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(L^p,L^q)-Boundedness of Hausdorff Operators with Power Weight on Euclidean Spaces

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Abstract. In this paper, we prove the (L^p,L^q)-boundedness of (fractional) Hausdorff operators with power weight on Euclidean spaces. As special cases, we can obtain some well known results about Hardy operators.

Key Words: Hausdorff operator, Hardy operator, Cesàro operator, Young's inequality.

AMS Subject Classifications: 42B99, 46E30, 47B38

1 Introduction

Hausdorff operators (Hausdorff summability methods) play important roles in the study of one dimensional Fourier analysis, particularly the summability of classical Fourier series. Modern theory of Hausdorff operators started with the work of Siskakis [30] in complex analysis setting and with the work of Georgakis [16] and Liflyand and Móricz [25] in the Fourier transform setting. One can see [24] for a brief overview of Hausdorff operators. Some recent developments for Hausdorff operators can be founded in [1–8, 12–15, 20–27, 29, 30, 33]. Now we recall the one-dimensional Hausdorff operator

$$h_\Phi f(x)=\int_0^\infty \frac{\Phi(t)}{t} f\left(\frac{x}{t}\right)dt,$$

where Φ is a locally integrable function on $(0,\infty)$. Liflyand and Móricz [25] proved that h_Φ generated by a function $\Phi\in L^1(\mathbb{R})$ is a bounded linear operator on the real Hardy space $H^1(\mathbb{R})$. Following this, the boundedness of h_Φ was considered in various spaces, for example, see [3, 20, 26].

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The one-dimensional Hausdorff operator contains the classical Hardy operator h and its adjoint operator h^* if we choose $\Phi(t)$ as $t^{-1}\chi_{(1,\infty)}(t)$ and $\chi_{(0,1]}(t)$ respectively, i.e.,

$$hf(x):=\frac{1}{x}\int_0^x f(t)dt \quad \text{and} \quad h^*f(x):=\int_x^\infty \frac{f(t)}{t}dt.$$

It is well known that Hardy operators are important operators in Harmonic analysis, for instance, see [18, 19]. On the other hand, if we choose $\Phi(t)=\alpha(1-t)^{\alpha-1}\chi(0,1)(t)$ for $\alpha=1,2,\cdots$, then $H_\Phi=C_\alpha$ is called the Cesàro operator of order α. A brief history of the study of the Cesàro operator can be found in [20].

For multidimensional Hausdorff operators, there are many kinds of definitions [1, 3, 4, 16–18, 21, 22]. One of the interesting definitions of the Hausdorff operators is

$$H_{\Omega,\Phi}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(y')}{|y|^n}\Phi\Big(\frac{x}{|y|}\Big)f(y)dy,$$

where $\Omega(y')$ is an integrable function defined on the unit sphere S^{n-1}. Similar to h_Φ, $H_{\Omega,\Phi}$ contains the high dimensional Hardy operator H and its adjoint operator H^* (see the below definitions).

Recently, Chen, Fan and Li [5] obtained that if Φ is a radial function and $1\le p\le\infty$, then

$$\|H_{\Omega,\Phi}f\|_{L^p(\mathbb{R}^n)}\le\|\Omega\|_{L^{p'}(S^{n-1})}|S^{n-1}|^{\frac{1}{p}}\int_0^\infty\frac{|\Phi(t)|}{t}t^{\frac{n}{p}}dt\cdot\|f\|_{L^p(\mathbb{R}^n)}. \tag{1.1}$$

Particularly,

$$\|H_\Phi f\|_{L^p(\mathbb{R}^n)}\le|S^{n-1}|\int_0^\infty\frac{|\Phi(t)|}{t}t^{\frac{n}{p}}dt\cdot\|f\|_{L^p(\mathbb{R}^n)}.$$

As applications, they obtained the known results about boundedness of Hardy operators H and H^* on $L^p(\mathbb{R}^n)$ (see [9, 10]). For a general function Φ, Wang [31] proved

$$\|H_\Phi f\|_{L^p(\mathbb{R}^n)}\le|S^{n-1}|^{\frac{1}{p'}}\int_0^\infty\Big(\int_{S^{n-1}}|\Phi(t\varphi)|^p d\varphi\Big)^{\frac{1}{p}}t^{-1+\frac{n}{p}}dt\cdot\|f\|_{L^p(\mathbb{R}^n)}. \tag{1.2}$$

In [29], Lin and Sun defined the n-dimensional fractional Hausdorff operator for a radial function Φ as follows

$$H_{\Phi,\beta}f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|y|^{n-\beta}}\Phi\Big(\frac{|x|}{|y|}\Big)dy, \quad 0\le\beta<n.$$

They gave the sufficient condition on Φ about the boundedness of $H_{\Phi,\beta}$ on $L^p(|x|^\gamma)$, where $0<\beta<n$. If we choose Φ as $|t|^{\beta-n}\chi_{(1,\infty)}(|t|)$ and $\chi_{(0,1]}(|t|)$, $H_{\Phi,\beta}f$ becomes the fractional Hardy operator H_β and its adjoint operator H^*_β respectively, where

$$H_\beta f(x)=\frac{1}{|x|^{n-\beta}}\int_{|y|<|x|}f(y)dy, \quad x\in\mathbb{R}^n\backslash\{0\} \quad \text{and} \quad H^*_\beta f(x)=\int_{|y|\ge|x|}\frac{f(y)}{|y|^{n-\beta}}dy.$$

For some results about the boundedness of H_β and H^*_β, please refer to [10] and [32]. We denote $H_0=H$, $H^*_0=H^*$ and $H_{\Phi,0}=H_\Phi$. Next we also denote $H^0_{\Omega,\Phi}=H_{\Omega,\Phi}$.

In this paper, we consider the Hausdorff operator $H^\beta_{\Omega,\Phi}$ for a general function Φ, i.e.,

$$H^\beta_{\Omega,\Phi}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(y')}{|y|^{n-\beta}}\Phi\Big(\frac{x}{|y|}\Big)f(y)dy,\quad 0\le\beta<n,$$

and obtain that

Theorem 1.1. *Let $1\le p,\, q\le\infty$, $0\le\beta<n$ and $\alpha,\gamma\in\mathbb{R}$ satisfy*

$$\frac{\alpha+n}{p}-\frac{\gamma+n}{q}=\beta.$$

If

$$K=\|\Omega\|_{L^{p'}(S^{n-1})}\Big(\int_{S^{n-1}}\Big(\int_0^\infty|\Phi(t\varphi)|^s t^{-1+\frac{(\gamma+n)s}{q}}dt\Big)^{\frac{q}{s}}d\varphi\Big)^{\frac{1}{q}}<\infty,$$

where s satisfies

$$\frac{1}{q}=\frac{1}{p}+\frac{1}{s}-1,$$

then the operator $H^\beta_{\Omega,\Phi}$ is bounded, from $L^p(|x|^\alpha)$ into $L^q(|x|^\gamma)$, i.e.,

$$\|H^\beta_{\Omega,\Phi}f\|_{L^q(|x|^\gamma)}\le K\|f\|_{L^p(|x|^\alpha)}.$$

Remark 1.1. Let $\alpha=\beta=\gamma=0$. If Φ is a radial function, then we obtain the inequality (1.1) from Theorem 1.1; if $\Omega=1$, by Minkowski's inequality, we have

$$K\le|S^{n-1}|^{\frac{1}{p'}}\int_0^\infty\Big(\int_{S^{n-1}}|\Phi(t\varphi)|^p d\varphi\Big)^{\frac{1}{p}}t^{-1+\frac{n}{p}}dt.$$

So we can obtain the inequality (1.2). Please see [15] for more details.

Remark 1.2. Let $\Omega=1$, $\gamma=\alpha$ and $\beta=0$. For $\Phi\ge0$,

$$K=|S^{n-1}|^{\frac{1}{p'}}\Big(\int_{S^{n-1}}\Big(\int_0^\infty\Phi(t\varphi)t^{-1+\frac{\gamma+n}{p}}dt\Big)^p d\varphi\Big)^{\frac{1}{p}}<\infty$$

is the sufficient and necessary condition for the operator H_Φ on $L^p(|x|^\gamma)$. See [14].

Remark 1.3. When choose $\Phi(t)$ as $|t|^{\beta-n}\chi_{(1,\infty)}(|t|)$, if $\alpha<(p-1)n$, then by Theorem 1.1, we have

$$\|H_\beta f\|_{L^q(|x|^\gamma)}\le\Big(\frac{p|S^{n-1}|}{s(pn-\alpha-n)}\Big)^{\frac{1}{s}}\|f\|_{L^p(|x|^\alpha)}.$$

Let $\Phi(t)=\chi_{(0,1]}(|t|)$. If $\alpha>\beta p-n$, then

$$\|H_\beta^* f\|_{L^q(|x|^\gamma)}\leq\Big(\frac{q|S^{n-1}|}{s(\gamma+n)}\Big)^{\frac{1}{s}}\|f\|_{L^p(|x|^\alpha)}.$$

On the one hand, if $\beta=0$, then we can obtain some related results in [28, 35] from the above inequalities. Moreover, if $\alpha=\gamma=0$, we can obtain a better bound than that of [32]. See the detail in [15].

Remark 1.4. Obviously, by Theorem 1.1, we also give a sufficient condition on Φ for the boundedness of $H_{\Phi,\beta}$ from $L^p(|x|^\gamma)$ to $L^p(|x|^\gamma)$. Unluckily, our result is not comparable to that of Lin and Sun [29].

2 Proof of theorem

Proof of Theorem 1.1. The first method of proof. We adapt some ideas used in [11]. Let

$$g(x)=\frac{\int_{S^{n-1}}\Omega(\theta)f(|x|\theta)d\theta}{\int_{S^{n-1}}\Omega(\theta)d\theta}.$$

Obviously, $g(x)$ is a radial function. Using Minkowski's inequality, polar coordinates and Hölder's inequality, we have

$$\begin{aligned}\|g\|_{L^p(|x|^\alpha)}&\leq\Big|\int_{S^{n-1}}\Omega(\theta)d\theta\Big|^{-1}\int_{S^{n-1}}|\Omega(\theta)|\Big(\int_{\mathbb{R}^n}|f(|x|\theta)|^p|x|^\alpha dx\Big)^{\frac{1}{p}}d\theta\\&=\|f\|_{L^p(|x|^\alpha)}\cdot\frac{|S^{n-1}|^{\frac{1}{p}}\|\Omega\|_{L^{p'}(S^{n-1})}}{|\int_{S^{n-1}}\Omega(\theta)d\theta|}.\end{aligned}$$

On the other hand, by Fubini's Theorem, we have

$$\begin{aligned}H^\beta_{\Omega,\Phi}(g)(x)&=\Big(\int_{S^{n-1}}\Omega(\theta)d\theta\Big)^{-1}\int_{S^{n-1}}\Omega(\theta)\int_{\mathbb{R}^n}\frac{\Omega(y')}{|y|^{n-\beta}}\Phi\Big(\frac{x}{|y|}\Big)f(|y|\theta)dyd\theta\\&=\Big(\int_{S^{n-1}}\Omega(\theta)d\theta\Big)^{-1}\int_{S^{n-1}}\Omega(\theta)\int_0^\infty\int_{S^{n-1}}\frac{\Omega(\sigma)}{r^{n-\beta}}\Phi\Big(\frac{x}{r}\Big)f(r\theta)r^{n-1}d\sigma drd\theta\\&=H^\beta_{\Omega,\Phi}(f)(x).\end{aligned}$$

In consequence,

$$\frac{\|H^\beta_{\Omega,\Phi}(f)\|_{L^q(|x|^\gamma)}}{\|f\|_{L^p(|x|^\alpha)}}\leq\frac{|S^{n-1}|^{\frac{1}{p}}\|\Omega\|_{L^{p'}(S^{n-1})}}{|\int_{S^{n-1}}\Omega(\theta)d\theta|}\frac{\|H^\beta_{\Omega,\Phi}(g)\|_{L^q(|x|^\gamma)}}{\|g\|_{L^p(|x|^\alpha)}}.\tag{2.1}$$

Therefore, this implies that if we want to obtain the operator norm of $H^{\beta}_{\Omega,\Phi}$ from $L^p(|x|^\alpha)$ to $L^q(|x|^\gamma)$, we can restrict f to radial functions. So we can assume that $f(x)$ is a radial function in the following proof

$$\begin{aligned}\|H^{\beta}_{\Omega,\Phi}(f)\|_{L^q(|x|^\gamma)}=&\Big(\int_0^\infty\int_{S^{n-1}}\Big|\int_0^\infty\int_{S^{n-1}}\frac{\Omega(\sigma)}{r^{1-\beta}}\Phi(\rho\varphi r^{-1})f(r)d\sigma dr\Big|^q\rho^{\gamma+n-1}d\varphi d\rho\Big)^{\frac{1}{q}}\\ =&\Big|\int_{S^{n-1}}\Omega(\sigma)d\sigma\Big|\Big(\int_{S^{n-1}}\int_0^\infty\Big|\int_0^\infty\frac{\Phi(\rho\varphi r^{-1})}{r^{1-\beta}}f(r)dr\Big|^q\rho^{\gamma+n-1}d\rho d\varphi\Big)^{\frac{1}{q}}.\end{aligned}$$

Let

$$I=\Big(\int_0^\infty\Big|\int_0^\infty\frac{\Phi(\rho\varphi r^{-1})}{r^{1-\beta}}f(r)dr\Big|^q\rho^{\gamma+n-1}d\rho\Big)^{\frac{1}{q}}.$$

Noticing that $\frac{\gamma+n}{q}=\frac{\alpha+n}{p}-\beta$, so

$$I=\Big(\int_0^\infty\Big|\int_0^\infty\Phi(\rho\varphi r^{-1})(\rho r^{-1})^{\frac{\gamma+n}{q}}f(r)r^{\frac{\alpha+n}{p}}\frac{dr}{r}\Big|^q\frac{d\rho}{\rho}\Big)^{\frac{1}{q}}.$$

Using Young's inequality (see [17, Theorem 1.2.12]) for $\frac{1}{q}=\frac{1}{p}+\frac{1}{s}-1$, we obtain that

$$I\le\Big(\int_0^\infty|\Phi(\rho\varphi)|^s\rho^{\frac{(\gamma+n)s}{q}-1}d\rho\Big)^{\frac{1}{s}}\Big(\int_0^\infty|f(\rho)|^p\rho^{\alpha+n-1}d\rho\Big)^{\frac{1}{p}}.$$

Therefore, combining the above inequalities, we have

$$\begin{aligned}\|H^{\beta}_{\Omega,\Phi}(f)\|_{L^q(|x|^\gamma)}\le&\frac{|\int_{S^{n-1}}\Omega(\sigma)d\sigma|}{|S^{n-1}|^{\frac{1}{p}}}\Big(\int_{S^{n-1}}\Big(\int_0^\infty|\Phi(\rho\varphi)|^s\rho^{-1+\frac{(\gamma+n)s}{q}}d\rho\Big)^{\frac{q}{s}}d\varphi\Big)^{\frac{1}{q}}\\ &\times\Big(\int_0^\infty\int_{S^{n-1}}|f(\rho)|^p\rho^{\alpha+n-1}d\varphi d\rho\Big)^{\frac{1}{p}}\\ =&\frac{|\int_{S^{n-1}}\Omega(\sigma)d\sigma|}{|S^{n-1}|^{\frac{1}{p}}}\Big(\int_{S^{n-1}}\Big(\int_0^\infty|\Phi(\rho\varphi)|^s\rho^{-1+\frac{(\gamma+n)s}{q}}d\rho\Big)^{\frac{q}{s}}d\varphi\Big)^{\frac{1}{q}}\|f\|_{L^p(|x|^\alpha)}.\end{aligned}$$

For any general functional function $f(x)$, using the inequality (2.1), we obtain

$$\|H^{\beta}_{\Omega,\Phi}(f)\|_{L^q(|x|^\gamma)}\le K\|f\|_{L^p(|x|^\alpha)}.$$

This completes the proof of Theorem 1.1. □

The second method of proof. We express $\|H^{\beta}_{\Omega,\Phi}f\|_{L^q(|x|^\gamma)}$ in polar coordinates by writing $x=\rho\varphi$ and $y=t\theta$. Then we apply

$$\frac{\gamma+n}{q}=\frac{\alpha+n}{p}-\beta$$

and then Fubini's theorem to interchange the integrals in ρ and φ. Then

$$\begin{aligned}\|H^{\beta}_{\Omega,\Phi}f\|^q_{L^q(|x|^\gamma)}&=\int_0^\infty\int_{S^{n-1}}\Big|\int_0^\infty\int_{S^{n-1}}\Omega(\theta)\Phi(\rho\varphi t^{-1})(\rho t^{-1})^{\frac{\gamma+n}{q}}f(t\theta)t^{\frac{n+\alpha}{p}}d\theta\frac{dt}{t}\Big|^q d\varphi\frac{d\rho}{\rho}\\&\le\int_{S^{n-1}}\int_0^\infty\Big(\int_{S^{n-1}}\int_0^\infty\big|\Phi(\rho\varphi t^{-1})\big|(\rho t^{-1})^{\frac{\gamma+n}{q}}\big|\Omega(\theta)f(t\theta)\big|t^{\frac{n+\alpha}{p}}\frac{dt}{t}d\theta\Big)^q\frac{d\rho}{\rho}d\varphi.\end{aligned}$$

By Minkowski's inequality, we obtain

$$\begin{aligned}&\Big(\int_0^\infty\Big(\int_{S^{n-1}}\int_0^\infty\big|\Phi(\rho\varphi t^{-1})\big|(\rho t^{-1})^{\frac{\gamma+n}{q}}\big|\Omega(\theta)f(t\theta)\big|t^{\frac{n+\alpha}{p}}\frac{dt}{t}d\theta\Big)^q\frac{d\rho}{\rho}\Big)^{\frac{1}{q}}\\\le&\int_{S^{n-1}}\big|\Omega(\theta)\big|\Big(\int_0^\infty\Big|\int_0^\infty\big|\Phi(\rho\varphi t^{-1})\big|(\rho t^{-1})^{\frac{\gamma+n}{q}}\big|f(t\theta)\big|t^{\frac{n+\alpha}{p}}\frac{dt}{t}\Big|^q\frac{d\rho}{\rho}\Big)^{\frac{1}{q}}d\theta.\end{aligned}$$

For

$$\int_0^\infty|\Phi(\rho\varphi t^{-1})|(\rho t^{-1})^{\frac{\gamma+n}{q}}|f(t\theta)|t^{\frac{n+\alpha}{p}}\frac{dt}{t},$$

we can regard it as a convolution on the multiplicative group $\mathbb{R}^+$ with Haar measure $\frac{dx}{x}$. Applying Young's inequality (see [17, Theorem 1.2.12]) for $\frac{1}{q}=\frac{1}{s}+\frac{1}{p}-1$, we have

$$\begin{aligned}&\Big(\int_0^\infty\Big(\int_0^\infty|\Phi(\rho\varphi t^{-1})|(\rho t^{-1})^{\frac{\gamma+n}{q}}|f(t\theta)|t^{\frac{n+\alpha}{p}}\frac{dt}{t}\Big)^q\frac{d\rho}{\rho}\Big)^{\frac{1}{q}}\\\le&\Big(\int_0^\infty\big|\Phi(t\varphi)\big|^s t^{-1+\frac{(\gamma+n)s}{q}}dt\Big)^{\frac{1}{s}}\Big(\int_0^\infty\big|f(t\theta)\big|^p t^{\alpha+n-1}dt\Big)^{\frac{1}{p}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|H^{\beta}_{\Omega,\Phi}f\|^q_{L^q(|x|^\gamma)}\le&\int_{S^{n-1}}\Big(\int_{S^{n-1}}\big|\Omega(\theta)\big|\Big(\int_0^\infty\big|\Phi(t\varphi)\big|^s t^{-1+\frac{(\gamma+n)s}{q}}dt\Big)^{\frac{1}{s}}\\&\times\Big(\int_0^\infty\big|f(t\theta)\big|^p t^{\alpha+n-1}dt\Big)^{\frac{1}{p}}d\theta\Big)^q d\varphi.\end{aligned}$$

Because applying Hölder's inequality, we deduce that

$$\begin{aligned}&\int_{S^{n-1}}\big|\Omega(\theta)\big|\Big(\int_0^\infty\big|f(t\theta)\big|^p t^{\alpha+n-1}dt\Big)^{\frac{1}{p}}d\theta\\\le&\|\Omega\|_{L^{p'}(S^{n-1})}\Big(\int_{S^{n-1}}\int_0^\infty\big|f(t\theta)\big|^p t^{\alpha+n-1}dtd\theta\Big)^{\frac{1}{p}}\\=&\|\Omega\|_{L^{p'}(S^{n-1})}\|f\|_{L^p(|x|^\alpha)}.\end{aligned}$$

Hence we have

$$\|H^{\beta}_{\Omega,\Phi}f\|_{L^q(|x|^\gamma)}\le K\|f\|_{L^p(|x|^\alpha)}.$$

Thus, we complete the proof. □

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