}{[n]_q+\beta}\Big),\end{aligned} \tag{1.8}$$

where $s_{n,k}^q(x)$ and $p_{n,k}^q(t)$ are given by (1.3). For $\alpha=\beta=0$, we get (1.2). In the present paper, we study the direct theorem, rate of approximation and Voronovskaja type asymptotic formula for the operators $S_{n,q}^{\alpha,\beta}$.

2 Moment estimates

Lemma 2.1 (see [9]). *For $S_{n,q}(t^m;x)$, $m=0,1,2$, one has*

(i) $S_{n,q}(1;x)=1$;

(ii) $S_{n,q}(t;x)=x$;

(iii) $S_{n,q}(t^2;x)=\frac{[n]_qx^2+[2]_qx}{q[n-1]_q}$ *for* $n>1$.

Lemma 2.2. *The following equalities hold:*

(i) $S^{\alpha,\beta}_{n,q}(1;x)=1$;

(ii) $S^{\alpha,\beta}_{n,q}(t;x)=\frac{[n]_qx+\alpha}{[n]_q+\beta}$;

(iii) $S^{\alpha,\beta}_{n,q}(t^2;x)=\frac{[n]^3_qx^2}{q[n-1]_q([n]_q+\beta)^2}+\big(\frac{[2]_q[n]^2_q}{q[n-1]_q}+2\alpha[n]_q\big)\frac{x}{([n]_q+\beta)^2}+\frac{\alpha^2}{([n]_q+\beta)^2}$ *for* $n>1$.

Proof. Obviously the $S^{\alpha,\beta}_{n,q}$ are well defined on the function 1, t, t^2. By Lemma 2.1, we estimate the moments as follows: First, for $f(t)=1$, we have

$$S^{\alpha,\beta}_{n,q}(1;x)=\sum_{k=1}^{\infty}q^{\frac{3k^2-3k}{2}}s^q_{n,k}(x)\int_0^{\infty/A}p^q_{n,k}(t)d_qt+E_q(-[n]_qx)=S_{n,q}(1;x)=1.$$

Next, we estimate the first order moment

$$\begin{aligned}
S^{\alpha,\beta}_{n,q}(t;x)=&\sum_{k=1}^{\infty}q^{\frac{3k^2-3k}{2}}s^q_{n,k}(x)\int_0^{\infty/A}p^q_{n,k}(t)\frac{q[n]_qt+\alpha}{[n]_q+\beta}d_qt+E_q(-[n]_qx)\frac{\alpha}{[n]_q+\beta}\\
=&\frac{[n]_q}{[n]_q+\beta}\sum_{k=1}^{\infty}q^{\frac{3k^2-3k}{2}}s^q_{n,k}(x)\int_0^{\infty/A}p^q_{n,k}(t)qtd_qt\\
&+\frac{\alpha}{[n]_q+\beta}\Big(\sum_{k=1}^{\infty}q^{\frac{3k^2-3k}{2}}s^q_{n,k}(x)\int_0^{\infty/A}p^q_{n,k}(t)d_qt+E_q(-[n]_qx)\Big)\\
=&\frac{[n]_q}{[n]_q+\beta}S_{n,q}(t;x)+\frac{\alpha}{[n]_q+\beta}S_{n,q}(1;x)\\
=&\frac{[n]_qx+\alpha}{[n]_q+\beta}.
\end{aligned}$$

Finally, for $n>1$,

$$\begin{aligned}
S^{\alpha,\beta}_{n,q}(t^2;x)=&\sum_{k=1}^{\infty}q^{\frac{3k^2-3k}{2}}s^q_{n,k}(x)\int_0^{\infty/A}p^q_{n,k}(t)\Big(\frac{q[n]_qt+\alpha}{[n]_q+\beta}\Big)^2d_qt+E_q(-[n]_qx)\Big(\frac{\alpha}{[n]_q+\beta}\Big)^2\\
=&\frac{[n]^2_q}{([n]_q+\beta)^2}\sum_{k=1}^{\infty}q^{\frac{3k^2-3k}{2}}s^q_{n,k}(x)\int_0^{\infty/A}p^q_{n,k}(t)q^2t^2d_qt
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2\alpha[n]_q}{([n]_q+\beta)^2}\sum_{k=1}^{\infty}q^{\frac{3k^2-3k}{2}}s_{n,k}^q(x)\int_0^{\infty/A}p_{n,k}^q(t)qtd_qt\\
&+\frac{\alpha^2}{([n]_q+\beta)^2}\Big(\sum_{k=1}^{\infty}q^{\frac{3k^2-3k}{2}}s_{n,k}^q(x)\int_0^{\infty/A}p_{n,k}^q(t)d_qt+E_q(-[n]_qx)\Big)\\
=&\frac{[n]_q^2}{([n]_q+\beta)^2}S_{n,q}(t^2;x)+\frac{2\alpha[n]_q}{([n]_q+\beta)^2}S_{n,q}(t;x)+\frac{\alpha^2}{([n]_q+\beta)^2}S_{n,q}(1;x)\\
=&\frac{[n]_q^3x^2}{q[n-1]_q([n]_q+\beta)^2}+\Big(\frac{[2]_q[n]_q^2}{q[n-1]_q}+2\alpha[n]_q\Big)\frac{x}{([n]_q+\beta)^2}+\frac{\alpha^2}{([n]_q+\beta)^2}.
\end{aligned}
$$

Thus, we complete the proof. □

Remark 2.1. Let $n>1$ and $x\in[0,\infty)$, then for every $q\in(0,1)$, we have

$$
\begin{aligned}
&S_{n,q}^{\alpha,\beta}((t-x);x)=\frac{\alpha-\beta x}{[n]_q+\beta},\\
&I_{n,\alpha,\beta}=S_{n,q}^{\alpha,\beta}((t-x)^2;x)=\frac{[n]_q^3x^2}{q[n-1]_q([n]_q+\beta)^2}+\Big(\frac{[2]_q[n]_q^2}{q[n-1]_q}+2\alpha[n]_q\Big)\frac{x}{([n]_q+\beta)^2}\\
&\qquad+\frac{\alpha^2}{([n]_q+\beta)^2}-\frac{2x([n]_qx+\alpha)}{[n]_q+\beta}+x^2\\
&=\Big(\frac{[n]_q^3}{q[n-1]_q([n]_q+\beta)^2}+1-\frac{2[n]_q}{[n]_q+\beta}\Big)x^2+\Big[\Big(\frac{[2]_q[n]_q^2}{q[n-1]_q}+2\alpha[n]_q\Big)\frac{1}{[n]_q+\beta}-2\alpha\Big]\frac{x}{[n]_q+\beta}\\
&+\frac{\alpha^2}{([n]_q+\beta)^2}.
\end{aligned}
$$

Therefore,

$$
S_{n,q}^{\alpha,\beta}((t-x)^2;x)\le\Big(\frac{[n]_q}{q[n-1]_q}-1\Big)x^2+\frac{[2]_q}{q[n-1]_q}x+\frac{\alpha^2}{([n]_q+\beta)^2}. \tag{2.1}
$$

3 Direct theorem

By $C_B[0,\infty)$ we denote the class of all real valued continuous bounded functions f defined on $[0,\infty)$. The norm on this space is given by

$$
\|f\|=\sup_{x\in[0,\infty)}|f|.
$$

We denote the usual modulus of continuity of $f\in C_B[0,\infty)$ as

$$
\omega(f,\delta)=\sup_{0<h\le\delta}\ \sup_{x\in[0,\infty)}|f(x+h)-f(x)|
$$

and the second order modulus of smoothness of f as

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h\leq\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|.$$

The K-functional are defined as

$$K_2(f,\delta)=\inf\{\|f-g\|+\delta\|g''\|\},$$

where $\delta>0$ and $W^2=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}$. By [14, pp. 177, Theorem 2.4], there exist an absolute constant $C>0$ such that

$$K_2(f,\delta)\leq C\omega_2(f,\sqrt{\delta}).$$

Theorem 3.1. *Let $f\in C_B[0,\infty)$ and $q\in[0,1)$. Then for every $x\in[0,\infty)$ and $n>1$, there exists an absolute constant $C>0$ such that*

$$|S_{n,q}^{\alpha,\beta}(f;x)-f(x)|\leq C\omega_2\Big(f,\frac{\delta_n(x)}{2}\Big)+\omega\Big(f,\frac{\alpha-\beta x}{[n]_q+\beta}\Big),$$

where

$$\delta_n(x)=\Big[I_{n,\alpha,\beta}+\Big(\frac{\alpha-\beta x}{[n]_q+\beta}\Big)^2\Big]^{\frac{1}{2}}.$$

Proof. For $x\in[0,\infty)$, we consider the auxiliary operators $\overline{S_{n,q}^{\alpha,\beta}}$, which is defined by

$$\overline{S_{n,q}^{\alpha,\beta}(f;x)}=S_{n,q}^{\alpha,\beta}(f;x)-f\Big(\frac{[n]_qx+\alpha}{[n]_q+\beta}\Big)+f(x). \tag{3.1}$$

We can learn from Lemma 2.2 that these operators $\overline{S_{n,q}}$ are linear and vanish the linear function:

$$\overline{S_{n,q}^{\alpha,\beta}}(t-x;x)=0. \tag{3.2}$$

Let $g\in W^2$ and $x,t\in[0,\infty)$, by Taylor's expansion, we have

$$g(t)=g(x)+g(x)'(t-x)+\int_x^t(t-u)g(u)''du.$$

Applying (3.2), we get

$$\overline{S_{n,q}^{\alpha,\beta}}(g;x)=g(x)+\overline{S_{n,q}^{\alpha,\beta}}\Big(\int_x^t(t-u)g(u)''du;x\Big).$$

Hence, by (3.1), we have

$$\begin{aligned}
|\overline{S_{n,q}^{\alpha,\beta}}(g;x)-g(x)| &\le \Big|S_{n,q}^{\alpha,\beta}\Big(\int_x^t (t-u)g(u)''du;x\Big)\Big| + \Big|\int_x^{\frac{[n]_qx+\alpha}{[n]_q+\beta}} \Big(\frac{[n]_qx+\alpha}{[n]_q+\beta}-u\Big)g(u)''du\Big| \\
&\le S_{n,q}^{\alpha,\beta}\Big(\Big|\int_x^t (t-u)g(t)''du\Big|;x\Big) + \int_x^{\frac{[n]_qx+\alpha}{[n]_q+\beta}} \Big|\frac{[n]_qx+\alpha}{[n]_q+\beta}-u\Big||g(u)''|du \\
&\le \Big[S_{n,q}^{\alpha,\beta}((t-x)^2;x)+\Big(\frac{\alpha-\beta x}{[n]_qx+\beta}\Big)^2\Big]\|g''\| \\
&= \Big(I_{n,\alpha,\beta}+\Big(\frac{\alpha-\beta x}{[n]_qx+\beta}\Big)^2\Big)\|g''\| \\
&= \delta_n(x)^2\|g''\|.
\end{aligned} \tag{3.3}$$

On the other hand, from (1.8) we know

$$\begin{aligned}
&|S_{n,q}^{\alpha,\beta}(f(t);x)| \\
\le &\sum_{k=1}^{\infty} q^{\frac{3k^2-3k}{2}} s_{n,k}^q(x)\int_0^{\infty/A} p_{n,k}^q(t)\Big|f\Big(\frac{q[n]_qt+\alpha}{[n]_q+\beta}\Big)\Big|d_qt + E_q(-[n]_qx)\Big|f\Big(\frac{\alpha}{[n]_q+\beta}\Big)\Big| \\
\le &\|f\|,
\end{aligned} \tag{3.4}$$

then by (3.1), we have

$$|\overline{S_{n,q}^{\alpha,\beta}}(f;x)| \le |S_{n,q}^{\alpha,\beta}(f;x)|+2\|f\| \le 3\|f\|. \tag{3.5}$$

Now using (3.1), (3.3) and (3.5), we have

$$\begin{aligned}
|S_{n,q}^{\alpha,\beta}(f(t);x)-f(x)| \le& |\overline{S_{n,q}^{\alpha,\beta}}(f-g;x)-(f-g)(x)| + |\overline{S_{n,q}^{\alpha,\beta}}(g;x)-g(x)| \\
&+\Big|f\Big(\frac{[n]_qx+\alpha}{[n]_q+\beta}\Big)-f(x)\Big| \\
\le& 4\|f-g\| + \delta_n(x)^2\|g''\| + \Big|f\Big(\frac{[n]_qx+\alpha}{[n]_q+\beta}\Big)-f(x)\Big|.
\end{aligned}$$

Thus, taking the infimum on the right hand over all $g\in W^2$, we get

$$|S_{n,q}^{\alpha,\beta}(f(t);x)-f(x)| \le CK_2\Big(f,\frac{\delta_n(x)^2}{4}\Big)+\omega\Big(f,\frac{\alpha-\beta x}{[n]_q+\beta}\Big).$$

In view of $K_2(f,\delta_n(x)) \le C\omega_2(f,\sqrt{\delta})$, we get

$$|S_{n,q}^{\alpha,\beta}(f(t);x)-f(x)| \le C\omega_2\Big(f,\frac{\delta_n(x)}{2}\Big)+\omega\Big(f,\frac{\alpha-\beta x}{[n]_q+\beta}\Big).$$

This completes the proof of the theorem. □

4 Rate of approximation

Let $H_{x^2}[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)|\leq M_f(1+x^2)$, where M_f is a constant depending only on f. By $C_{x^2}[0,\infty)$ we denote the subspace of all continuous functions belonging to $H_{x^2}[0,\infty)$. Let $C^*_{x^2}[0,\infty)$ be the subspace of all functions $f\in C_{x^2}[0,\infty)$, for which $\lim_{|x|\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is defined by

$$\|f\|_{x^2}=\sup_{x\in[0,\infty)}\frac{f(x)}{1+x^2}.$$

The modulus of continuity of f on the closed interval $[0,a]$, $a>0$ is

$$\omega_a(f,\delta)=\sup_{|t-x|\leq\delta}\sup_{x,t\in[0,a)}|f(x+h)-f(x)|.$$

We can see that for a function $f\in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero.

Theorem 4.1. *Let $q=q_n$ satisfies $0<q_n<1$ and let $q_n\to 1$ as $n\to\infty$. For each $f\in C^*_{x^2}[0,\infty)$, we have*

$$\lim_{n\to\infty}\|S^{\alpha,\beta}_{n,q_n}(f)-f(x)\|_{x^2}=0.$$

Proof. Using the Korovkin's theorem in [15], it is sufficient to verify the following three conditions

$$\lim_{n\to\infty}\|S^{\alpha,\beta}_{n,q_n}(t^v;x)-x^v\|_{x^2}=0 \quad \text{for } v=0,1,2, \tag{4.1}$$

since $S^{\alpha,\beta}_{n,q_n}(1,x)=1$, (4.1) holds for $v=0$. By Lemma 2.2, we have for $n>1$

$$\begin{aligned}\|S^{\alpha,\beta}_{n,q_n}(t;x)-x\|_{x^2}&=\sup_{x\in[0,\infty)}\Big(\frac{[n]_{q_n}x+\alpha}{[n]_{q_n}+\beta}-x\Big)\frac{1}{1+x^2}\\&=\frac{\alpha}{[n]_{q_n}+\beta}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}-\frac{\beta}{[n]_{q_n}+\beta}\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\\&\leq\frac{\alpha}{[n]_{q_n}+\beta}.\end{aligned}\tag{4.2}$$

Thus

$$\lim_{n\to\infty}\|S^{\alpha,\beta}_{n,q_n}(t;x)-x\|_{x^2}=0.$$

Similarly, we have

$$\begin{aligned}
&\|S^{\alpha,\beta}_{n,q_n}(t^2;x)-x^2\|_{x^2}\\
=&\sup_{x\in[0,\infty)}\Big[\Big(\frac{[n]^3_{q_n}}{q_n[n-1]_{q_n}([n]_{q_n}+\beta)^2}-1\Big)x^2+\Big(\frac{[2]_{q_n}[n]^2_{q_n}}{q_n[n-1]_{q_n}}+2\alpha[n]_{q_n}\Big)\frac{x}{([n]_{q_n}+\beta)^2}\\
&+\frac{\alpha^2}{([n]_{q_n}+\beta)^2}\Big]\frac{1}{1+x^2}\\
\le&\Big(\frac{[n]^3_{q_n}}{q_n[n-1]_{q_n}([n]_{q_n}+\beta)^2}-1\Big)\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}\\
&+\Big(\frac{[2]_{q_n}[n]^2_{q_n}}{q_n[n-1]_{q_n}}+2\alpha[n]_{q_n}\Big)\frac{1}{([n]_{q_n}+\beta)^2}\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\frac{\alpha^2}{([n]_{q_n}+\beta)^2}\sup_{x\in[0,\infty)}\frac{1}{1+x^2},
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|S^{\alpha,\beta}_{n,q_n}(t^2;x)-x^2\|_{x^2}=0.$$

This completes the proof of the Theorem 4.1. □

Theorem 4.2. *Let $f\in C_{x^2}[0,\infty)$, $q\in(0,1)$ and $\omega_{a+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,a+1]\subset[0,\infty)$, where $a>0$, then for every $n>1$, we have*

$$\begin{aligned}
&\|S^{\alpha,\beta}_{n,q}(f;x)-f(x)\|_{c[0,a]}\\
\le&6M_f(1+a^2)\Big[\Big(\frac{[n]_q}{q[n-1]_q}-1\Big)x^2+\frac{[2]_q}{q[n-1]_q}x+\frac{\alpha^2}{([n]_q+\beta)^2}\Big]\\
&+2\omega_{a+1}\Big(f,\Big[\Big(\frac{[n]_q}{q[n-1]_q}-1\Big)x^2+\frac{[2]_q}{q[n-1]_q}x+\frac{\alpha^2}{([n]_q+\beta)^2}\Big]^{\frac{1}{2}}\Big).
\end{aligned}$$

Proof. For $x\in[0,a]$, when $t\le a+1$, we have

$$|f(t)-f(x)|\le\omega_{a+1}(f,|t-x|)\le\Big(1+\frac{|t-x|}{\delta}\Big)\omega_{a+1}(f,\delta) \tag{4.3}$$

with $\delta>0$. When $t>a+1$, since $t-x>1$, we have

$$|f(t)-f(x)|\le M_f(2+x^2+t^2)\le 6M_f(1+a^2)(t-x)^2. \tag{4.4}$$

From (4.3) and (4.4), for $x\in[0,a]$ and $t>0$, we have

$$|f(t)-f(x)|\le 6M_f(1+a^2)(t-x)^2+\Big(1+\frac{|t-x|}{\delta}\Big)\omega_{a+1}(f,\delta).$$

Hence, by Lemma 2.2 and Schwartz's inequality, we have for every $q\in(0,1)$, $x\in[0,a]$

$$\begin{aligned}
&|S_{n,q}^{\alpha,\beta}(f;x)-f(x)|\\
\le& S_{n,q}^{\alpha,\beta}(|f(t)-f(x)|;x)\\
\le& 6M_f(1+a^2)S_{n,q}^{\alpha,\beta}((t-x)^2;x)+\omega_{a+1}(f,\delta)\Big(1+\frac{1}{\delta}(S_{n,q}^{\alpha,\beta}((t-x)^2;x))^{\frac{1}{2}}\Big)\\
\le& 6M_f(1+a^2)\Big[\Big(\frac{[n]_q}{q[n-1]_q}-1\Big)x^2+\frac{[2]_q}{q[n-1]_q}x+\frac{\alpha^2}{([n]_q+\beta)^2}\Big]\\
&+\omega_{a+1}(f,\delta)\Big(1+\frac{1}{\delta}\Big[\Big(\frac{[n]_q}{q[n-1]_q}-1\Big)x^2+\frac{[2]_q}{q[n-1]_q}x+\frac{\alpha^2}{([n]_q+\beta)^2}\Big]^{\frac{1}{2}}\Big).
\end{aligned}$$

By taking

$$\delta=\Big[\Big(\frac{[n]_q}{q[n-1]_q}-1\Big)x^2+\frac{[2]_q}{q[n-1]_q}x+\frac{\alpha^2}{([n]_q+\beta)^2}\Big]^{\frac{1}{2}},$$

we have the desired result. □

5 Pointwise estimates

We say a function $f\in C[0,\infty)$ is in $Lip\ \alpha$ on D, $D\subset[0,\infty)$, $\alpha\in(0,1]$, if f satisfies the condition

$$|f(t)-f(x)|\le M_f|t-x|^\alpha,\quad t\in[0,\infty)\quad\text{and}\quad x\in D,$$

where M_f is a constant depending only on α and f. Now, we give some pointwise estimates for the rate of convergence of the q analogues of Szász-Beta-Stancu operators.

Theorem 5.1. *Let $f\in Lip\ \alpha$, $\alpha\in(0,1]$, $D\subset[0,\infty)$, then*

$$|S_{n,q}^{\alpha,\beta}(f;x)-f(x)|\le M_f\Big(\Big[\Big(\frac{[n]_q}{q[n-1]_q}-1\Big)x^2+\frac{[2]_q}{q[n-1]_q}x+\frac{\alpha^2}{([n]_q+\beta)^2}\Big]^{\frac{\alpha}{2}}+2d^\alpha(x,D)\Big),$$

where $d(x,D)$ represents the distance between x and D.

Proof. For $x_0\in\overline{D}$, the closure of the set D in $[0,\infty)$, we have

$$|f(t)-f(x)|\le|f(t)-f(x_0)|+|f(x_0)-f(x)|,\quad x\in[0,\infty),$$

so we have

$$\begin{aligned}
|S_{n,q}^{\alpha,\beta}(f;x)-f(x)|&\le S_{n,q}^{\alpha,\beta}(|f(t)-f(x_0)|;x)+|f(x_0)-f(x)|\\
&\le M_fS_{n,q}^{\alpha,\beta}(|t-x_0|^\alpha;x)+M_f|x_0-x|^\alpha. \qquad (5.1)
\end{aligned}$$

On the other hand,

$$S_{n,q}^{\alpha,\beta}(|t-x|^\alpha;x)\le(S_{n,q}^{\alpha,\beta}(|t-x|^2;x))^{\frac{\alpha}{2}}(S_{n,q}^{\alpha,\beta}(1;x))^{1-\frac{\alpha}{2}} \qquad (5.2)$$

and

$$S_{n,q}^{\alpha,\beta}(|t-x_0|^{\alpha};x)\le (S_{n,q}^{\alpha,\beta}(|t-x|^2;x))^{\frac{\alpha}{2}}+|x_0-x|^{\alpha}.$$

Using (5.1), (5.2) and the inequality (2.1), we have

$$\begin{aligned}|S_{n,q}^{\alpha,\beta}(f;x)-f(x)| &\le M_f(S_{n,q}^{\alpha,\beta}(|t-x|^2;x))^{\frac{\alpha}{2}}+2M_f|x_0-x|^{\alpha}\\ &\le M_f[(S_{n,q}^{\alpha,\beta}(|t-x|^2;x))^{\frac{\alpha}{2}}+2d^{\alpha}(x,D)]\\ &\le M_f\Big(\Big[\Big(\frac{[n]_q}{q[n-1]_q}-1\Big)x^2+\frac{[2]_q}{q[n-1]_q}x+\frac{\alpha^2}{([n]_q+\beta)^2}\Big]^{\frac{\alpha}{2}}+2d^{\alpha}(x,D)\Big).\end{aligned}$$

Thus the result holds. □

6 Voronovskaja type theorem

Lemma 6.1. *Let $q=q_n$ satisfies $0<q_n<1$ and let $q_n\to 1$, $q_n^n\to\lambda$ as $n\to\infty$. For each $x\in[0,\infty)$, we have*

$$\begin{aligned}&\lim_{n\to\infty}[n]_{q_n}S_{n,q_n}^{\alpha,\beta}((t-x);x)=\alpha-\beta x,\\ &\lim_{n\to\infty}[n]_{q_n}S_{n,q_n}^{\alpha,\beta}((t-x)^2;x)=2x.\end{aligned}$$

Using Lemma 2.2 and making necessary process we can easily get the proof of this Lemma so we omit it.

Theorem 6.1. *Let $q_n\to 1$, $q_n^n\to\lambda$, as $n\to\infty$, $f\in C_{x^2}^*[0,\infty)$, and $f',f''\in C_{x^2}^*[0,\infty)$, then we have*

$$\lim_{n\to\infty}[n]_{q_n}(S_{n,q_n}^{\alpha,\beta}(f;x)-f(x))=(\alpha-\beta x)f'(x)+xf''(x).$$

Proof. Using Taylor's expansion, we get

$$f(t)-f(x)=f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2,$$

where $r(t,x)$ is Peano form of the remainder, and $r(t,x)\to 0$ as $t\to x$. By applying the operator $S_{n,q_n}^{\alpha,\beta}(f;x)$ to the above relation, we obtain

$$\begin{aligned}&S_{n,q_n}^{\alpha,\beta}(f;x)-f(x)\\ =&f'(x)S_{n,q_n}^{\alpha,\beta}((t-x);x)+\frac{1}{2}f''(x)S_{n,q_n}^{\alpha,\beta}((t-x)^2;x)+S_{n,q_n}^{\alpha,\beta}(r(t,x)(t-x)^2;x).\end{aligned}$$

Applying Cauchy-Schwarz inequality, we have

$$[n]_{q_n}S_{n,q_n}^{\alpha,\beta}(r(t,x)(t-x)^2;x)\le\sqrt{S_{n,q_n}^{\alpha,\beta}(r(t,x)^2;x)}\sqrt{[n]_{q_n}^2S_{n,q_n}^{\alpha,\beta}((t-x)^4;x)}.$$

It's easy to observe that

$$\lim_{n\to\infty} S_{n,q_n}^{\alpha,\beta}(r(t,x)^2,x)=0,$$

and using Lemma 2.2 and making necessary process, we know $\lim_{n\to\infty}[n]_{q_n}^2 S_{n,q_n}^{\alpha,\beta}((t-x)^4,x)$ is finite. So we get

$$\lim_{n\to\infty}[n]_{q_n} S_{n,q_n}^{\alpha,\beta}(r(t,x)(t-x)^2;x)=0.$$

Therefore, using Lemma 6.1, we yield

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{q_n}(S_{n,q_n}^{\alpha,\beta}(f;x)-f(x))\\
=&f'(x)\lim_{n\to\infty}[n]_{q_n}S_{n,q_n}^{\alpha,\beta}((t-x);x)+\frac{1}{2}f''(x)\lim_{n\to\infty}[n]_{q_n}S_{n,q_n}^{\alpha,\beta}((t-x)^2;x)\\
&+\lim_{n\to\infty}[n]_{q_n}S_{n,q_n}^{\alpha,\beta}(r(t,x)(t-x)^2;x)\\
=&(\alpha-\beta x)f'(x)+xf''(x),
\end{aligned}$$

which complete the proof. □

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