Analysis in Theory and Applications
Anal. Theory Appl., Vol. **31**, No. 1 (2015), pp. 81-91

DOI: 10.4208/ata.2015.v31.n1.7

Some Integral Mean Estimates for Polynomials with Restricted Zeros

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Received 19 January 2014; Accepted (in revised version) 11 March 2015

Abstract. Let $P(z)$ be a polynomial of degree n having all its zeros in $|z|\le k$. For $k=1$, it is known that for each $r>0$ and $|\alpha|\ge 1$,

$$n(|\alpha|-1)\Big\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\Big\}^{\frac{1}{r}} \le \Big\{\int_0^{2\pi}|1+e^{i\theta}|^r d\theta\Big\}^{\frac{1}{r}}\max_{|z|=1}|D_\alpha P(z)|.$$

In this paper, we shall first consider the case when $k\ge 1$ and present certain generalizations of this inequality. Also for $k\le 1$, we shall prove an interesting result for Lacunary type of polynomials from which many results can be easily deduced.

Key Words: Polynomial, zeros, polar derivative.

AMS Subject Classifications: 30A10, 30C10, 30D15

1 Introduction and statement of results

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative. It was shown by Turan [21] that if $P(z)$ has all its zeros in $|z|\le 1$, then

$$\max_{|z|=1}|P'(z)|\ge \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1.1}$$

More generally, if the polynomial $P(z)$ has all its zeros in $|z|\le k\le 1$, it was proved by Malik [12] that the inequality (1.1) can be replaced by

$$\max_{|z|=1}|P'(z)|\ge \frac{n}{1+k}\max_{|z|=1}|P(z)|, \tag{1.2}$$

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while as Govil [6] proved that if all the zeros of $P(z)$ lie in $|z|\le k$ where $k\ge 1$, then

$$\max_{|z|=1}|P'(z)|\ge \frac{n}{1+k^n}\max_{|z|=1}|P(z)|. \tag{1.3}$$

As an improvement of (1.3), Govil [7] proved that if $P(z)$ has all its zeros in $|z|\le k$ where $k\ge 1$, then

$$\max_{|z|=1}|P'(z)|\ge \frac{n}{1+k^n}\Big(\max_{|z|=1}|P(z)|+\min_{|z|=k}|P(z)|\Big). \tag{1.4}$$

Let $D_\alpha P(z)$ denotes the polar derivative of the polynomial $P(z)$ of degree n with respect to the point α. Then

$$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\Big\{\frac{D_\alpha P(z)}{\alpha}\Big\}=P'(z). \tag{1.5}$$

Shah [18] extended (1.1) to the polar derivative of $P(z)$ and proved that if all the zeros of the polynomial $P(z)$ lie in $|z|\le 1$, then

$$\max_{|z|=1}\big|D_\alpha P(z)\big|\ge \frac{n}{2}\big(|\alpha|-1\big)\max_{|z|=1}|P(z)|, \quad |\alpha|\ge 1. \tag{1.6}$$

Aziz and Rather [3] generalised (1.6) which also extends (1.2) to the polar derivative of a polynomial. In fact, they proved that if all the zeros of $P(z)$ lie in $|z|\le k$ where $k\le 1$, then for every real or complex number α with $|\alpha|\ge k$,

$$\max_{|z|=1}\big|D_\alpha P(z)\big|\ge n\Big(\frac{|\alpha|-k}{1+k}\Big)\max_{|z|=1}|P(z)|. \tag{1.7}$$

Further as a generalization of (1.3) to the polar derivative of a polynomial, Aziz and Rather [3] proved that if all the zeros of $P(z)$ lie in $|z|\le k$ where $k\ge 1$, then for every real or complex number α with $|\alpha|\ge k$,

$$\max_{|z|=1}\big|D_\alpha P(z)\big|\ge n\Big(\frac{|\alpha|-k}{1+k^n}\Big)\max_{|z|=1}|P(z)|. \tag{1.8}$$

Recently Govil and McTume [8] sharpened (1.8) and proved that if all the zeros of $P(z)$ lie in $|z|\le k$, $k\ge 1$, then for every real or complex number α with $|\alpha|\ge 1+k+k^n$,

$$\begin{aligned}\max_{|z|=1}\big|D_\alpha P(z)\big|\ge & n\Big(\frac{|\alpha|-k}{1+k^n}\Big)\max_{|z|=1}|P(z)|\\ &+n\Big(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\Big)\min_{|z|=k}|P(z)|.\end{aligned} \tag{1.9}$$

On the other hand, Malik [13] obtained an L^r analogue of (1.1) by proving that if $P(z)$ has all its zeros in $|z|\le 1$, then for each $r>0$,

$$n\Big\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\Big\}^{\frac{1}{r}}\le\Big\{\int_0^{2\pi}|1+e^{i\theta}|^r d\theta\Big\}^{\frac{1}{r}}\max_{|z|=1}|P'(z)|. \tag{1.10}$$

As an extension of (1.3), Aziz [1] proved that if $P(z)$ has all its zeros in $|z|\le k$, $k\ge 1$, then for each $r\ge 1$,

$$n\Big\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\Big\}^{\frac{1}{r}}\le\Big\{\int_0^{2\pi}|1+k^n e^{i\theta}|^r d\theta\Big\}^{\frac{1}{r}}\max_{|z|=1}|P'(z)|. \tag{1.11}$$

More recently, Dewan, Singh, Mir and Bhat [5] generalized (1.6) by obtaining an L^r analogue of it. More precisely, they proved that if $P(z)$ has all its zeros in $|z|\le 1$, then for every real or complex number α with $|\alpha|\ge 1$ and for each $r>0$,

$$n(|\alpha|-1)\Big\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\Big\}^{\frac{1}{r}}\le\Big\{\int_0^{2\pi}|1+e^{i\theta}|^r d\theta\Big\}^{\frac{1}{r}}\max_{|z|=1}\big|D_\alpha P(z)\big|. \tag{1.12}$$

If we let $r\to\infty$ in (1.12) and make use of the well-known fact from analysis (see for example [17, pp. 73] or [20, pp. 91]) that

$$\Big\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\Big\}^{\frac{1}{r}}\to\max_{0\le\theta<2\pi}|P(e^{i\theta})|,$$

we get (1.6).

In this paper, we shall first present certain generalizations of the inequality (1.12) by considering polynomials having all zeros in $|z|\le k$, $k\ge 1$. We shall also prove a result for Lacunary type of polynomials having all zeros in $|z|\le k$, $k\le 1$ from which many results can be easily deduced.

Theorem 1.1. *If $P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z|\le k$ where $k\ge 1$, then for every complex number α with $|\alpha|\ge k$ and for each $r>0$, $p>1$, $q>1$ with $p^{-1}+q^{-1}=1$, we have*

$$n(|\alpha|-k)\Big\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\Big\}^{\frac{1}{r}}\le C_r\Big\{\int_0^{2\pi}|1+e^{i\theta}|^{qr} d\theta\Big\}^{\frac{1}{qr}}\Big\{\int_0^{2\pi}\big|D_\alpha P(e^{i\theta})\big|^{pr} d\theta\Big\}^{\frac{1}{pr}}, \tag{1.13}$$

where

$$C_r=\frac{\big\{\int_0^{2\pi}|1+k^n e^{i\theta}|^r d\theta\big\}^{\frac{1}{r}}}{\big\{\int_0^{2\pi}|1+e^{i\theta}|^r d\theta\big\}^{\frac{1}{r}}}.$$

Remark 1.1. If we let $r\to\infty$ and $p\to\infty$ (so that $q\to 1$) in (1.13) we get (1.8). If we divide both sides of (1.13) by $|\alpha|$ and let $|\alpha|\to\infty$, we get a result recently proved Mir and Dar [15]. If we take $k=1$ in (1.13) and note that $C_r=1$, we obtain a generalization of (1.12) in the sense that the right hand side of (1.12) is replaced by a factor involving the integral mean of $|D_\alpha P(z)|$ on $|z|=1$.

The following corollary immediately follows by letting $p\to\infty$ (so that $q\to 1$) in Theorem 1.1.

Corollary 1.1. If $P(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z|\le k$ where $k\ge 1$, then for every complex number α with $|\alpha|\ge k$ and for each $r>0$,

$$n(|\alpha|-k)\left\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}}\le\left\{\int_0^{2\pi}|1+k^n e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}\max_{|z|=1}\big|D_\alpha P(z)\big|. \tag{1.14}$$

Remark 1.2. Dividing both sides of (1.14) by $|\alpha|$ and let $|\alpha|\to\infty$, we get (1.11) and also extends it to the values $r\in(0,1)$. For $k=1$, Corollary 1.1 reduces to inequality (1.12).

Our next result is a generalization of Theorem 1.1 which in turn provides extensions and generalizations of results of Aziz and Ahemad [2]. We will see that as a special case Theorem 1.2 gives a result of Govil and McTume [8, Theorem 3].

Theorem 1.2. *If $P(z)=\sum_{v=0}^{n} a_v z^v$ is a polynomial of degree n having all its zeros in $|z|\le k$, where $k\ge 1$, then for every complex numbers α, λ with $|\alpha|\ge k$, $|\lambda|<1$ and for each $r>0$, $p>1$, $q>1$ with $p^{-1}+q^{-1}=1$, we have*

$$\begin{aligned}&n(|\alpha|-k)\left\{\int_0^{2\pi}|P(e^{i\theta})+\lambda m|^r d\theta\right\}^{\frac{1}{r}}\\ \le& C_r\left\{\int_0^{2\pi}|1+e^{i\theta}|^{qr}d\theta\right\}^{\frac{1}{qr}}\left\{\int_0^{2\pi}\big|D_\alpha P(e^{i\theta})+\lambda mn\big|^{pr}d\theta\right\}^{\frac{1}{pr}},\end{aligned} \tag{1.15}$$

where $m=\min_{|z|=k}|P(z)|$ and C_r is same as defined in Theorem 1.1.

Remark 1.3. A variety of interesting results can be easily deduced from Theorem 1.2 in the same way as we have deduced from Theorem 1.1. Here we mention a few of these. Dividing the two sides of (1.15) by $|\alpha|$ and let $|\alpha|\to\infty$, we get a result recently proved Mir and Dar [15]. Moreover, if we take $k=1$ in (1.15) (noting that $C_r=1$) and then divide both sides of it by $|\alpha|$ and let $|\alpha|\to\infty$, we get a result of Aziz and Ahemad [2, Theorem 2].

If in (1.15), we let $p\to\infty$ (so that $q\to 1$), we get

$$n(|\alpha|-k)\left\{\int_0^{2\pi}|P(e^{i\theta})+\lambda m|^r d\theta\right\}^{\frac{1}{r}}\le\left\{\int_0^{2\pi}|1+k^n e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}}\max_{|z|=1}\big|D_\alpha P(z)+\lambda mn\big|. \tag{1.16}$$

If we divide both sides of (1.16) by $|\alpha|$ and let $|\alpha|\to\infty$, we get a result of Aziz and Ahemad [2, Theorem 4] and also extends it for $0<r<1$ as well. For $\lambda=0$, (1.16) reduces to (1.14). Further, if we let $r\to\infty$ in (1.16) and assume $|\alpha|\geq 1+k+k^n$, we get

$$\max_{|z|=1}\big|D_\alpha P(z)+\lambda mn\big|\geq n\Big(\frac{|\alpha|-k}{1+k^n}\Big)\max_{|z|=1}|P(z)+\lambda m|. \tag{1.17}$$

Let z_0 be a point on $|z|=1$ such that $|P(z_0)|=\max_{|z|=1}|P(z)|$, then from (1.17), we get

$$\max_{|z|=1}\big|D_\alpha P(z)+\lambda mn\big|\geq n\Big(\frac{|\alpha|-k}{1+k^n}\Big)|P(z_0)+\lambda m|. \tag{1.18}$$

If we choose the argument of λ such that

$$|P(z_0)+\lambda m|=|P(z_0)|+|\lambda|m,$$

then from (1.18), we get

$$\max_{|z|=1}\big|D_\alpha P(z)\big|+|\lambda|mn\geq n\Big(\frac{|\alpha|-k}{1+k^n}\Big)\Big(|P(z_0)|+|\lambda|m\Big),$$

which is equivalent to

$$\max_{|z|=1}\big|D_\alpha P(z)\big|\geq n\Big(\frac{|\alpha|-k}{1+k^n}\Big)\max_{|z|=1}|P(z)|+n|\lambda|\Big(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\Big)m. \tag{1.19}$$

If in (1.19) we make $|\lambda|\to 1$, we get

$$\max_{|z|=1}\big|D_\alpha P(z)\big|\geq n\Big(\frac{|\alpha|-k}{1+k^n}\Big)\max_{|z|=1}|P(z)|+n\Big(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\Big)m, \tag{1.20}$$

which is exactly inequality (1.9).

Remark 1.4. Inequality (1.20) sharpens inequality (1.8). Also it generalise inequality (1.4) and to obtain (1.4) from (1.20) simply divide both sides of (1.20) by $|\alpha|$ and let $|\alpha|\to\infty$.

Finally, we prove the following result from which a variety of interesting results follows as special cases.

Theorem 1.3. *If $P(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu}$, $1\leq\mu\leq n$, is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, then for every complex numbers α, β with $|\alpha|\geq k$ and $|\beta|\leq 1$, we have*

$$\min_{|z|=1}\Big|zD_\alpha P(z)+n\beta\Big(\frac{|\alpha|-k^\mu}{1+k^\mu}\Big)P(z)\Big|\geq\frac{n}{k^n}\Big|\alpha+\beta\Big(\frac{|\alpha|-k^\mu}{1+k^\mu}\Big)\Big|\min_{|z|=k}|P(z)|. \tag{1.21}$$

The result is best possible and equality holds in (1.21) *for $P(z)=\gamma z^n$, $\gamma\in C$.*

Remark 1.5. For $\mu=k=1$, Theorem 1.3 reduces to a result of Liman, Mohapatra and Shah [11, Lemma 3]. If we divide both sides of inequality (1.21) by $|\alpha|$ and let $|\alpha|\to\infty$, we get

$$\min_{|z|=1}\Big|zP'(z)+\frac{n\beta}{1+k^{\mu}}P(z)\Big|\geq\frac{n}{k^n}\Big|1+\frac{\beta}{1+k^{\mu}}\Big|\min_{|z|=k}|P(z)|. \tag{1.22}$$

For $\mu=k=1$, inequality (1.22) reduces to a result of Jain [10, Lemma 3] and for $\mu=1$, inequality (1.22) reduces to a result of Soleiman et al. [19, Lemma 3].

2 Lemmas

For the proof of these theorems we shall make use of the following lemmas.

Lemma 2.1. *If $P(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu}$, $1\leq\mu\leq n$, is a polynomial of degree n having all its zeros in $|z|\leq k\leq 1$ and $Q(z)=z^n\overline{P(1/\overline{z})}$, then*

$$|Q'(z)|\leq k^{\mu}|P'(z)| \quad \textit{for} \ \ |z|=1. \tag{2.1}$$

The above lemma is due to Aziz and Shah [4].

Lemma 2.2. *If $P(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu}$, $1\leq\mu\leq n$, is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, then for every complex number α with $|\alpha|\geq k$ and $|z|=1$, we have*

$$\big|D_{\alpha}P(z)\big|\geq\frac{n(|\alpha|-k^{\mu})}{1+k^{\mu}}|P(z)|. \tag{2.2}$$

Proof. If $Q(z)=z^n\overline{P(1/\overline{z})}$, then $P(z)=z^n\overline{Q(1/\overline{z})}$ and one can easily verify that for $|z|=1$,

$$|Q'(z)|=|nP(z)-zP'(z)|\geq n|P(z)|-|P'(z)|,$$

which implies

$$|P'(z)|+|Q'(z)|\geq n|P(z)| \quad \text{for} \ \ |z|=1. \tag{2.3}$$

By combining (2.1) and (2.3), we obtain

$$|P'(z)|\geq\frac{n}{1+k^{\mu}}|P(z)| \quad \text{for} \ \ |z|=1. \tag{2.4}$$

Now for every complex number α with $|\alpha|\geq k$ $(\geq k^{\mu})$,

$$\big|D_{\alpha}P(z)\big|=|nP(z)+(\alpha-z)P'(z)|\geq|\alpha||P'(z)|-|nP(z)-zP'(z)|,$$

which implies that for $|z|=1$,

$$\big|D_{\alpha}P(z)\big|\geq|\alpha||P'(z)|-|Q'(z)|. \tag{2.5}$$

Inequality (2.5) when combined with Lemma 2.1 gives

$$\big|D_\alpha P(z)\big| \geq (|\alpha|-k^\mu)|P'(z)| \quad \text{for} \ \ |z|=1. \tag{2.6}$$

Inequality (2.6) in conjunction with (2.4) gives

$$\big|D_\alpha P(z)\big| \geq \frac{n(|\alpha|-k^\mu)}{1+k^\mu}|P(z)| \quad \text{for} \ \ |z|=1,$$

which proves Lemma 2.2 completely. □

3 Proof of theorems

Proof of Theorem 1.1. Since $P(z)$ has all its zeros in $|z|\leq k$, $k\geq 1$, it follows that the polynomial $G(z)=P(kz)$ has all its zeros in $|z|\leq 1$. Hence the polynomial $H(z)=z^n\overline{G(1/\overline{z})}$ has all its zeros in $|z|\geq 1$ and $|G(z)|=|H(z)|$ for $|z|=1$. Also it is easy to verify that for $|z|=1$,

$$|G'(z)|=|nH(z)-zH'(z)| \tag{3.1}$$

and

$$|H'(z)|=|nG(z)-zG'(z)|. \tag{3.2}$$

Again since $G(z)$ has all its zeros in $|z|\leq 1$, we have by Lemma 2.1 (for $k=\mu=1$),

$$|H'(z)|\leq |G'(z)| \quad \text{for} \ \ |z|=1. \tag{3.3}$$

Using (3.1) in (3.3), we get

$$|H'(z)|\leq |nH(z)-zH'(z)| \quad \text{for} \ \ |z|=1. \tag{3.4}$$

Now for every complex number α with $|\alpha|\geq k$, we have

$$\big|D_{\frac{\alpha}{k}}G(z)\big| = \Big|nG(z)+\Big(\frac{\alpha}{k}-z\Big)G'(z)\Big| \geq \frac{|\alpha|}{k}|G'(z)|-|nG(z)-zG'(z)|,$$

which gives by (3.2) and (3.3) for $|z|=1$, that

$$\big|D_{\frac{\alpha}{k}}G(z)\big| \geq \Big(\frac{|\alpha|}{k}-1\Big)|G'(z)|,$$

or

$$k\big|D_{\frac{\alpha}{k}}G(z)\big| \geq (|\alpha|-k)|G'(z)|. \tag{3.5}$$

Also, by the Guass-Lucas theorem, all the zeros of $G'(z)$ lie in $|z|\le 1$. This implies that the polynomial

$$z^{n-1}\overline{G'(1/\overline{z})}\equiv nH(z)-zH'(z)$$

does not vanish in $|z|<1$. Therefore, it follows from (3.4) that the function

$$W(z)=\frac{zH'(z)}{nH(z)-zH'(z)}$$

is analytic for $|z|\le 1$ and $|W(z)|\le 1$ for $|z|\le 1$. Furthermore, $W(0)=0$ and so the function $1+W(z)$ is subordinate to the function $1+z$ for $|z|\le 1$. Hence by a well-known property of sub ordination [9], we have for each $r>0$,

$$\int_0^{2\pi}|1+W(e^{i\theta})|^r d\theta\le\int_0^{2\pi}|1+e^{i\theta}|^r d\theta. \tag{3.6}$$

Now

$$1+W(z)=\frac{nH(z)}{nH(z)-zH'(z)},$$

which gives with the help of (3.1) that for $|z|=1$,

$$n|H(z)|=|1+W(z)||G'(z)|. \tag{3.7}$$

From (3.5), (3.6) and (3.7), we deduce for each $r>0$,

$$n^r(|\alpha|-k)^r\int_0^{2\pi}|H(e^{i\theta})|^r d\theta\le k^r\int_0^{2\pi}|1+e^{i\theta}|^r\big|D_{\frac{\alpha}{k}}G(e^{i\theta})\big|^r d\theta. \tag{3.8}$$

If $F(z)$ is a polynomial of degree n which does not vanish in $|z|<1$, then according to a result of Rahman and Schemeisser [16], we have for every $R\ge 1$ and $r>0$,

$$\int_0^{2\pi}|F(Re^{i\theta})|^r d\theta\le B_r\int_0^{2\pi}|F(e^{i\theta})|^r d\theta, \tag{3.9}$$

where

$$B_r=\frac{\int_0^{2\pi}|1+R^n e^{i\theta}|^r d\theta}{\int_0^{2\pi}|1+e^{i\theta}|^r d\theta}.$$

Since $H(z)$ is a polynomial of degree n and $H(z)\neq 0$ in $|z|<1$, we apply (3.9) with $R=k\ge 1$ to $H(z)$ and obtain

$$\int_0^{2\pi}|H(ke^{i\theta})|^r d\theta\le(C_r)^r\int_0^{2\pi}|H(e^{i\theta})|^r d\theta. \tag{3.10}$$

Also, since $H(z)=z^n\overline{G(1/\overline{z})}=z^n\overline{P(k/\overline{z})}$, therefore, for $0\le\theta<2\pi$, we have

$$|H(ke^{i\theta})|=|k^n e^{in\theta}\overline{P(e^{i\theta})}|=k^n|P(e^{i\theta})|. \tag{3.11}$$

Hence, from (3.8), (3.10) and (3.11), it follows for each $r>0$,

$$\begin{aligned}&n^r(|\alpha|-k)^r k^{nr}\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\\ =&n^r(|\alpha|-k)^r\int_0^{2\pi}|H(ke^{i\theta})|^r d\theta\\ \le&n^r(|\alpha|-k)^r(C_r)^r\int_0^{2\pi}|H(e^{i\theta})|^r d\theta\\ \le&k^r(C_r)^r\int_0^{2\pi}|1+e^{i\theta}|^r\big|D_{\frac{\alpha}{k}}G(e^{i\theta})\big|^r d\theta,\end{aligned}$$

which gives with the help of Holder's inequality for each $r>0$, $p>1$, $q>1$ with $p^{-1}+q^{-1}=1$,

$$\begin{aligned}&n^r(|\alpha|-k)^r k^{nr}\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\\ \le&k^r(C_r)^r\Big\{\int_0^{2\pi}|1+e^{i\theta}|^{qr}d\theta\Big\}^{\frac{1}{q}}\Big\{\int_0^{2\pi}\Big|D_{\frac{\alpha}{k}}G(e^{i\theta})\Big|^{pr}d\theta\Big\}^{\frac{1}{p}}.\end{aligned}$$

Equivalently,

$$\begin{aligned}&n(|\alpha|-k)k^{n-1}\Big\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\Big\}^{\frac{1}{r}}\\ \le&C_r\Big\{\int_0^{2\pi}|1+e^{i\theta}|^{qr}d\theta\Big\}^{\frac{1}{qr}}\Big\{\int_0^{2\pi}\Big|D_{\frac{\alpha}{k}}G(e^{i\theta})\Big|^{pr}d\theta\Big\}^{\frac{1}{pr}}.\end{aligned} \tag{3.12}$$

Since

$$\begin{aligned}D_{\frac{\alpha}{k}}G(z)=&nG(z)+\Big(\frac{\alpha}{k}-z\Big)G'(z)=nP(kz)+\Big(\frac{\alpha}{k}-z\Big)kP'(kz)\\ =&nP(kz)+(\alpha-kz)P'(kz)=D_\alpha P(kz)\end{aligned}$$

is a polynomial of degree $n-1$, therefore for each $t>0$ and $R\ge1$, we have by an inequality (see [16]) that

$$\Big\{\int_0^{2\pi}|D_\alpha P(Re^{i\theta})|^t d\theta\Big\}^{\frac{1}{t}}\le R^{n-1}\Big\{\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^t d\theta\Big\}^{\frac{1}{t}}.$$

Applying this in (3.12) with R replaced by k and t by pr, we obtain for each $r>0$,

$$\begin{aligned}&n(|\alpha|-k)\Big\{\int_0^{2\pi}|P(e^{i\theta})|^r d\theta\Big\}^{\frac{1}{r}}\\ \le&C_r\Big\{\int_0^{2\pi}|1+e^{i\theta}|^{qr}d\theta\Big\}^{\frac{1}{qr}}\Big\{\int_0^{2\pi}|D_\alpha P(e^{i\theta})|^{pr}d\theta\Big\}^{\frac{1}{pr}},\end{aligned}$$

which proves Theorem 1.1. □

Proof of Theorem 1.2. We assume with out loss of generality that $P(z)$ has all its zeros in $|z|<k$, $k\geq 1$, for if $P(z)$ has a zero on $|z|=k$, then $m=0$ and in view of Theorem 1.1, the theorem holds trivially. Since $P(z)$ has all its zeros in $|z|<k$ where $k\geq 1$, so that $\min_{|z|=k}|P(z)|=m>0$ and for every $\lambda\in\mathbb{C}$ with $|\lambda|<1$, we have $|\lambda m|<m\leq|P(z)|$, for $|z|=k$. By Rouche's theorem the polynomial $P(z)+\lambda m$ also has all its zeros in $|z|<k$ where $k\geq 1$. Applying Theorem 1.1 to the polynomial $P(z)+\lambda m$ and noting that $D_\alpha(P(z)+\lambda m)=D_\alpha P(z)+\lambda mn$, Theorem 1.2 follows. □

Proof of Theorem 1.3. If $P(z)$ has a zero on $|z|=k$, then the theorem is trivial. So, we assume that $P(z)$ has all its zeros in $|z|<k$, therefore $\min_{|z|=k}|P(z)|=m>0$ and hence for every complex number γ with $|\gamma|<1$, we have $|\gamma mz^n/k^n|<|P(z)|$, for $|z|=k$. It follows by Rouche's theorem that the polynomial $P(z)-\gamma mz^n/k^n$ of degree n has all its zeros in $|z|<k$, $k\leq 1$. On applying Lemma 2.2 to $P(z)-\gamma mz^n/k^n$, we have for every complex number α with $|\alpha|\geq k$,

$$\Big|D_\alpha\Big(P(z)-\frac{\gamma mz^n}{k^n}\Big)\Big|\geq\frac{n}{1+k^\mu}\big(|\alpha|-k^\mu\big)\Big|P(z)-\frac{\gamma mz^n}{k^n}\Big| \quad \text{for } |z|=1.$$

Equivalently,

$$\Big|zD_\alpha P(z)-\frac{\alpha\gamma mnz^n}{k^n}\Big|\geq\frac{n\big(|\alpha|-k^\mu\big)}{1+k^\mu}\Big|P(z)-\frac{\gamma mz^n}{k^n}\Big| \quad \text{for } |z|=1. \tag{3.13}$$

Since by Laguerre's theorem (see [14, pp. 52]), the polynomial

$$D_\alpha\big(P(z)-\frac{\gamma mz^n}{k^n}\big)=D_\alpha P(z)-\frac{\alpha\gamma mnz^{n-1}}{k^n}$$

has all zeros in $|z|<k$ for every complex number α with $|\alpha|\geq k$, therefore, for any complex β with $|\beta|<1$, the polynomial

$$\begin{aligned}T(z)=&zD_\alpha P(z)-\frac{\gamma mn\alpha z^n}{k^n}+n\beta\frac{|\alpha|-k^\mu}{1+k^\mu}\Big\{P(z)-\frac{\gamma mz^n}{k^n}\Big\}\\ =&\Big\{zD_\alpha P(z)+n\beta\frac{|\alpha|-k^\mu}{1+k^\mu}P(z)\Big\}-\frac{\gamma mnz^n}{k^n}\Big\{\alpha+\beta\frac{|\alpha|-k^\mu}{1+k^\mu}\Big\}\\ \neq& 0 \quad \text{for } |z|\geq k.\end{aligned} \tag{3.14}$$

Since $k\leq 1$, we have $T(z)\neq 0$ for $|z|\geq 1$ also.

Now choosing the argument of γ in (3.14) suitably and letting $|\gamma|\to 1$, we get for $|z|=1$ and $|\beta|<1$,

$$\Big|zD_\alpha P(z)+n\beta\frac{|\alpha|-k^\mu}{1+k^\mu}P(z)\Big|\geq\Big|\frac{mnz^n}{k^n}\Big\{\alpha+\beta\frac{|\alpha|-k^\mu}{1+k^\mu}\Big\}\Big|,$$

or

$$\Big|zD_\alpha P(z)+n\beta\frac{|\alpha|-k^\mu}{1+k^\mu}P(z)\Big|\geq\frac{mn}{k^n}\Big|\alpha+\beta\frac{|\alpha|-k^\mu}{1+k^\mu}\Big| \quad \text{for } |z|=1.$$

For β, with $|\beta|=1$, above inequality holds by continuity. □

Acknowledgements

The work of the first author is supported by UGC under major research project scheme vide No. MRP-MAJOR-MATH-2013-29143.

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