

Construction of Multivariate Tight Framelet Packets Associated with Dilation Matrix

Firdous A. Shah^{1,*} and Abdullah²

¹ *Department of Mathematics, University of Kashmir, South Campus, Anantnag-192
101, Jammu and Kashmir, India*

² *Department of Mathematics, ZHDC, University of Delhi, New Delhi-110002, India*

Received 24 October 2014; Accepted (in revised version) 10 March 2015

Abstract. In this paper, we present a method for constructing multivariate tight framelet packets associated with an arbitrary dilation matrix using unitary extension principles. We also prove how to construct various tight frames for $L^2(\mathbb{R}^d)$ by replacing some mother framelets.

Key Words: Wavelet, tight frame, framelet packet, matrix dilation, extension principle, Fourier transform.

AMS Subject Classifications: 42C40, 42C15, 65T60

1 Introduction

A tight wavelet frame is a generalization of an orthonormal wavelet basis by introducing redundancy into a wavelet system. By sacrificing orthonormality and allowing redundancy, the tight wavelet frames become much easier to construct than the orthonormal wavelets. Tight wavelet frames provide representations of signals and images in applications, where redundancy of the representation is preferred and the perfect reconstruction property of the associated filter bank algorithm, as in the case of orthonormal wavelets, is kept (see [6]). The main tools for construction and characterization of wavelet frames are the several extension principles, the unitary extension principle (UEP) and oblique extension principle (OEP) as well as their generalized versions, the mixed unitary extension principle (MUEP) and the mixed oblique extension principle (MOEP). They give sufficient conditions for constructing tight and dual wavelet frames for any given refinable function which generates a multiresolution analysis (MRA). These essential methods were firstly introduced by Ron and Shen in [11] and in the fundamental work of Daubechies et al. [4] for scalar refinable functions. The resulting tight wavelet frames are

*Corresponding author. *Email addresses:* fashah79@gmail.com (F. A. Shah), abd.zhc.du@gmail.com (Abdullah)

based on a multiresolution analysis, and the generators are often called framelets. The advantages of MRA-based tight wavelet frames and their promising features in applications have attracted a great deal of interest and effort in recent years to extensively study them. To mention only a few references on tight wavelet frames, the reader is referred to [4–9] and many references therein.

However, wavelet frames provide poor frequency localization in many applications as they are not suitable for signals whose domain frequency channels are focused only on the middle frequency region. Therefore, in order to make more kinds of signals suited for analyzing by wavelet frames, it is necessary to extend the concept of wavelet frames to a library of wavelet frames, called framelet packets or wavelet frame packets. The original idea of framelet packets was introduced by Coifman et al. in [3] to provide more efficient decomposition of signals containing both transient and stationary components. Chui and Li [2] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be applied to the spline wavelets and so on. Shen [18] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate orthogonal wavelets such that they may be used in a wider field. Other notable generalizations are the wavelet packets and framelet packets related to the Walsh polynomials [12, 15], wave packets and tight framelet packets on local fields of positive characteristic [13, 14], the vector-valued wavelet packets [1], the vector-valued multivariate wavelet frame packets [17] and the tight framelet packets on \mathbb{R}^d for dilation factor 2 [10].

Recently, Shah and Debnath [16] have introduced a general construction scheme for a class of stationary M -band tight framelet packets in $L^2(\mathbb{R})$ via extension principles. In this paper, we generalize the concept of univariate framelet packets to the case of multivariate tight framelet packets associated with an expansive dilation matrix using unitary extension principles and our approach is quite different as described in [16].

This paper is organized as follows. In Section 2 we review some basic facts about tight wavelet frames associated with dilation matrix using extension principles. In Section 3, we prove our main results regarding the construction of multivariate tight framelet packets in $L^2(\mathbb{R}^d)$.

2 Notations and preliminaries

Throughout this paper, we use the following notations. Let \mathbb{R} and \mathbb{C} be all real and complex numbers, respectively. \mathbb{N} and \mathbb{N}_0 denote all natural and non-negative integers, respectively. \mathbb{Z}^d and \mathbb{R}^d denote the set of all d -tuples integers and d -tuples of reals, respectively. Assume that we have an expansive dilation matrix A , i.e., all its eigenvalues lie outside the unit circle, the matrix A is known as the dilation matrix. For a $d \times d$ real matrix A , we denote by A^* the transpose of A and B be a $d \times d$ non-singular matrix.

For a $d \times d$ expansive matrix A , define the dilation operator D and the shift operator

T_k for each $k \in \mathbb{Z}^d$ on $L^2(\mathbb{R}^d)$ by

$$Df(\cdot) = |\det A|^{1/2} f(A\cdot), \quad T_k f(\cdot) = f(\cdot - k),$$

for $f \in L^2(\mathbb{R}^d)$. Obviously, they are both unitary operators on $L^2(\mathbb{R}^d)$. The Fourier transform of an arbitrary $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx,$$

and the Fourier transforms of the functions in $L^2(\mathbb{R}^d)$ are understood as its unitary extension.

Definition 2.1. A countable family $\{f_\alpha\}_{\alpha \in \mathcal{J}}$ of elements in a separable Hilbert space \mathbb{H} is called a frame if there exist constants C_1 and C_2 , $0 < C_1 \leq C_2 < \infty$, such that

$$C_1 \|f\|_2^2 \leq \sum_{\alpha \in \mathcal{J}} |\langle f, f_\alpha \rangle|^2 \leq C_2 \|f\|_2^2 \quad \text{for all } f \in \mathbb{H}. \tag{2.1}$$

The constants C_1 and C_2 independent of f for which (2.1) holds are called frame bounds. A frame is a tight frame if C_1 and C_2 can be chosen so that $C_1 = C_2$ and is a normalized tight frame if $C_1 = C_2 = 1$.

Let $\Psi = \{\psi_1, \psi_2, \dots, \psi_L\} \subset L^2(\mathbb{R}^d)$. The wavelet system generated by Ψ and associated with A is the collection

$$X(\Psi) = \{\psi_{\ell,j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^d, \ell = 1, 2, \dots, L\}, \tag{2.2}$$

where

$$\psi_{\ell,j,k}(x) = |\det A|^{j/2} \psi_\ell(A^j x - Bk). \tag{2.3}$$

Taking Fourier transform on both sides of (2.3), we have

$$\hat{\psi}_{\ell,j,k}(\xi) = |\det A|^{-j/2} e^{-2\pi i(A^{-j} Bk)\xi} \hat{\psi}_\ell(A^{*-j}\xi). \tag{2.4}$$

As we know that the common method for construction and characterization of tight wavelet frames relies on the so-called extension principles. They give sufficient conditions for constructing tight and dual wavelet frames for any given refinable function which generates a multiresolution analysis (MRA). Therefore, we first recall the extension principles which uses the multiresolution analysis. The MRA starts from a refinable function φ . A function φ is said to be A -refinable if it satisfies the refinement equation

$$\varphi(x) = \sum_{k \in \mathbb{Z}^d} h_0[k] \varphi(Ax - Bk), \tag{2.5}$$

for some $h_0 \in l^2(\mathbb{Z}^d)$. Taking Fourier transform on both sides of (2.5), we obtain

$$\hat{\varphi}(A^*\xi) = H_0(\xi)\hat{\varphi}(\xi), \tag{2.6}$$

where H_0 is a $B^{*-1}\mathbb{Z}^d$ -periodic measurable function in $L^\infty(B^{*-1}([0,1]^d))$ and is often called the refinement symbol of φ .

Furthermore, we make the following assumptions on φ :

- (i) $\hat{\varphi}(0) = \lim_{\xi \rightarrow 0} \hat{\varphi}(\xi) = 1$,
- (ii) $\sum_{k \in \mathbb{Z}^d} |\hat{\varphi}(\xi + B^{*-1}k)|^2 \neq 0, \quad \xi \in B^{*-1}([0,1]^d)$.

Under assumptions (i) and (ii) along with the condition (2.6), the family of subspaces

$$V_j = \overline{\text{span}}\{D^j T_k \varphi : k \in \mathbb{Z}^d\}, \quad j \in \mathbb{Z}, \tag{2.7}$$

forms an MRA for $L^2(\mathbb{R}^d)$ (see [6,8]).

Theorem 2.1 (Unitary Extension Principle, see [4]). *Let $\varphi = \psi_0$ be a scaling function, and let H_0 be the corresponding refinement filter. For each $\ell = 1, 2, \dots, L$, let H_ℓ be a $B^{*-1}\mathbb{Z}^d$ -periodic measurable function. Define ψ_ℓ by*

$$\hat{\psi}_\ell(A^*\xi) = H_\ell(\xi)\hat{\psi}_0(\xi). \tag{2.8}$$

If

$$H_0(\xi)\overline{H_0(\xi + B^{*-1}\nu)} + \sum_{\ell=1}^L H_\ell(\xi)\overline{H_\ell(\xi + B^{*-1}\nu)} = \begin{cases} 1, & \text{if } \nu = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{2.9}$$

then the wavelet system $X(\Psi)$ generates a normalized tight frame for $L^2(\mathbb{R}^d)$.

If the wavelet system $X(\Psi)$ is a frame for $L^2(\mathbb{R}^d)$, then $X(\Psi)$ is called a wavelet frame and elements in Ψ are called mother framelets. Moreover, the functions $H_\ell, \ell = 1, \dots, L$ are called the framelet symbols or wavelet masks.

We will also consider the set \mathfrak{D} as a dense subset of $L^2(\mathbb{R}^d)$ defined by

$$\mathfrak{D} = \left\{ f \in L^2(\mathbb{R}^d) : \hat{f} \text{ is continuous and has compact support} \right\}.$$

3 Construction of multivariate tight framelet packets

In this section, we shall show the construction of the basic framelet packets for $L^2(\mathbb{R}^d)$ via multiresolution generated by the framelet symbols. To do this, let $\{\psi_\ell, H_\ell\}_{\ell=0}^L$ satisfy the conditions of the unitary extension principle and $\omega_0 = \varphi$. Define the functions $\omega_n(x), n = 0, 1, 2, \dots$, associated with the refinable function φ recursively by

$$\hat{\omega}_n(\xi) = \hat{\omega}_{(L+1)r+\ell}(A^*\xi) = H_\ell(\xi)\omega_r(\xi), \quad \ell = 0, 1, \dots, L, \quad r \in \mathbb{N}_0. \tag{3.1}$$

Note that for $r = 0$ and $\ell = 1, \dots, L$, we have

$$\hat{\omega}_\ell(A^* \zeta) = H_\ell(\zeta) \omega_0(\zeta) = H_\ell(\zeta) \varphi(\zeta), \tag{3.2}$$

which shows that $\omega_\ell(\cdot) = \psi_\ell(\cdot)$, $\ell = 1, \dots, L$.

Theorem 3.1. *Let ω_n , $n = 0, 1, \dots$, be as in Eq. (3.1). Then, for all $n \in \mathbb{N}_0$ and $j \in \mathbb{Z}$, we have*

$$\sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \omega_n \rangle|^2 = \sum_{\ell=0}^L \sum_{k \in \mathbb{Z}^d} |\langle f, D^{j-1} T_k \omega_{n(L+1)+\ell} \rangle|^2 \quad \text{for all } f \in L^2(\mathbb{R}^d). \tag{3.3}$$

Proof. For all $f \in \mathfrak{D}$, we have

$$\begin{aligned} & \langle f, D^{j-1} T_k \omega_{n(L+1)+\ell} \rangle \\ &= |\det A|^{-\frac{(j-1)}{2}} \int_{\mathbb{R}^d} \hat{f}(\zeta) \overline{\hat{\omega}_{n(L+1)+\ell}(A^{*(j-1)} \zeta)} e^{2\pi i A^{-(j-1)} \cdot Bk \zeta} d\zeta \\ &= |\det A|^{-\frac{(j-1)}{2}} \int_{B^{*-1}([0,1]^d)} \sum_{s \in \mathbb{Z}^d} \hat{f}(\zeta + B^{*-1}s) \\ & \quad \times \overline{\hat{\omega}_{n(L+1)+\ell}(A^{*(j-1)}(\zeta + B^{*-1}s))} e^{2\pi i A^{-(j-1)} \cdot Bk \zeta} d\zeta. \end{aligned}$$

Using Eq. (3.1) and the fact that

$$\{ |\det A|^{-\frac{(j-1)}{2}} e^{2\pi i A^{-(j-1)} \cdot Bk \zeta} : k \in \mathbb{Z}^d \}$$

is an orthonormal basis for $L^2(B^{*-1}([0,1]^d))$, we have

$$\begin{aligned} I &= \sum_{\ell=0}^L \sum_{k \in \mathbb{Z}^d} \left| \langle f, D^{j-1} T_k \omega_{n(L+1)+\ell} \rangle \right|^2 \\ &= |\det A|^{-(j-1)} \sum_{\ell=0}^L \int_{B^{*-1}([0,1]^d)} \left| \sum_{s \in \mathbb{Z}^d} \hat{f}(\zeta + B^{*-1}s) \overline{\hat{\omega}_{n(L+1)+\ell}(A^{*(j-1)}(\zeta + B^{*-1}s))} \right|^2 d\zeta \\ &= \frac{1}{|\det B|} \sum_{\ell=0}^L \int_{A^{*(j-1)} B^{*-1}([0,1]^d)} \left| \sum_{s \in \mathbb{Z}^d} \hat{f}(\zeta + A^{*(j-1)} B^{*-1}s) \right. \\ & \quad \times \overline{\hat{\omega}_{n(L+1)+\ell}(A^{*(j-1)} \zeta + B^{*-1}s)} \left. \right|^2 d\zeta \\ &= \frac{1}{|\det B|} \sum_{\ell=0}^L \int_{A^{*(j-1)} B^{*-1}([0,1]^d)} \left| \sum_{s \in \mathbb{Z}^d} \hat{f}(\zeta + A^{*(j-1)} B^{*-1}s) \right. \\ & \quad \times \overline{\hat{\omega}_n(A^{*-j} \zeta + A^{*-1} B^{*-1}s)} H_\ell(A^{*-j} \zeta + A^{*-1} B^{*-1}s)} \left. \right|^2 d\zeta \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{|\det B|} \sum_{\ell=0}^L \int_{A^{*(j-1)}B^{*-1}([0,1]^d)} \left| \sum_{s' \in \mathbb{Z}^d} \sum_{v \in \{0,1\}^d} \hat{f}(\xi + A^{*(j-1)}B^{*-1}(A^*s' + v)) \right. \\
 &\quad \left. \times \overline{\hat{\omega}_n(A^{*-j}\xi + A^{*-1}B^{*-1}(A^*s' + v))} H_\ell(A^{*-j}\xi + A^{*-1}B^{*-1}(A^*s' + v)) \right|^2 d\xi \\
 &= \frac{1}{|\det B|} \sum_{\ell=0}^L \int_{A^{*(j-1)}B^{*-1}([0,1]^d)} \left| \sum_{s' \in \mathbb{Z}^d} \sum_{v \in \{0,1\}^d} \hat{f}(\xi + A^{*(j-1)}B^{*-1}(A^*s' + v)) \right. \\
 &\quad \left. \times \overline{\hat{\omega}_n(A^{*-j}\xi + A^{*-1}B^{*-1}(A^*s' + v))} H_\ell(A^{*-j}\xi + A^{*-1}B^{*-1}\nu) \right|^2 d\xi \\
 &= \frac{1}{|\det B|} \sum_{\ell=0}^L \int_{A^{*(j-1)}B^{*-1}([0,1]^d)} \left| \sum_{v \in \{0,1\}^d} P_{f,\omega_n}^j(\xi, \nu) \overline{H_\ell(A^{*-j}\xi + A^{*-1}B^{*-1}\nu)} \right|^2 d\xi,
 \end{aligned}$$

where

$$P_{f,\omega_n}^j(\xi, \nu) = \sum_{s' \in \mathbb{Z}^d} \hat{f}(\xi + A^{*(j-1)}B^{*-1}(A^*s' + \nu)) \overline{\hat{\omega}_n(A^{*-j}\xi + A^{*-1}B^{*-1}(A^*s' + \nu))}.$$

By the implementing relation (2.9) for $\xi \in B^{*-1}([0,1]^d)$ and $\nu \in \{0,1\}^d$, we obtain

$$\begin{aligned}
 I &= \frac{1}{|\det B|} \sum_{\ell=0}^L \int_{A^{*(j-1)}B^{*-1}([0,1]^d)} \sum_{v \in \{0,1\}^d} P_{f,\omega_n}^j(\xi, \nu) \overline{H_\ell(A^{*-j}\xi + A^{*-1}B^{*-1}\nu)} \\
 &\quad \times \sum_{v' \in \{0,1\}^d} \overline{P_{f,\omega_n}^j(\xi, \nu')} H_\ell(A^{*-j}\xi + A^{*-1}B^{*-1}\nu') d\xi \\
 &= \frac{1}{|\det B|} \int_{A^{*(j-1)}B^{*-1}([0,1]^d)} \sum_{v, v' \in \{0,1\}^d} P_{f,\omega_n}^j(\xi, \nu) \overline{P_{f,\omega_n}^j(\xi, \nu')} \\
 &\quad \times \sum_{\ell=0}^L H_\ell(A^{*-j}\xi + A^{*-1}B^{*-1}\nu') \overline{H_\ell(A^{*-j}\xi + A^{*-1}B^{*-1}\nu)} d\xi \\
 &= \frac{1}{|\det B|} \int_{A^{*(j-1)}B^{*-1}([0,1]^d)} \sum_{v \in \{0,1\}^d} |P_{f,\omega_n}^j(\xi, \nu)|^2 d\xi.
 \end{aligned}$$

Since $P_{f,\omega_n}^j(\xi, \nu)$ are $A^{*j}B^{*-1}\mathbb{Z}^d$ -periodic functions and

$$\bigcup_{\nu \in \{0,1\}^d} [A^{*(j-1)}B^{*-1}([0,1]^d) + A^{*(j-1)}B^{*-1}\nu] = A^{*j}B^{*-1}([0,1]^d).$$

Therefore, we have

$$\begin{aligned}
 I &= \frac{1}{|\det B|} \sum_{v \in \{0,1\}^d} \int_{A^{*(j-1)}B^{*-1}([0,1]^d)} \left| \sum_{s' \in \mathbb{Z}^d} \hat{f}(\xi + A^{*(j-1)}B^{*-1}(A^*s' + \nu)) \right. \\
 &\quad \left. \times \overline{\hat{\omega}_n(A^{*-j}\xi + A^{*-1}B^{*-1}(A^*s' + \nu))} \right|^2 d\xi
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{|\det B|} \sum_{\nu \in \{0,1\}^d} \int_{A^{*(j-1)B^{*-1}}([0,1]^d) + A^{*(j-1)B^{*-1}\nu}} \left| \sum_{s' \in \mathbb{Z}^d} \hat{f}(\xi + A^{*j}B^{*-1}s') \right. \\
 &\quad \left. \times \overline{\hat{\omega}_n(A^{*-j}\xi + B^{*-1}s')} \right|^2 d\xi \\
 &= \frac{1}{|\det B|} \sum_{\nu \in \{0,1\}^d} \int_{A^{*j}B^{*-1}([0,1]^d)} \left| \sum_{s' \in \mathbb{Z}^d} \hat{f}(\xi + A^{*j}B^{*-1}s') \overline{\hat{\omega}_n(A^{*-j}\xi + B^{*-1}s')} \right|^2 d\xi.
 \end{aligned}$$

Similarly, with the same method we can obtain

$$\begin{aligned}
 \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \omega_n \rangle|^2 &= \frac{1}{|\det B|} \sum_{\nu \in \{0,1\}^d} \int_{A^{*j}B^{*-1}([0,1]^d)} \left| \sum_{s' \in \mathbb{Z}^d} \hat{f}(\xi + A^{*j}B^{*-1}s') \right. \\
 &\quad \left. \times \overline{\hat{\omega}_n(A^{*-j}\xi + B^{*-1}s')} \right|^2 d\xi.
 \end{aligned}$$

Hence Eq. (3.3) holds for all $f \in \mathfrak{D}$. This completes the proof of the theorem as \mathfrak{D} is dense in $L^2(\mathbb{R}^d)$. □

Define a family of subspaces of $L^2(\mathbb{R}^d)$ by

$$U_j^n = \overline{\text{span}}\{D^j T_k \omega_n : k \in \mathbb{Z}^d\}, \quad j \in \mathbb{Z}, \quad n = 0, 1, 2, \dots \tag{3.4}$$

Then, it is immediate from (3.1) that

$$U_j^{n(L+1)+L} \subseteq U_{j+1}^n, \quad \ell = 0, 1, \dots, L,$$

or equivalently

$$\Delta_j = U_j^{n(L+1)} + U_j^{n(L+1)+1} + \dots + U_j^{n(L+1)+L} \subseteq U_{j+1}^n.$$

The reverse inequality is also obvious because if $0 \neq f \in U_{j+1}^n$ such that $f \in \Delta_j$, then clearly,

$$\sum_{\ell=0}^L \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \omega_{n(L+1)+\ell} \rangle|^2 = 0,$$

which contradicts Theorem 3.1 as $\sum_{k \in \mathbb{Z}^d} |\langle f, D^{j+1} T_k \omega_n \rangle|^2 \neq 0$. Thus, we conclude that for $n = 0, 1, 2, \dots$, we have

$$U_{j+1}^n = U_j^{n(L+1)} + U_j^{n(L+1)+1} + \dots + U_j^{n(L+1)+L}, \quad j \in \mathbb{Z}. \tag{3.5}$$

Clearly, $V_j = U_j^0$. Moreover, using (3.5) repeatedly with $j = 1, 2, \dots$, we have

$$U_j^0 = \sum_{\ell=0}^L U_{j-1}^\ell = \sum_{\ell=0}^{(L+1)^2-1} U_{j-2}^\ell = \dots = \sum_{\ell=0}^{(L+1)^{k-1}} U_{j-k}^\ell = \dots = \sum_{\ell=0}^{(L+1)^{j-1}} U_0^\ell. \tag{3.6}$$

Lemma 3.1. *Let $f \in \mathfrak{D}$ and $\varphi \in L^2(\mathbb{R}^d)$. Let φ satisfy the assumption (i). Then for any $\varepsilon > 0$, there exists $J \in \mathbb{Z}$ such that*

$$(1-\varepsilon)\|f\|_2^2 \leq \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \varphi \rangle|^2 \leq (1+\varepsilon)\|f\|_2^2 \quad \text{for all } j \geq J. \tag{3.7}$$

Proof. For $f \in \mathfrak{D}$ and $j \in \mathbb{Z}$, we have

$$\begin{aligned} & \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \varphi \rangle|^2 \\ &= \sum_{k \in \mathbb{Z}^d} |\langle D^{-j} f, T_k \varphi \rangle|^2 \\ &= \sum_{k \in \mathbb{Z}^d} \left| \left| \det A \right|^{j/2} \int_{\mathbb{R}^d} \hat{f}(A^{*j} \xi) \overline{\hat{\varphi}(\xi)} e^{2\pi i B k \xi} d\xi \right|^2 \\ &= \sum_{k \in \mathbb{Z}^d} \left| \det A \right|^j \left| \int_{B^{*-1}([0,1]^d)} \sum_{s \in \mathbb{Z}^d} \hat{f}(A^{*j}(\xi + B^{*-1}s)) \times \overline{\hat{\varphi}(\xi + B^{*-1}s)} e^{2\pi i B k \xi} d\xi \right|^2. \end{aligned}$$

It is easy to verify that the function $F(\xi)$ defined by

$$F(\xi) = \sum_{k \in \mathbb{Z}^d} \hat{f}(A^{*j}(\xi + B^{*-1}k)) \overline{\hat{\varphi}(\xi + B^{*-1}k)}$$

is well defined. Moreover, $F(\xi)$ can be bounded by a finite linear combination of translates of $\hat{\varphi}$ if $\xi \in B^{*-1}([0,1]^d)$. Since $\{e^{2\pi i B k \xi} : k \in \mathbb{Z}^d\}$ is an orthonormal basis for $L^2(B^{*-1}([0,1]^d))$. Therefore, we have

$$\sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \varphi \rangle|^2 = \left| \det A \right|^j \int_{B^{*-1}([0,1]^d)} \left| \sum_{k \in \mathbb{Z}^d} \hat{f}(A^{*j}(\xi + B^{*-1}k)) \overline{\hat{\varphi}(\xi + B^{*-1}k)} \right|^2 d\xi.$$

Now, let $\varepsilon > 0$ be given. By assumption, we can choose $b \in (0,1)$ such that $1-\varepsilon \leq |\hat{\varphi}(\xi)|^2 \leq 1+\varepsilon$ whenever ξ with $\|\xi\|_{\mathbb{R}^d} \leq b|\det B|$. Assume that $J \in \mathbb{Z}$ such that $D^j \hat{f}$ has support in $B^{*-1}([0,b]^d)$ for $j \geq J$. Then, with this choice of $j \geq J$, we obtain

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \varphi \rangle|^2 &= \left| \det A \right|^j \int_{B^{*-1}([0,b]^d)} |\hat{f}(A^{*j} \xi) \overline{\hat{\varphi}(\xi)}|^2 d\xi \\ &\leq (1+\varepsilon) \int_{B^{*-1}([0,b]^d)} |D^j \hat{f}(\xi)|^2 d\xi \\ &= (1+\varepsilon) \int_{\mathbb{R}^d} |D^j \hat{f}(\xi)|^2 d\xi \\ &= (1+\varepsilon)\|f\|_2^2. \end{aligned}$$

On the other hand, we have

$$(1-\varepsilon)\|f\|_2^2 \leq \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \varphi \rangle|^2.$$

So the Eq. (3.7) hold for all $f \in \mathfrak{D}$. □

Theorem 3.2. Let U_j^n be as in the Eq. (3.4). Then

$$L^2(\mathbb{R}^d) = \sum_{\ell=0}^{\infty} U_0^\ell, \tag{3.8}$$

and the collection $\{\omega_n(x-k): k \in \mathbb{Z}^d, n=0,1,\dots\}$ generates a normalized tight frame for $L^2(\mathbb{R}^d)$.

Proof. Since φ generates an MRA for $L^2(\mathbb{R}^d)$, therefore by using density property of an MRA and the Eq. (2.6), we obtain

$$L^2(\mathbb{R}^d) = \overline{\bigcup_{j \in \mathbb{Z}} V_j} = \lim_{j \rightarrow \infty} V_j = \lim_{j \rightarrow \infty} \sum_{\ell=0}^{(L+1)^j-1} U_0^\ell = \sum_{\ell=0}^{\infty} U_0^\ell.$$

Let $\varepsilon > 0$ be given, and $f \in \mathfrak{D}$. By Lemma 3.1, we can choose $J > 0$ such that for all $j > J$,

$$(1-\varepsilon)\|f\|_2^2 \leq \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \varphi \rangle|^2 \leq (1+\varepsilon)\|f\|_2^2.$$

Moreover, for any $j \geq 0$, Theorem 3.1 implies that

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \varphi \rangle|^2 &= \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \omega_0 \rangle|^2 = \sum_{\ell=0}^L \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \omega_\ell \rangle|^2 \\ &= \sum_{\ell=0}^L \sum_{m=0}^L \sum_{k \in \mathbb{Z}^d} |\langle f, D^{j-2} T_k \omega_{(L+1)\ell+m} \rangle|^2 \\ &= \sum_{n=0}^{(L+1)^2-1} \sum_{k \in \mathbb{Z}^d} |\langle f, D^{j-2} T_k \omega_n \rangle|^2 \dots \\ &= \sum_{n=0}^{(L+1)^{j-1}} \sum_{k \in \mathbb{Z}^d} |\langle f, T_k \omega_n \rangle|^2. \end{aligned}$$

Therefore, it follows that for all $j > J$, we have

$$(1-\varepsilon)\|f\|_2^2 \leq \sum_{n=0}^{(L+1)^{j-1}} \sum_{k \in \mathbb{Z}^d} |\langle f, T_k \omega_n \rangle|^2 \leq (1+\varepsilon)\|f\|_2^2.$$

Let $j \rightarrow \infty$, we obtain

$$(1-\varepsilon)\|f\|_2^2 \leq \sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}^d} |\langle f, T_k \omega_n \rangle|^2 \leq (1+\varepsilon)\|f\|_2^2.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\sum_{n=0}^{\infty} \sum_{k \in \mathbb{Z}^d} |\langle f, T_k \omega_n \rangle|^2 = \|f\|_2^2.$$

The proof is complete as \mathfrak{D} is dense in $L^2(\mathbb{R}^d)$. □

Definition 3.1. The functions $\{\omega_n : n=0,1,2,\dots\}$ as defined in (3.1) will be called the basic framelet packets associated with the refinable function φ .

In the following we find an expression for the Fourier transforms of the basic framelet packets in terms of $\hat{\varphi}$.

For $\omega_0 = \varphi$ with $\lim_{\xi \rightarrow 0} \hat{\omega}_0(\xi) = 1$, we have

$$\hat{\omega}_0(\xi) = H_0(A^{*-1}\xi)\hat{\omega}_0(A^{*-1}\xi).$$

Replace ξ by $A^{*-1}\xi$, we have

$$\hat{\omega}_0(A^{*-1}\xi) = H_0(A^{*-2}\xi)\hat{\omega}_0(A^{*-2}\xi)$$

and then

$$\hat{\omega}_0(\xi) = H_0(A^{*-1}\xi)H_0(A^{*-2}\xi)\hat{\omega}_0(A^{*-2}\xi).$$

We can continue this and obtain, for any $n \in \mathbb{N}$,

$$\begin{aligned} \hat{\omega}_0(\xi) &= (A^{*-1}\xi)H_0(A^{*-2}\xi)\cdots H_0(A^{*-n}\xi)\hat{\omega}_0(A^{*-n}\xi) \\ &= \hat{\omega}_0(A^{*-n}\xi) \prod_{j=1}^n H_0(A^{*-j}\xi). \end{aligned}$$

If the finite product $\prod_{j=1}^n H_0(A^{*-j}\xi)$ is convergent as $n \rightarrow \infty$ for each $\xi \in \mathbb{R}^d$, then

$$\hat{\omega}_0(\xi) = \prod_{j=1}^{\infty} H_0(A^{*-j}\xi). \quad (3.9)$$

In order to generalize this result to all framelet packets, we need to consider the unique " p -adic expansion" for an integer $n \geq 1$:

$$n = \sum_{j=1}^k \varepsilon_j p^{j-1}, \quad (3.10)$$

where $0 \leq \varepsilon_j \leq p-1$ for all $j=1,2,\dots,k$ and $\varepsilon_k \neq 0$. Let $p=L+1$. Suppose that $\varepsilon_j=0$ if $j \geq k+1$. Then, we have

$$n = \sum_{j=1}^{\infty} \varepsilon_j (L+1)^{j-1}, \quad (3.11)$$

for all $n \geq 0$, where $\varepsilon_j \in \{0,1,\dots,L\}$.

Theorem 3.3. Let n be a non-negative integer with $(L+1)$ -adic expansion given by (3.11). Then, the Fourier transform of the basic framelet packets given by (3.1) satisfies

$$\hat{\omega}_n(\xi) = \prod_{j=1}^{\infty} H_{\varepsilon_j}(A^{*-j}\xi) = \left\{ \prod_{j=1}^k H_{\varepsilon_j}(A^{*-j}\xi) \right\} \hat{\varphi}(A^{*-j}\xi) \quad (3.12)$$

provided $H_0(\xi)$ is continuously differentiable.

Proof. The infinite product

$$\prod_{j=1}^{\infty} H_{\varepsilon_j}(A^{*-j}\xi)$$

clearly converges for each $\xi \in \mathbb{R}^d$. In fact, from the definition of the basic framelet packets we know that when k is sufficiently large, $H_{\varepsilon_k} = H_0$. Since $\hat{\varphi}(0) = 1$, it is immediate from (2.6) that $H_0(0) = 1$. For $k \in \mathbb{N}$, let

$$G_k(\xi) = \prod_{j=1}^k H_{\varepsilon_j}(A^{*-j}\xi).$$

Note that Eq. (2.9) implies that $|H_{\varepsilon_j}(\xi)| \leq 1$ for all ξ , which shows that $|G_k(\xi)| \leq 1$ for all $k \in \mathbb{N}$. Consequently,

$$\begin{aligned} |G_{k+1}(\xi) - G_k(\xi)| &= |G_k(\xi)(H_0(A^{*-(k+1)}\xi) - 1)| \\ &\leq |H_0(A^{*-(k+1)}\xi) - H_0(0)| \\ &\leq \|H'_0\|_{L^\infty(\mathbb{R}^d)} A^{*-(k+1)} |\xi|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} |G_{k+1}(\xi) - G_k(\xi)| &\leq \|H'_0\|_{L^\infty(\mathbb{R}^d)} |\xi| (A^{*-(k+1)} + \dots + A^{*-(k+m)}) \\ &\leq \|H'_0\|_{L^\infty(\mathbb{R}^d)} |\xi| A^{*-k} \end{aligned}$$

for all $m \in \mathbb{N}$ and all sufficiently large $k \in \mathbb{N}$. Thus we conclude that the sequence $\{G_k(\xi) : k \in \mathbb{N}\}$ converges uniformly on every bounded subset of \mathbb{R}^d .

Assume that $n \in \{0, 1, \dots, L\}$. Then, the $(L+1)$ -adic expansion of n is $\varepsilon_1 = n, \varepsilon_j = 0, j \geq 2$ and hence, by Eq. (3.1), we have

$$\begin{aligned} \hat{\omega}_n(\xi) &= H_n(A^{*-1}\xi)\hat{\omega}_0(A^{*-1}\xi) \\ &= H_n(A^{*-1}\xi)\prod_{j=1}^{\infty} H_0(A^{*-(j+1)}\xi) \\ &= H_n(A^{*-1}\xi)\prod_{j=2}^{\infty} H_0(A^{*-j}\xi) \\ &= \prod_{j=1}^{\infty} H_{\varepsilon_j}(A^{*-j}\xi). \end{aligned}$$

Hence (3.12) holds for $n = 0, 1, \dots, L$.

Suppose (3.12) holds for every non-negative integer $m < n$, where $n \geq L+1$, then we can rewrite n as $n = (L+1)r + s$, where $r \in \mathbb{N}_0, 0 \leq s \leq L$. Suppose the $(L+1)$ -adic expansion of r is

$$r = \sum_{j=1}^{\infty} \varepsilon_j (L+1)^{j-1}.$$

Then, by our assumption, we can write

$$\hat{\omega}_r(\xi) = \prod_{j=1}^{\infty} H_{\varepsilon_j}(A^{*-j}\xi).$$

Furthermore, we have

$$n = (L+1)r + s = \sum_{j=1}^{\infty} \varepsilon_j(L+1)^j + s = \sum_{j=2}^{\infty} \varepsilon_{j-1}(L+1)^{j-1} + s = \sum_{j=1}^{\infty} \varepsilon'_j(L+1)^{j-1},$$

where $\varepsilon'_1 = \mu, \varepsilon'_j = \varepsilon_{j-1}, j \geq 2$. Consequently, the Fourier transform of ω_n becomes

$$\begin{aligned} \hat{\omega}_n(\xi) &= \hat{\omega}_{(L+1)r+s}(\xi) = H_s(A^{*-1}\xi) \hat{\omega}_r(A^{*-1}\xi) = H_s(A^{*-1}\xi) \prod_{j=1}^{\infty} H_{\varepsilon_j}(A^{*-(j+1)}\xi) \\ &= H_s(A^{*-1}\xi) \prod_{j=2}^{\infty} H_{\varepsilon_{j-1}}(A^{*-j}\xi) = \prod_{j=1}^{\infty} H_{\varepsilon'_j}(A^{*-j}\xi). \end{aligned}$$

This completes the proof of the theorem. □

The idea of the basic framelet packets enables us to construct various tight frames for $L^2(\mathbb{R}^d)$ by replacing some mother framelets. To do so, we define the functions $\psi^r_{m_0}, 0 \leq r \leq L, 1 \leq m_0 \leq L$ by

$$\psi^r_{m_0}(A^*\xi) = H_r(\xi) \hat{\psi}_{m_0}(\xi). \tag{3.13}$$

Theorem 3.4. *Suppose $X(\Psi)$ is a normalized tight wavelet frame constructed via UEP in an MRA and H_0, H_1, \dots, H_L are the refinement mask and framelet masks, respectively. Let $\omega_n, n = 0, 1, 2, \dots$ be as in the Eq. (3.1). For any $m_0 \in \{1, 2, \dots, L\}$, let $\psi^r_{m_0}, 0 \leq r \leq L$, are the functions given by (3.13). Then the collection*

$$\mathcal{F} = \{D^j T_k \psi_m, D^{j-1} T_k \psi^r_{m_0} : m = 1, \dots, L, m \neq m_0, j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$$

generates a new normalized tight frame for $L^2(\mathbb{R}^d)$.

Proof. By taking $\omega_n = \psi_{m_0}$, Theorem 3.1 implies that

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \psi_{m_0} \rangle|^2 &= \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \omega_n \rangle|^2 = \sum_{r=0}^L \sum_{k \in \mathbb{Z}^d} |\langle f, D^{j-1} T_k \omega_{(L+1)n+r} \rangle|^2 \\ &= \sum_{r=0}^L \sum_{k \in \mathbb{Z}^d} |\langle f, D^{j-1} T_k \psi^r_{m_0} \rangle|^2. \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 & \sum_{\substack{m=1 \\ m \neq m_0}}^L \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \psi_m \rangle|^2 + \sum_{r=0}^L \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |\langle f, D^{j-1} T_k \psi_{m_0}^r \rangle|^2 \\
 &= \sum_{\substack{m=1 \\ m \neq m_0}}^L \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \psi_m \rangle|^2 + \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \psi_{m_0} \rangle|^2 \\
 &= \sum_{m=1}^L \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |\langle f, D^j T_k \psi_m \rangle|^2 \\
 &= \|f\|_2^2.
 \end{aligned}$$

This completes the proof. \square

References

- [1] Q. Chen and Z. Chang, A study on compactly supported orthogonal vector-valued wavelets and wavelet packets, *Chaos, Soliton. Fract.*, 31 (2007), 1024–1034.
- [2] C. K. Chui and C. Li, Non-orthogonal wavelet packets, *SIAM J. Math. Anal.*, 24 (1993), 712–738.
- [3] R. R. Coifman, Y. Meyer, S. Quake and M. V. Wickerhauser, *Signal Processing and Compression with Wavelet Packets*, Technical Report, Yale University, 1990.
- [4] I. Daubechies, B. Han, A. Ron and Z. Shen, Framelets: MRA-based constructions of wavelet frames, *Appl. Comput. Harmon. Anal.*, 14 (2003), 1–46.
- [5] L. Debnath and F. A. Shah, *Wavelet Transforms and Their Applications*, Birkhäuser, Boston, 2015.
- [6] B. Dong, H. Ji, J. Li, Z. Shen and Y. Xu, Wavelet frame based blind image inpainting, *Appl. Comput. Harmonic Anal.*, 32 (2012), 268–279.
- [7] B. Han, Compactly supported tight wavelet frames and orthonormal wavelets of exponential decay with a general dilation matrix, *J. Comput. Appl. Math.*, 155 (2003), 43.
- [8] B. Han, Wavelets and framelets within the framework of non-homogeneous wavelet systems, In: *Approximation Theory XIII.*, Neamtu, M., Schumaker, L., Eds.; Springer, 2012., pp. 121–161.
- [9] M. J. Lai and J. Stöckler, Construction of multivariate compactly supported tight wavelet frames, *Appl. Comput. Harmonic Anal.*, 21 (2006), 324–348.
- [10] D. Lu and Q. Fan, A class of tight framelet packets, *Czechoslovak Math. J.*, 61 (2011), 623–639.
- [11] A. Ron and Z. Shen, Affine systems in $L^2(\mathbb{R}^d)$: the analysis of the analysis operator, *J. Funct. Anal.*, 148 (1997), 408–447.
- [12] F. A. Shah, Construction of wavelet packets on p -adic field, *Int. J. Wavelets, Multiresolut. Inf. Process.*, 7(5) (2009), 553–565.
- [13] F. A. Shah and Abdullah, Wave packet frames on local fields of positive characteristic, *Appl. Math. Comput.*, 249 (2014), 133–141.
- [14] F. A. Shah and M. Y. Bhat, Tight framelet packets on local fields of positive characteristic, *J. Classical Anal.*, 6 (2015), 85–101.

- [15] F. A. Shah and L. Debnath, p -Wavelet frame packets on a half-line using the Walsh-Fourier transform, *Integ. Trans. Special Funct.*, 22(12) (2011), 907–917.
- [16] F. A. Shah and L. Debnath, Explicit construction of M -band tight framelet packets, *Analysis*, 32 (2012), 281–294.
- [17] F. A. Shah and N. A. Sheikh, Construction of vector-valued multivariate wavelet frame packets, *Thai J. Math.*, 10 (2012), 401–414.
- [18] Z. Shen, Non-tensor product wavelet packets in $L^2(\mathbb{R}^s)$, *SIAM J. Math. Anal.*, 26(4) (1995), 1061–1074.