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Some Remarks on the Restriction Theorems for the Maximal Operators on $\mathbb{R}^d$

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Abstract. The aim of this paper is to give a simple proof of the restriction theorem for the maximal operators on the d-dimensional Euclidean space $\mathbb{R}^d$, whose theorem was proved by Carro-Rodriguez in 2012. Moreover, we shall give some remarks of the restriction theorem for the linear and the multilinear operators by Carro-Rodriguez and Rodriguez, too.

Key Words: Weighted L^p spaces, Fourier multipliers, multilinear operators.

AMS Subject Classifications: 42B15,42B35

1 Introduction and results

Let p be in $1\le p<\infty$, $w(x)$ a nonnegative 2π periodic function in $L^1_{loc}(\mathbb{R}^d)$ which is called a weight. First we define weighted L^p spaces on the d-dimensional Euclidean space $\mathbb{R}^d$ or on the d-dimensional torus $\mathbb{T}^d$.

Definition 1.1. Let $1\le p<\infty$, $0<q<\infty$, and $w(x)$ a non-negative 2π periodic function in $L^1_{loc}(\mathbb{R}^d)$

$$L^{p,q}(\mathbb{R}^d,w)=\Big\{f\,|\,\|f\|_{L^{p,q}(\mathbb{R}^d,w)}=\Big(\int_0^\infty (tw(\{|f|>t\})^{1/p})^q\frac{dt}{t}\Big)^{1/q}<\infty\Big\},$$

$$L^{p,\infty}(\mathbb{R}^d,w)=\Big\{f\,|\,\|f\|_{L^{p,\infty}(\mathbb{R}^d,w)}=\inf\big\{M\,|\,tw(\{x\in\mathbb{R}^d\,|\,|f(x)|>t\})^{1/p}<M\ \text{ for }\ t>0\big\}<\infty\Big\},$$

$$L^{p,q}(\mathbb{T}^d,w)=\Big\{F\,|\,\|F\|_{L^{p,q}(\mathbb{R}^d,w)}=\Big(\int_0^\infty (tw(\{|F|>t\})^{1/p})^q\frac{dt}{t}\Big)^{1/q}<\infty\Big\},$$

$$L^{p,\infty}(\mathbb{T}^d,w)=\Big\{F\,|\,\|F\|_{L^{p,\infty}(\mathbb{T}^d,w)}=\inf\big\{M\,|\,tw(\{x\in\mathbb{R}^d\,|\,|F(x)|>t\})^{1/p}<M\ \text{ for }\ t>0\big\}<\infty\Big\},$$

where $w(E)=\int_E w(x)dx$ for $E\subset\mathbb{R}^d$ or $E\subset\mathbb{T}^d$.

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Also let $\{\phi_j\}_{j=1}^{\infty}$ be in $C_b(\mathbb{R}^d)$ which is the set of all bounded continuous functions on $\mathbb{R}^d$, and $\phi_{j|_{\mathbb{Z}^d}}$ the restriction function of ϕ_j on the d-dimensional integer space $\mathbb{Z}^d$. When $w(x)=1$ $(x\in\mathbb{R}^d)$, $L^p(\mathbb{R}^d,w)$, $L^{p,\infty}(\mathbb{R}^d,w)$ (resp. $L^p(\mathbb{T}^d,w)$, $L^{p,\infty}(\mathbb{T}^d,w)$) are denoted by $L^p(\mathbb{R}^d)$, $L^{p,\infty}(\mathbb{R}^d)$ (resp. $L^p(\mathbb{T}^d)$, $L^{p,\infty}(\mathbb{T}^d)$), respectively. Moreover, we define some operators T_{ϕ_j}, T^*, $\widetilde{T_{\phi_j|_{\mathbb{Z}^d}}}$, and $\widetilde{T^*}$:

Definition 1.2. For

$$T_{\phi_j}f(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\phi_j(\xi)\hat{f}(\xi)e^{ix\xi}d\xi, \qquad T^*f(x)=\sup_j|T_{\phi_j}f(x)|,$$

$$\widetilde{T_{\phi_{j|\mathbb{Z}^d}}}F(x)=\sum_{k\in\mathbb{Z}^d}\phi_j(k)\hat{F}(k)e^{ikx}, \qquad \widetilde{T^*}F(x)=\sup_j|\widetilde{T_{\phi_j|\mathbb{Z}^d}}F(x)|,$$

where f is in Schwartz spaces $\mathcal{S}(\mathbb{R}^d)$, and F in trigonometric polynomials $P(\mathbb{T}^d)$ on $\mathbb{T}^d$,

$$\hat{f}(\xi)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}f(x)e^{-i\xi x}dx \text{ and } \widehat{F}(k)=\frac{1}{(2\pi)^d}\int_{[0,2\pi)^d}F(x)e^{-ikx}dx\Big(=\int_{\mathbb{T}^d}F(x)e^{-ikx}dx\Big).$$

Now in 1960, K. de Leeuw [5] proved that if T_ϕ is bounded on $L^p(\mathbb{R}^d)$ for $\phi\in C_b(\mathbb{R}^d)$, $\widetilde{T_{\phi_{|\mathbb{Z}^d}}}$ is bounded on $L^p(\mathbb{T}^d)$. In 1985, Kenig-Tomas [14] showed the same result between T^* and $\widetilde{T^*}$ for $1<p<\infty$. Moreover, in 1994, Asmar-Berkson-Bourgain [2] (cf. [1,12]) proved that if T^* is bounded from $L^p(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$, $\widetilde{T^*}$ is bounded from $L^p(\mathbb{T}^d)$ to $L^{p,\infty}(\mathbb{T}^d)$ for $1\le p<\infty$. After that, there are many papers related to this property [6,7] (cf. [8,9,17]). Also in 2003, Berkson-Gillispie [3] proved that if T_ϕ is bounded on $L^p(\mathbb{R}^d,w)$ for $\phi\in C_b(\mathbb{R}^d)$ and $1<p<\infty$ with $w\in A_p(\mathbb{T}^d)$, $\widetilde{T_{\phi|Z^d}}$ is bounded on $L^p(\mathbb{T}^d,w)$, where

$$A_p(\mathbb{T}^d)=\Big\{w(x)\ge 0|w(x) \text{ is a } 2\pi \text{ periodic function on } \mathbb{R}^d \\ \text{with } \sup_{Q,\text{cube}}\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\Big)^{p-1}<\infty\Big\},$$

where $|Q|$ is the Lebesgue measure of $Q\subset\mathbb{T}^d$. In 2009, Anderson-Mohanty [1] proved Berkson-Gillispie's result without A_p condition. In 2012, Carro-Rodriguez [4] which is summing up to the restriction theorems of multipliers in weighted setting showed that if T^* is bounded from $L^p(\mathbb{R}^d,w)$ to $L^{p,\infty}(\mathbb{R}^d,w)$ for $1\le p<\infty$ with a non-negative 2π periodic function $w(x)\in L^1_{loc}(\mathbb{R}^d)$, $\widetilde{T^*}$ is bounded from $L^p(\mathbb{T}^d,w)$ to $L^{p,\infty}(\mathbb{T}^d,w)$ (cf. [13]). Their results are proved by applying Kolmogorov's condition with vector valued argument (cf. [10]).

Recently by the same method, Rodriguez [15] gives the analogy with respect to the multilinear operators, whose result is as follows: Let $1\le p_j<\infty$ $(j=1,\cdots,m)$ for $m\in\mathbb{N}$ with $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$, and $w(x),w_1(x),\cdots,w_m(x)$ 2π periodic non-negative functions. Also let

$\{\Phi_j\}_j$ be in $C_b(\mathbb{R}^{md})$, and let $T_{\Phi_j}(f_1,\cdots,f_m)$, $T^*(f_1,\cdots,f_m)$, $\widetilde{T_{\Phi_{j|\mathbb{Z}^{md}}}}(F_1,\cdots,F_m)$, $\widetilde{T^*}(F_1,\cdots,F_m)$ be such that

$$T_{\Phi_j}(f_1,\cdots,f_m)(x)=\frac{1}{(2\pi)^{md}}\int_{\mathbb{R}^{md}}\Phi_j(\xi_1,\cdots,\xi_m)\widehat{f_1}(\xi_1)\cdots\widehat{f_m}(\xi_m)e^{-i(\xi_1+\cdots+\xi_m)x}d\xi_1\cdots,d\xi_m,$$

$$\widetilde{T_{\Phi_{j|_{\mathbb{Z}^d}}}}(F_1,\cdots,F_m)(x)=\sum_{k_1,\cdots,k_m\in\mathbb{Z}^d}\Phi_j(k_1,\cdots,k_m)\widehat{F_1}(k_1)\cdots\widehat{F_m}(k_m)e^{-i(k_1+\cdots+k_m)x},$$

$$T^*(f_1,\cdots,f_m)(x)]=\sup_{j\geq 1}|T_{\Phi_j}(f_1,\cdots,f_m)(x)|,$$

$$\widetilde{T^*}(F_1,\cdots,F_m)(x)=\sup_{j\geq 1}|\widetilde{T_{\Phi_{j|\mathbb{Z}^d}}}(F_1,\cdots,F_m)(x)|.$$

Then, Rodoriguez [11] shows that if

$$\|T^*(f_1,\cdots,f_m)\|_{L^{p,\infty}(\mathbb{R}^d,w)}\leq C\Pi_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^d,w_j)},\quad (f_j\in\mathcal{S}(\mathbb{R}^d)),$$

for some constant $C>0$, $\widetilde{T^*}(F_1,\cdots,F_m)$ is bounded from $\Pi_{j=1}^m L^{p_j}(\mathbb{T}^d,w_j)$ to $L^{p,\infty}(\mathbb{T}^d,w)$, i.e.,

$$\|\widetilde{T^*}(F_1,\cdots,F_m)\|_{L^{p,\infty}(\mathbb{T}^d,w)}\leq C\Pi_{j=1}^m\|F_j\|_{L^{p_j}(\mathbb{T}^d,w_j)},\quad (F_j\in P(\mathbb{T}^d)),$$

for some constant $C>0$.

In this paper, we shall give simple proofs of Carro-Rodriguez [4, Theorem 1.3] and Rodriguez [15, Theorem 2.3] without their argument for the restriction theorems.

Our results are as follows:

Theorem 1.1. *Let $1\leq p<\infty$, $0<q\leq\infty$ and $w(x)$ a non-negative 2π periodic function in $L^1_{loc}(\mathbb{R}^d)$. Also assume $\{\phi_j\}$ in $L^\infty(\mathbb{R}^d)$ are continuous at every point of $\mathbb{Z}^d$. Then if T^* is bounded from $L^p(\mathbb{R}^d,w)$ to $L^{p,q}(\mathbb{R}^d,w)$,$\widetilde{T^*}$ is bounded from $L^p(\mathbb{T}^d,w)$ to $L^{p,q}(\mathbb{T}^d,w)$.*

Proposition 1.1 (see [4]). Let $1\leq p<\infty$, $0<q\leq\infty$ and $w(x)$ a non-negative 2π periodic function in $L^1_{loc}(\mathbb{R}^d)$. Also let $\{\phi_j\}_j$ be in $C_b(\mathbb{R}^d)$. Then if T^* is bounded from $L^p(\mathbb{R}^d,w)$ to $L^{p,q}(\mathbb{R}^d,w)$, $\widetilde{T^*}$ is bounded from $L^p(\mathbb{T}^d,w)$ to $L^{p,q}(\mathbb{T}^d,w)$.

Theorem 1.2. *Let $0<p<\infty$, $1\leq p_j<\infty$ with $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$, $0<q\leq\infty$ and $w(x)$, $w_j(x)$, non-negative 2π periodic functions in $L^1_{loc}(\mathbb{R}^d)$. Also assume $\{\Phi_j\}$ in $L^\infty(\mathbb{R}^{md})$ are continuous at every points of $\mathbb{Z}^{md}$. Then if $T^*(f_1,\cdots,f_m)$ is bounded from $\Pi_{j=1}^m L^{p_j}(\mathbb{R}^d,w_j)$ to $L^{p,q}(\mathbb{R}^d,w)$, $\widetilde{T^*}(F_1,\cdots,F_m)$ is bounded from $\Pi_{j=1}^m L^{p_j}(\mathbb{T}^d,w_j)$ to $L^{p,q}(\mathbb{T}^d,w)$.*

Proposition 1.2 (see [15]). Let $0<p<\infty$, $1\leq p_j<\infty$ with $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$, $0<q\leq\infty$ and $w(x)$, $w_j(x)$, 2π periodic non-negative functions in $L^1_{loc}(\mathbb{R}^d)$. Also let $\{\Phi_j\}$ be in $C_b(\mathbb{R}^{md})$. Then if $T^*(f_1,\cdots,f_m)$ is bounded from $\Pi_{j=1}^m L^{p_j}(\mathbb{R}^d,w_j)$ to $L^{p,q}(\mathbb{R}^d,w)$, $\widetilde{T^*}(F_1,\cdots,F_m)$ is bounded from $\Pi_{j=1}^m L^{p_j}(\mathbb{T}^d,w_j)$ to $L^{p,q}(\mathbb{T}^d,w)$.

In Section 2, we give an alternative proof of Carro-Rodriguezy [4, Theorem 1.3] that if T^* is bounded from $L^p(\mathbb{R}^d,w)$ to $L^{p,\infty}(\mathbb{R}^d,w)$ for $1\le p<\infty$ with a nonnegative 2π periodic function $w(x)\in L^1_{loc}(\mathbb{R}^d)$, $\widetilde{T^*}$ is bounded from $L^p(\mathbb{T}^d,w)$ to $L^{p,\infty}(\mathbb{T}^d,w)$. The investigation of the properties of Gauss kernel will bring an elementary proof of their result (cf. [1,13]). Moreover, we give a simple proof of the result with respect to multilinear operators in Rodriguez [15, Theorem 2.3] by applying the idea in Proposition 1.1. In Section 3, we treat the generalization of Propositions 1.1 and 1.2. For the sake of this, we define regulated functions for Fourier multipliers whose concept is well-known (cf. [5,11]). Those results of the generalization are shown in [4] and [15], but our proofs are easily rather than them. Throughout this paper, C denotes varying constant in that occasion.

2 The proofs of Propositions 1.1 and 1.2

First, we give the proof of Proposition 1.1, which is based on the idea in [13]. It will be applied to the proof of Proposition 1.2, too. For the proof of Proposition 1.1, we have two lemmas. Proposition 1.1 will easily bring Theorem 1.1. So we omit the proof of Theorem 1.1.

Lemma 2.1 (see [1]). *Let*

$$W_\delta(x)=e^{-\frac{\delta}{4\pi}|x|^2},\quad (x\in\mathbb{R}^d),\quad \text{and}\quad \widetilde{T^*_J}F(x)=\sup_{1\le j\le J}|\widetilde{T_{\phi_j|\mathbb{Z}^d}}F(x)|$$

for a natural number $J\in\mathbb{N}$. Then we have the following:
(1) For a 2π periodic function $f(x)$ in $L^1_{loc}(\mathbb{R}^d)$,

$$\lim_{\varepsilon\to 0+}\frac{\varepsilon^{d/2}}{(2\pi)^d}\int_{\mathbb{R}^d}f(x)W_\varepsilon(x)dx=\int_{\mathbb{T}^d}f(x)dx.$$

(2)

$$\lim_{\varepsilon\to 0+}\varepsilon^{\frac{d}{2p}}\|W_\varepsilon F\|_{L^p(\mathbb{R}^d,w)}=\Big(\frac{1}{\sqrt{p}}\Big)^{d/p}\|F\|_{L^p(\mathbb{T}^d,w)},\quad (F\in L^p(\mathbb{T}^d,w)).$$

(3) There exists a constant $C>0$ such that

$$\liminf_{\varepsilon\to 0+}\varepsilon^{d/2}w(\{x\in\mathbb{R}^d|W_\varepsilon(x)\widetilde{T^*}_JF(x)>t\})\ge Cw(\{x\in\mathbb{T}^d|\widetilde{T^*_J}F(x)>te\}),\quad (F\in P(\mathbb{T}^d)).$$

Proof. (1) This result is proved by [1, Lemma 2.2]. (2) By (1), we have that

$$\varepsilon^{d/2}\|W_\varepsilon F\|^p_{L^p(\mathbb{R}^d,w)}=\Big(\frac{1}{\sqrt{p}}\Big)^d\int_{\mathbb{T}^d}|F(x)|^pw(x)dx,$$

and

$$\lim_{\varepsilon\to 0+}\varepsilon^{\frac{d}{2p}}\|W_\varepsilon F\|_{L^p(\mathbb{R}^d,w)}=\Big(\frac{1}{\sqrt{p}}\Big)^{d/p}\|F\|_{L^p(\mathbb{T}^d,w)},\quad (F\in L^p(\mathbb{T}^d,w)).$$

(3) Let $G(x)$ be a function on$\mathbb{T}^d$. For $t>0$, we have

$$\begin{aligned} w(\{x\in\mathbb{R}^d|W_\delta(x)|G(x)|>t\}) &= \sum_{j\in\mathbb{Z}^d} w(\{x\in[2\pi j,2\pi(j+1))|e^{-\frac{\delta}{4\pi}|x|^2}|G(x)|>t\}) \\ &\geq \sum_{j=(j_1,\cdots,j_d)\in\mathbb{Z}^d,\ j_k\geq 0} w(\{x\in[0,2\pi)^d|e^{-\frac{\delta}{4\pi}|x+2\pi j|^2}|G(x)|>t\}), \end{aligned}$$

where $j+1=(j_1+1,\cdots,j_d+1)$. Since we have $|s+2\pi(j+1)|\geq|u+2\pi j|$ for $s\in[0,2\pi)^d$ and $u\in[0,2\pi)^d$, we obtain that

$$\begin{aligned} &w(\{x\in\mathbb{R}^d|W_\delta(x)|G(x)|>t\}) \\ \geq &\sum_{j=(j_1,\cdots,j_d)\in\mathbb{Z}^d,\ j_k\geq 0} \int_{\mathbb{T}^d} w(\{x\in[0,2\pi)^d|e^{-\frac{\delta}{4\pi}|x+2\pi j|^2}|G(x)|>t\})ds \\ \geq &C\Big(\frac{1}{\sqrt{\pi\delta}}\Big)^d w(\{u\in\mathbb{T}^d\||G(u)|>te\})dx, \quad (\sqrt{\pi\delta}<1/2), \end{aligned}$$

for some constant $C>0$. Therefore, we have

$$(\sqrt{\pi\delta})^d w(\{x\in\mathbb{R}^d|W_\delta(x)|G(x)|>t\})\geq Cw(\{u\in\mathbb{T}^d\||G(u)|>te\})$$

for sufficiently small $\delta>0$. After all, we have that for $G(x)=\widetilde{T_J^*}F(x)$ there exists a constant $C>0$ such that

$$\liminf_{\varepsilon\to 0+} \varepsilon^{d/2} w(\{x\in\mathbb{R}^d|W_\varepsilon(x)\widetilde{T_J^*}F(x)>t\})\geq Cw(\{x\in\mathbb{T}^d|\widetilde{T_J^*}F(x)>te\}).$$

So, we complete the proof. □

Proof of Proposition 1.1. In this part, we check some points as same as Kanjin-Kanno-Sato [13]. Let $\{\phi_j\}_{j=1}^\infty$ be in $C_b(\mathbb{R}^d)$, and T^*, $\widetilde{T^*}$ the assocated operators with $\{\phi_j\}_j$. We denote T_J^* and $\widetilde{T_J^*}$ by $T_J^*f(x)=\sup_{1\leq j\leq J}|T_{\phi_j}f(x)|$ for $f\in\mathcal{S}(\mathbb{R}^d)$ and $\widetilde{T_J^*}F(x)=\sup_{1\leq j\leq J}|\widetilde{T_{\phi_{j|\mathbb{Z}^d}}}F(x)|$ for $F\in P(\mathbb{T}^d)$. Moreover, we define that

$$\gamma_{j,\delta}(x)=W_\delta(x)\widetilde{T_{\phi_{j|\mathbb{Z}^d}}}F(x)-T_{\phi_j}(W_\delta(x)F(x)),$$

and

$$\Delta_{J,\delta}=\max_{1\leq j\leq J}\|\widehat{\gamma_{j,\delta}}\|_{L^1(\mathbb{R}^d)}$$

for $1\leq j\leq J$ and $F\in P(\mathbb{T}^d)$, where

$$\widehat{\gamma_{j,\delta}}(\xi)=\sum_k \hat{F}(k)\widehat{W_\delta}(\xi-k)\phi_j(k)-\phi_j(\xi))\widehat{W_\delta F}(\xi).$$

Then it is easy to prove the following by the method of [13] and we omit the details.

Lemma 2.2.
$$\lim_{\delta\to 0}\Delta_{J,\delta}=0.$$

Now we proceed to the proof of Proposition 1.1. First we show the case $0<q<\infty$. Since $W_\delta(x)\widetilde{T_{\phi_{j|\mathbb{Z}^d}}}F(x)=\gamma_{j,\delta}+T_{\phi_j}(W_\delta F)(x)$ for $F\in P(\mathbb{T}^d)$, we have

$$W_\delta(x)\widetilde{T_J^*}F(x)\le(2\pi)^d\Delta_{J,\delta}+T_J^*(W_\delta F)(x).$$

Since

$$\{x\in\mathbb{R}^d\,|\,W_\delta(x)\widetilde{T_J^*}F(x)>t\}\subset\{x\in\mathbb{R}^d\,|\,T_J^*(W_\delta F)(x)>t-(2\pi)^d\Delta_{J,\delta}\}$$

for $t>0$, we obtain that for $a>2(2\pi)^d\Delta_{J,\delta}$,

$$\begin{aligned}&\Big(\int_a^\infty(tw(\{x\in\mathbb{R}^d\,|\,W_\delta(x)\widetilde{T_J^*}F(x)>t\})^{1/p})^q\frac{dt}{t}\Big)^{1/q}\\ \le&C\Big(\int_{a-(2\pi)^d\Delta_{J,\delta}}^\infty(uw(\{x\in\mathbb{R}^d\,|\,T_J^*(W_\delta F)(x)>u\})^{1/p})^q\frac{du}{u}\Big)^{1/q}\\ \le&C\|W_\delta F\|_{L^p(\mathbb{R}^d,w)},\end{aligned}$$

by

$$\frac{u+(2\pi)^d\Delta_{J,\delta}}{u}\le 2$$

and the assumption of T^*. After all, we have

$$\Big(\int_a^\infty(tw(\{x\in\mathbb{R}^d\,|\,W_\delta(x)\widetilde{T_J^*}F(x)>t\})^{1/p})^q\frac{dt}{t}\Big)^{1/q}\le C\|W_\delta F\|_{L^p(\mathbb{R}^d,w)}$$

for $a>2(2\pi)^d\Delta_{J,\delta}$. Hence, by (2), (3) of Lemma 2.1 and Fatou's lemma

$$\Big(\int_a^\infty(tw(\{x\in\mathbb{R}^d\,|\,\tilde{T}_J^*F(x)>te\})^{1/p})^q\frac{dt}{t}\Big)^{1/q}\le C\|F\|_{L^p(\mathbb{R}^d,w)},$$

and by $a\downarrow 0$ we obtain

$$\Big(\int_0^\infty(tw(\{x\in\mathbb{R}^d\,|\,\widetilde{T_J^*}F(x)>te\})^{1/p})^q\frac{dt}{t}\Big)^{1/q}\le C\|F\|_{L^p(\mathbb{R}^d,w)}.$$

By applying Fatou's lemma again, we get the desired result:

$$\|\widetilde{T^*}F\|_{L^{p,q}(\mathbb{T}^d,w)}\le C\|F\|_{L^p(\mathbb{T}^d,w)},\quad (F\in P(\mathbb{T}^d)),$$

for $1\le p<\infty$ and $0<q<\infty$. In the case $q=\infty$, we can show it in the same way as the case $0<q<\infty$. We omit the details. □

Corollary 2.1 (see [11,16]). Let $1\le p<\infty$, $0<q\le\infty$, and ϕ in $C_b(\mathbb{R}^d)$. Then we have

$$\|T_\phi f\|_{L^{p,q}(\mathbb{R}^d)}\le C\|f\|_{L^p(\mathbb{R}^d)},\quad (f\in\mathcal{S}(\mathbb{R}^d)),$$

if and only if there exists a constant $C>0$ independent of $\varepsilon>0$ such that

$$\|\widetilde{T_{\phi_\varepsilon}}F\|_{L^{p,q}(\mathbb{T}^d)}\le C\|F\|_{L^p(\mathbb{T}^d)},\quad (F\in P(\mathbb{T}^d)),$$

where $\phi_\varepsilon(x)=\phi(\varepsilon x)$.

Proof. We assume $\|T_\phi f\|_{L^{p,q}(\mathbb{R}^d)}\le C\|f\|_{L^p(\mathbb{R}^d)}$, $(f\in\mathcal{S}(\mathbb{R}^d))$. Since

$$T_{\phi_\varepsilon}f(x)=\int_{\mathbb{R}^d}\phi(\varepsilon\xi)\hat f(\xi)e^{ix\xi}d\xi,$$

we have that

$$T_{\phi_\varepsilon}f(x)=\int_{\mathbb{R}^d}\phi(\xi)\varepsilon^{-d}\hat f(\varepsilon\xi)e^{i\varepsilon^{-1}x\xi}d\xi=(T_\phi f_\varepsilon)(\varepsilon^{-1}x),$$

where $f_\varepsilon(x)=f(\varepsilon x)$. Hence, by the assumption we obtain the desired result:

$$\|\widetilde{T_{\phi_\varepsilon}}F\|_{L^{p,q}(\mathbb{T}^d)}\le C\|F\|_{L^p(\mathbb{T}^d)},\quad (F\in P(\mathbb{T}^d)),$$

for all $\varepsilon>0$ by Proposition 1.1. Conversely, we assume that

$$\|\widetilde{T_{\phi_\varepsilon}}F\|_{L^{p,q}(\mathbb{T}^d)}\le C\|F\|_{L^p(\mathbb{T}^d)},\quad (F\in P(\mathbb{T}^d)),$$

for all $\varepsilon>0$. Let f be in $\mathcal{S}(\mathbb{R}^d)$, and $F_\varepsilon(x)=\varepsilon^{-d}\sum_{k\in\mathbb{Z}^d}f(\varepsilon^{-1}(x+2k\pi))$. Then we have $F_\varepsilon\in L^1(\mathbb{T}^d)$. Also by $\widehat{F_\varepsilon}(j)=\hat f(\varepsilon j)$, we get $F_\varepsilon(x)=\sum_j\hat f(\varepsilon j)e^{ijx}$. Here, by the definition of Riemann integral, we remark that for all $\varepsilon>0$

$$T_\phi f(x)=\int_{\mathbb{R}^d}\phi(\xi)\hat f(\xi)e^{ix\xi}d\xi=\lim_{\varepsilon\to0}\sum_j\phi(\varepsilon j)\hat f(\varepsilon j)e^{ix\varepsilon j}\varepsilon^d=\lim_{\varepsilon\to0}\widetilde{T_{\phi_\varepsilon}}(F_\varepsilon)(\varepsilon x)\varepsilon^d.$$

Moreover, since we have

$$T_\phi f(x)=\lim_{\varepsilon\to0}\widetilde{T_{\phi_\varepsilon}}(\varepsilon^dF_\varepsilon)(\varepsilon x)\chi_{[-\pi,\pi)^d}(\varepsilon x)$$

by $\lim_{\varepsilon\to0}\chi_{[-\pi,\pi)^d}(\varepsilon x)=1$, we have

$$\|T_\phi f\|_{L^{p,q}(\mathbb{R}^d)}\le\liminf_{\varepsilon\to0}\|\widetilde{T_{\phi_\varepsilon}}(\varepsilon^dF_\varepsilon)\|_{L^{p,q}(\mathbb{T}^d)}\varepsilon^{-d/p}\le C\lim_{\varepsilon\to0}\|\varepsilon^dF_\varepsilon\|_{L^p(\mathbb{T}^d)}\varepsilon^{-d/p}.$$

On the other hand, by $\|\varepsilon^dF_\varepsilon\|^p_{L^p(\mathbb{T}^d)}=\varepsilon^{-d}\|f\|^p_{L^p(\mathbb{T}^d)}$, we have $\|\varepsilon^dF_\varepsilon\|_{L^p(\mathbb{T}^d)}=\varepsilon^{-d/p}\|f\|_{L^p(\mathbb{T}^d)}$. Therefore, we obtain

$$\|T_\phi f\|_{L^{p,q}(\mathbb{R}^d)}\le C\|f\|_{L^p(\mathbb{R}^d)},\quad (f\in\mathcal{S}(\mathbb{R}^d)).$$

So, we complete the proof. □

Next, we shall give the proof of Proposition 1.2 in the same manner of the proof of Proposition 1.1. Rodriguez [15] proved Proposition 1.2, but we give an altenative proof of [15, Theorem 2.1] without using Kolmogorov's condition. Proposition 1.2 will easily bring Theorem 1.2 whose proof is omitted.

Proof of Proposition 1.2. We may assume $m=2$. Let $\{\Phi_j\}_{j=1}^{\infty}$ be in $C_b(\mathbb{R}^{2d})$, and $T^*(f_1,f_2)$, $\widetilde{T^*}(F_1,F_2)$ the associated maximal operators with $\{\Phi_j\}$ and $\{\Phi_{j|\mathbb{Z}^{2d}}\}$. Also we denote $T_J^*(f_1,f_2)$, $\widetilde{T_J^*}(F_1,F_2)$ by

$$T^*(f_1,f_2)(x)=\sup_{1\le j\le J}|T_{\Phi_j}(f_1,f_2)(x)|(f_k\in\mathcal{S}(\mathbb{R}^d))$$

and

$$\widetilde{T^*}(F_1,F_2)(x)=\sup_{1\le j\le J}|\widetilde{T_{\Phi_{j|\mathbb{Z}^{2d}}}}(F_1,F_2)(x)|(F_k\in P(\mathbb{T}^d))$$

for any natural number $J\in\mathbb{N}$. Here, we define that

$$\Box_{j,\delta}(x)=W_\delta(x)\widetilde{T_{\Phi_{j|\mathbb{Z}^{2d}}}}(F_1,F_2)(x)-T_{\Phi_j}(W_{\delta_1}F_1,W_{\delta_2}F_2)(x),$$

for $\delta_1=\delta_2=\delta/2$ and $\delta>0$, and $\Box_{J,\delta}=\max_{1\le j\le J}\|\Box_{j,\delta}\|_{L^\infty(\mathbb{R}^d)}$ for $1\le j\le J$. Then we have the following:

Lemma 2.3.

$$\lim_{\delta\to 0}\Box_{J,\delta}=0.$$

Proof. Since $F_j(x)=\sum_k\widehat{F_j}(k)e^{ikx}$, we have that

$$W_\delta(x)\widetilde{T_{\Phi_{j|\mathbb{Z}^{2d}}}}(F_1,F_2)(x)=\sum_{m,n}\Phi_j(m,n)\widehat{F_1}(m)\widehat{F_2}(n)W_\delta(x)e^{i(m+n)x},$$

and

$$\begin{aligned}&T_{\Phi_j}(W_{\delta_1}F_1,W_{\delta_2}F_2)(x)\\ =&\iint\Phi_j(\xi,\eta)\widehat{W_{\delta_1}F_1}(\xi)\widehat{W_{\delta_2}F_2}(\eta)e^{i(\xi+\eta)x}d\xi d\eta\\ =&\sum_{m,n}\hat{F}_1(m)\hat{F}_2(n)\iint\Phi_j(\xi,\eta)\widehat{W_{\delta_1}}(\xi-m)\widehat{W_{\delta_2}}(\eta-n)e^{i(\xi+\eta)x}d\xi d\eta.\end{aligned}$$

Also since

$$\begin{aligned}W_\delta(x)e^{i(m+n)x}&=W_{\delta_1}(x)e^{imx}W_{\delta_2}(x)e^{inx}\\&=\int\widehat{W_{\delta_1}}(\xi-m)e^{i\xi x}d\xi\int\widehat{W_{\delta_2}}(\eta-n)e^{i\eta x}d\eta,\end{aligned}$$

we obtain the following:

$$\begin{aligned}\square_{j,\delta}(x)=&W_\delta(x)\widetilde{T_{\Phi_{j|\mathbb{Z}^{2d}}}}(F_1,F_2)(x)-T_{\Phi_j}(W_{\delta_1}F_1,W_{\delta_2}F_2)(x)\\ =&\sum_{m,n}\widehat{F_1}(m)\widehat{F_2}(n)\iint(\Phi_j(m,n)-\Phi_j(\xi,\eta))\widehat{W_{\delta_1}}(\xi-m)\widehat{W_{\delta_2}}(\eta-n)e^{i(\xi+\eta)x}d\xi d\eta.\end{aligned}$$

Here, for any $\varepsilon_0>0$ we define I and II such that

$$\mathrm{I}=\iint_{|\xi-m|<\varepsilon_0,|\eta-n|<\varepsilon_0}(\Phi_j(m,n)-\Phi_j(\xi,\eta))\widehat{W_{\delta_1}}(\xi-m)\widehat{W_{\delta_2}}(\eta-n)e^{i(\xi+\eta)x}d\xi d\eta,$$

and

$$\begin{aligned}\mathrm{II}=&\int_{\mathbb{R}^d}\int_{|\xi-m|>\varepsilon_0}(\Phi_j(m,n)-\Phi_j(\xi,\eta))\times\widehat{W_{\delta_1}}(\xi-m)\widehat{W_{\delta_2}}(\eta-n)e^{i(\xi+\eta)x}d\xi d\eta\\ &+\int_{\mathbb{R}^d}\int_{|\eta-n|>\varepsilon_0}(\Phi_j(m,n)-\Phi_j(\xi,\eta))\times\widehat{W_{\delta_1}}(\xi-m)\widehat{W_{\delta_2}}(\eta-n)e^{i(\xi+\eta)x}d\xi d\eta.\end{aligned}$$

Then we have

$$|\square_{j,\delta}(x)|\le\sum_{m,n}|\widehat{F_1}(m)||\widehat{F_2}(n)(\mathrm{I}+\mathrm{II})|.$$

By the continuity of $\Phi_j(\xi,\eta)$, for any $\varepsilon>0$ there exists $\varepsilon_0>0$ such that $|\Phi_j(m,n)-\Phi_j(\xi,\eta)|<\varepsilon$ for $|(\xi,\eta)-(m,n)|<\varepsilon_0$. Hence, we have

$$|\mathrm{I}|\le\varepsilon\iint\widehat{W_{\delta_1}}(\xi-m)\widehat{W_{\delta_2}}(\eta-n)d\xi d\eta=\varepsilon.$$

On the other hand, we have

$$\begin{aligned}|\mathrm{II}|\le&4\|\Phi_j\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d}\int_{|\xi-m|>\varepsilon_0}\widehat{W_{\delta_1}}(\xi-m)d\xi\widehat{W_{\delta_2}}(\eta)d\eta\\ =&4\|\Phi_j\|_{L^\infty(\mathbb{R}^d)}\int_{|\xi-m|>\varepsilon_0}\widehat{W_{\delta_1}}(\xi-m)d\xi\longrightarrow 0\quad\text{as }\ \delta\to 0.\end{aligned}$$

Therefore, we get $\limsup_{\delta\to0}\square_{J,\delta}\le\varepsilon$, and $\lim_{\delta\to0}\square_{J,\delta}=0$. $\square$

Now we proceed to the proof of Proposition 1.2 in the same way as Proposition 1.1. First we show the case $0<q<\infty$. Since

$$W_\delta(x)\widetilde{T_{\Phi_{j|\mathbb{Z}^{2d}}}}(F_1,F_2)(x)\le\square_{J,\delta}(x)+T_{\Phi_j}(W_{\delta_1}F_1,W_{\delta_2}F_2)(x)$$

for F_1, F_2 in $P(\mathbb{T}^d)$, we have

$$W_\delta(x)\widetilde{T_J^*}(F_1,F_2)(x)=\square_{J,\delta}+T_J^*(W_{\delta_1}F_1,W_{\delta_2}F_2)(x).$$

Also by

$$\{x\in\mathbb{R}^d\,|\,W_\delta(x)\widetilde{T}^*_J(F_1,F_2)(x)>t\}\subset\{x\in\mathbb{R}^d\,|\,T^*_J(W_{\delta_1}F_1,W_{\delta_2}F_2)(x)>t-\Box_{J,\delta}\}$$

for $t>0$, we obtain that $a>2\Box_{J,\delta}$,

$$\begin{aligned}&\Big(\int_a^\infty(tw(\{x\in\mathbb{R}^d|W_\delta(x)\widetilde{T^*_J}(F_1,F_2)(x)>t\})^{1/p})^q\frac{dt}{t}\Big)^{1/q}\\ \le& C\Big(\int_{a-\Box_{J,\delta}}^\infty(uw(\{x\in\mathbb{R}^d|T^*_J(W_{\delta_1}F_1,W_{\delta_2}F_2)(x)>u\})^{1/p})^q\frac{du}{u}\Big)^{1/q},\end{aligned}$$

by $\frac{u+\Box_{J,\delta}}{u}\le 2$. Hence, we have

$$\begin{aligned}&\Big(\int_a^\infty(tw(\{x\in\mathbb{R}^d|W_\delta(x)\tilde{T}^*_J(W_{\delta_1}F_1,W_{\delta_2}F_2)(x)>t\})^{1/p})^q\frac{dt}{t}\Big)^{1/q}\\ \le& C\|W_{\delta_1}F_1\|_{L^{p_1}(\mathbb{R}^d,w_1)}\|W_{\delta_2}F_2\|_{L^{p_2}(\mathbb{R}^d,w_2)}\end{aligned}$$

by the assumption of T^*. After all, we have

$$\begin{aligned}&\left(\int_a^\infty(tw(\{x\in\mathbb{R}^d|W_\delta(x)\widetilde{T^*}_J(F_1,F_2)(x)>t\})^{1/p})^q\frac{dt}{t}\right)^{1/q}\\ \le& C\|W_{\delta_1}F_1\|_{L^{p_1}(\mathbb{R}^d,w)}\|W_{\delta_2}F_2\|_{L^{p_2}(\mathbb{R}^d,w)}\end{aligned}$$

for $a>2\Box_{J,\delta}$. By (2), (3) of Lemma 2.1 and Fatou's lemma, we have

$$\begin{aligned}&\Big(\int_a^\infty(tw(\{x\in\mathbb{R}^d|\widetilde{T^*_J}(F_1,F_2)(x)>te\})^{1/p})^q\frac{dt}{t}\Big)^{1/q}\\ \le& C\|F_1\|_{L^{p_1}(\mathbb{T}^d,w_1)}\|F_2\|_{L^{p_2}(\mathbb{T}^d,w_2)}.\end{aligned}$$

Therefore, by $a\downarrow 0$ we obtain

$$\Big(\int_0^\infty(tw(\{x\in\mathbb{R}^d|\widetilde{T^*_J}(F_1,F_2)(x)>te\})^{1/p})^q\frac{dt}{t}\Big)^{1/q}\le C\|F_1\|_{L^{p_1}(\mathbb{R}^d,w_1)}\|F_2\|_{L^{p_2}(\mathbb{R}^d,w_2)}.$$

Moreover, by applying Fatou's lemma again, we get the desired result:

$$\|\widetilde{T^*}(F_1,F_2)\|_{L^{p,q}(\mathbb{T}^d,w)}\le C\|F_1\|_{L^{p_1}(\mathbb{T}^d,w_1)}\|F_2\|_{L^{p_2}(\mathbb{T}^d,w_2)},\quad (F_1,F_2\in P(\mathbb{T}^d)),$$

for $1\le p<\infty$ and $0<q<\infty$.

In the same way, we can show it in the case of $q=\infty$, and we omit the details. □

Corollary 2.2. (cf. [9]) For $m\in\mathbb{N}$, let $0<p<\infty, 1\le p_j<\infty$ with $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$, $0<q\le\infty$, and ϕ in $C_b(\mathbb{R}^d)$. Then we have

$$\|T_\phi(f_1,\cdots,f_m)\|_{L^{p,q}(\mathbb{R}^d)}\le C\Pi_{j=1}^m\|f_j\|_{L^{p_j}(\mathbb{R}^d)},\quad (f_j\in\mathcal{S}(\mathbb{R}^d)),$$

if and only if there exists a constant $C>0$ independent of $\varepsilon>0$ such that

$$\|\widetilde{T_{\phi_\varepsilon|\mathbb{Z}^{md}}}(F_1,\cdots,F_m)\|_{L^{p,q}(\mathbb{T}^d)}\le C\Pi_{j=1}^m\|F_j\|_{L^{p_j}(\mathbb{T}^d)},\quad (F_j\in P(\mathbb{T}^d)),$$

where $\phi_\varepsilon(x_1,\cdots,x_m)=\phi(\varepsilon x_1,\cdots,\varepsilon x_m)$.

Proof. We prove this result in the same way as the proof of Corollary 2.1. It is sufficient to prove it in the case of $m=2$. We assume $\|T_\phi(f_1,f_2)\|_{L^{p,q}(\mathbb{R}^d)}\le C\Pi_{j=1}^2\|f_j\|_{L^{p_j}(\mathbb{R}^d)}$, $(f_j\in\mathcal{S}(\mathbb{R}^d))$. Since

$$T_{\phi_\varepsilon}(f_1,f_2)(x)=\int_{\mathbb{R}^{2d}}\phi(\varepsilon\xi_1,\varepsilon\xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)e^{ix(\xi+\xi_2)}d\xi_1 d\xi_2,$$

we have that

$$\begin{aligned}T_{\phi_\varepsilon}(f_1,f_2)(x)&=\int_{\mathbb{R}^{2d}}\phi(\xi_1,\xi_2)\widehat{(f_1)_\varepsilon}(\varepsilon\xi_1)\widehat{(f_2)_\varepsilon}(\xi_2)e^{i\varepsilon^{-1}x(\xi_1+\xi_2)}d\xi_1 d\xi_2\\&=(T_\phi((f_1)_\varepsilon,(f_2)_\varepsilon)(\varepsilon^{-1}x_1,\varepsilon^{-1}x_2),\end{aligned}$$

where $(f_j)_\varepsilon(x)=f_j(\varepsilon x)$, $(j=1,2)$. Hence, by the assumption we get

$$\begin{aligned}\|T_{\phi_\varepsilon}(f_1,f_2)\|_{L^{p,q}(\mathbb{R}^d)}&=\|T_\phi((f_1)_\varepsilon,(f_2)_\varepsilon)\|_{L^{p,q}(\mathbb{R}^d)}\varepsilon^{-d/p}\\&\le C\Pi_{j=1}^2\|(f_j)_\varepsilon\|_{L^{p_j}(\mathbb{R}^d)}\varepsilon^{-d/p}=C\Pi_{j=1}^2\|(f_j)_\varepsilon\|_{L^{p_j}(\mathbb{R}^d)},\quad (f_j\in\mathcal{S}(\mathbb{R}^d)).\end{aligned}$$

Therefore, we obtain the desired result:

$$\|\widetilde{T_{\phi_\varepsilon|\mathbb{Z}^{2d}}}(F_1,F_2)\|_{L^{p,q}(\mathbb{T}^d)}\le C\Pi_{j=1}^2\|F_j\|_{L^{p_j}(\mathbb{T}^d)},\quad (F_j\in P(\mathbb{T}^d)),$$

for all $\varepsilon>0$ by Theorem 1.2. Conversely, we assume that

$$\|\widetilde{T_{\phi_\varepsilon|\mathbb{Z}^{2d}}}(F_1,F_2)\|_{L^{p,q}(\mathbb{T}^d)}\le C\Pi_{j=1}^2\|F_j\|_{L^{p}(\mathbb{T}^d)},\quad (F_j\in P(\mathbb{T}^d)),$$

for all $\varepsilon>0$. Let f_j be in $\mathcal{S}(\mathbb{R}^d)$, and $(F_j)_\varepsilon(x)=\varepsilon^{-d}\sum_{k\in\mathbb{Z}^d}f_j(\varepsilon^{-1}(x+2k\pi))$. Then we have $(F_j)_\varepsilon\in L^1(\mathbb{T}^d)$. Also by $\widehat{(F_j)}_\varepsilon(k)=\widehat{f_j}(\varepsilon k)$, we get $(F_j)_\varepsilon(x)=\sum_k\widehat{f_j}(\varepsilon k)e^{ikx}$. Here, by the definition of Riemann integral, we remark that for all $\varepsilon>0$

$$\begin{aligned}T_\phi(f_1,f_2)(x)&=\int_{\mathbb{R}^{2d}}\phi(\xi_1,\xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)e^{ix(\xi_1+\xi_2)}d\xi_1 d\xi_2\\&=\lim_{\varepsilon\to 0}\sum_{j=(j_1,j_2)\in\mathbb{Z}^{2d}}\phi(\varepsilon j_1,\varepsilon j_2)\hat{f}_1(\varepsilon\xi_1)\hat{f}_2(\varepsilon\xi_2)e^{ix(\varepsilon j_1+\varepsilon j_2)}\varepsilon^{2d}\\&=\lim_{\varepsilon\to 0}\widetilde{T_{\phi_\varepsilon|\mathbb{Z}^{2d}}}(\varepsilon^d(F_1)_\varepsilon,\varepsilon^d(F_2)_\varepsilon)(\varepsilon x).\end{aligned}$$

Moreover, since we have

$$T_\phi(f_1,f_2)(x)=\lim_{\varepsilon\to 0}\widetilde{T_{\phi_\varepsilon|\mathbb{Z}^{2d}}}(\varepsilon^d(F_1)_\varepsilon,\varepsilon^d(F_2)_\varepsilon)(\varepsilon x)\chi_{[-\pi,\pi)^d}(\varepsilon x)$$

by $\lim_{\varepsilon\to 0}\chi_{[-\pi,\pi)^d}(\varepsilon x)=1$, we have

$$\begin{aligned}\|T_\phi(f_1,f_2)\|_{L^{p,q}(\mathbb{R}^d)} &\le \liminf_{\varepsilon\to 0}\|\widetilde{T_{\phi_\varepsilon|\mathbb{Z}^d}}(\varepsilon^d(F_1)_\varepsilon,\varepsilon^d(F_2)_\varepsilon)\|_{L^{p,q}(\mathbb{T}^d)}\varepsilon^{-d/p}\\ &\le C\lim_{\varepsilon\to 0}\Pi_{j=1}^2\|\varepsilon^d(F_j)_\varepsilon\|_{L^{p_j}(\mathbb{T}^d)}\varepsilon^{-d/p}.\end{aligned}$$

On the other hand, by $\|\varepsilon^d(F_j)_\varepsilon\|^{p_j}_{L^{p_j}(\mathbb{T}^d)}=\varepsilon^{-d}\|f_j\|^{p_j}_{L^{p_j}(\mathbb{T}^d)}$, we have

$$\|\varepsilon^d(F_j)_\varepsilon\|_{L^{p_j}(\mathbb{T}^d)}=\varepsilon^{-d/p_j}\|f_j\|_{L^{p_j}(\mathbb{T}^d)}.$$

Therefore, we obtain

$$\|T_\phi(f_1,f_2)\|_{L^{p,q}(\mathbb{R}^d)}\le C\Pi_{j=1}^2\|f_j\|_{L^{p_j}(\mathbb{R}^d)},\quad (f_j\in\mathcal{S}(\mathbb{R}^d)).$$

□

3 Remarks of Theorems 1.1 and 1.2

In this section, we generalize Theorems 1.1 and 1.2 in Section 1, whose results are known in Carro-Rodriguez [4] and Rodriguez [15], but our proofs are easily rather than their proofs.

Definition 3.1. A bounded function ϕ defined on $\mathbb{R}^d$ is regulated if there exists a non-negative function φ on $\mathbb{R}^d$ in $\mathcal{S}(\mathbb{R}^d)$ with $\|\varphi\|_{L^1(\mathbb{R}^d)}=1$ such that $\lim_{n\to\infty}\varphi_n*\phi(m)=\phi(m)$ for any $m\in\mathbb{Z}^d$, where $\varphi_n(x)=n^d\varphi(nx)$, $(n=1,2,\cdots)$.

Then, we can show the following whose proof is given in similar to that of [8, Theorem 2.1]. We omit the proof.

Lemma 3.1 (cf. [12])**.** *Let $1\le p<\infty$. Suppose $k\in L^1(\mathbb{R}^d)$ and $\phi_j\in L^\infty(\mathbb{R}^d)$, $(j=1,2,\cdots)$. Also let w be a 2π periodic weighted function. Then there exists a constant $C>0$ such that*

$$\Big\|\sup_j|T_{k*\phi_j}f|\Big\|_{L^{p,\infty}(\mathbb{R}^d,w)}\le C\|k\|_{L^1(\mathbb{R}^d)}\|f\|_{L^p(\mathbb{R}^d,w)},$$

where C is independent of k.

The following result is a generalization of Theorem 1.1, whose proof is different from that of Carro-Rodriguez [4].

Theorem 3.1. *Let $\{\phi_j\}$ be regulated functions on $\mathbb{R}^d$, and w a 2π periodic weighted function. Also let $T^*f(x)=\sup_j|T_{\phi_j}f(x)|$. Then if there exists a constant $C>0$ such that*

$$\|T^*f\|_{L^{p,\infty}(\mathbb{R}^d,w)}\le C\|f\|_{L^p(\mathbb{R}^d,w)},\quad (f\in\mathcal{S}(\mathbb{R}^d)),$$

we have

$$\|\widetilde{T^*}F\|_{L^{p,\infty}(\mathbb{R}^d,w)}\le C\|F\|_{L^p(\mathbb{R}^d,w)}$$

for any trigonometric polynomial F, where $\widetilde{T^*}F(x)=\sup_j|\widetilde{T_{\phi_{j|\mathbb{Z}^d}}}F(x)|$.

Proof. Let $k_{n,j}=\varphi_n*\phi_j$. Then by Lemma 3.1, we have that for any $J\in\mathbb{N}$

$$\Big\|\sup_{1\le j\le J}|T_{k_{n,j}}f(x)|\Big\|_{L^{p,\infty}(\mathbb{R}^d,w)}\le C\|\varphi_n\|_{L^1(\mathbb{R}^d)}\Big\|\sup_{1\le j\le J}|T_{\phi_j}f(x)|\Big\|_{L^{p,\infty}(\mathbb{R}^d,w)}=C\|f\|_{L^p(\mathbb{R}^d,w)},$$

where $C>0$ is a constant independent of φ, J and f. Here, we remark $k_{n,j}\in C_b(\mathbb{R}^d)$. By applying Theorem 1.1, we obtain that for any $J\in\mathbb{N}$ and a trigonometric polynomial F,

$$\Big\|\sup_{1\le j\le J}\widetilde{T_{k_{n,j}|\mathbb{Z}^d}}F\Big\|_{L^{p,\infty}(\mathbb{T}^d,w)}\le C\|F\|_{L^p(\mathbb{T}^d,w)},$$

where $C>0$ is a constant independent of J and F. On the other hand, by the assumption of ϕ_j we have

$$\lim_{n\to\infty}\widetilde{T_{k_{n,j}}}F(x)=\lim_{n\to\infty}\sum_m\varphi_n*\phi_j(m)\widehat{F}(m)e^{imx}=\sum_m\phi_j(m)\widehat{F}(m)e^{imx}.$$

Then, by Fatou's lemma,

$$\Big\|\sup_{1\le j\le J}|\widetilde{T_{\phi_{j|\mathbb{Z}^d}}}F(x)|\Big\|_{L^{p,\infty}(\mathbb{T}^d,w)}\le\liminf_{n\to\infty}\Big\|\sup_{1\le j\le J}|\widetilde{T_{k_{n,j}|\mathbb{Z}^d}}F(x)|\Big\|_{L^{p,\infty}(\mathbb{T}^d,w)}\le C\|F\|_{L^p(\mathbb{T}^d,w)}$$

for any trigonometric polynomial F, and where $C>0$ is a constant independent of J and F. By applying Fatou's lemma again, we obtain that

$$\|\widetilde{T^*}F\|_{L^{p,\infty}(\mathbb{T}^d,w)}\le\liminf_{J\to\infty}\Big\|\sup_{1\le j\le J}|\widetilde{T_{\psi_{j|\mathbb{Z}^d}}}F(x)|\Big\|_{L^{p,\infty}(\mathbb{T}^d,w)}\le C\|F\|_{L^p(\mathbb{T}^d,w)},$$

where $C>0$ is a constant independent of F. Therefore, we get the desired result. □

Next we give a remark about the case of bilinear Fourier multipliers in the same way of Theorem 3.1. The result is given by Rodriguez-Lopez [15], but our proof is slightly easy than that in [15].

Definition 3.2. A bounded function $\phi(\xi,\eta)$ defined on $\mathbb{R}^{2d}$ is regulated if there exists a non-negative function φ on $\mathbb{R}^{2d}$ in $\mathcal{S}(\mathbb{R}^{2d})$ with $\|\varphi\|_{L^1(\mathbb{R}^{2d})}=1$ such that $\lim_{n\to\infty}\varphi_n*\phi(m)=\phi(m)$ for any $m\in\mathbb{Z}^{2d}$, where $\varphi_n(x)=n^{2d}\varphi(nx)$, $(n=1,2,\cdots)$.

Then, the following is proved by Rodriguez [15, Proposition 3.6].

Lemma 3.2 (see [15]). *Let $\varphi\in L^1(\mathbb{R}^d)$ with $\{\phi_j\}\subset L^\infty(\mathbb{R}^{2d})$, and p, p_j, w, w_j, $(j=1,2)$ in Section 1. Also let $(\varphi\otimes\varphi)(\xi,\eta)=\varphi(\xi)\varphi(\eta)$. Then $\{(\varphi\otimes\varphi)*\phi_j\}_j$ satisfies*

$$\|T_{(\varphi\otimes\varphi)*\phi_j}(f_1,f_2)\|_{L^{p,\infty}(\mathbb{R}^d,w)}\le C\|\varphi\|^2_{L^1(\mathbb{R}^d)}\Pi_{j=1}^2\|f_j\|_{L^{p_j}(\mathbb{R}^d,w_j)},$$

where $C>0$ is a constant independent of φ and f_j.

The following result is a generalization of Theorem 1.2, whose proof is different from that in Rodriguez [4], and is slightly simple, because we use our Theorem 1.2 for the proof without applying [11, Theorem3.1].

Theorem 3.2. *Let* $0<p<\infty$, $(j=1,2)$ *with* $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, *and* $w(x)$, $w_1(x)$, $w_2(x)$ *non-negative* 2π *periodic weighted functions in* $L^1_{loc}(\mathbb{R}^d)$. *Also let* $\{\Phi_j\}\subset L^\infty(\mathbb{R}^{2d})$ *be in regulated functions on* $\mathbb{R}^{2d}$. *Then if there exists a constant* $C>0$ *such that*

$$\|T^*(f_1,f_2)\|_{L^{p,\infty}(\mathbb{R}^d,w)}\le C\Pi_{j=1}^2\|f_j\|_{L^{p_j}(\mathbb{R}^d,w_j)},\quad (f_j\in\mathcal{S}(\mathbb{R}^d)),$$

we have that

$$\|\widetilde{T^*}(F_1,F_2)\|_{L^{p,\infty}(\mathbb{T}^d,w)}\le C\Pi_{j=1}^2\|F_j\|_{L^{p_j}(\mathbb{T}^d,w_j)}$$

for any trigonometric polynomial F_1, F_2, *where* $\widetilde{T^*}(F_1,F_2)(x)=\sup_j|\widetilde{T_{\Phi_{j|\mathbb{Z}^d}}}(F_1,F_2)(x)|$.

Proof. By Lemma 3.2, we have that for any $J\in\mathbb{N}$,

$$\begin{aligned}\Big\|\sup_{1\le j\le J}|T_{(\varphi_n\otimes\varphi_n)*\Phi_j}(f_1,f_2)(x)|\Big\|_{L^{p,\infty}(\mathbb{R}^d,w)}&\le C\|\varphi_n\|^2_{L^1(\mathbb{R}^d)}\Big\|\sup_{1\le j\le J}|T_{\Phi_j}(f_1,f_2)(x)|\Big\|_{L^{p,\infty}(\mathbb{R}^d,w)}\\&\le C\Pi_{j=1}^2\|f_j\|_{L^{p_j}(\mathbb{R}^d,w_j)},\end{aligned}$$

where $C>0$ is a constant independent of φ, J and f_j. Here, we remark $(\varphi_n\otimes\varphi_n)*\Phi_j\in C_b(\mathbb{R}^d)$. By applying Theorem 1.2, we obtain that for any $J\in\mathbb{N}$ and a trigonometric polynomial F_1, F_2,

$$\Big\|\sup_{1\le j\le J}\widetilde{T_{(\varphi_n\otimes\varphi_n)*\Phi_j|\mathbb{Z}^{2d}}}(F_1,F_2)\Big\|_{L^{p,\infty}(\mathbb{T}^d,w)}\le C\Pi_{j=1}^2\|F_j\|_{L^{p_j}(\mathbb{T}^d,w_j)},$$

where $C>0$ is a constant independent of φ, J and F_j. On the other hand, by the assumption of Φ_j we have

$$\lim_{n\to\infty}\widetilde{T_{(\varphi_n\otimes\varphi_n)*\Phi_j}}(F_1,F_2)(x)=\sum_{(m_1,m_2)\in\mathbb{Z}^{2d}}\Phi_j(m_1,m_2)\Pi_{j=1}^2\widehat{F}_j(m_j)e^{i(m_1+m_2)x}.$$

Then, by Fatou's lemma,

$$\begin{aligned}\Big\|\sup_{1\le j\le J}|\widetilde{T_{\Phi_{j|\mathbb{Z}^{2d}}}}(F_1,F_2)(x)|\Big\|_{L^{p,\infty}(\mathbb{T}^d,w)}&\le\liminf_{n\to\infty}\Big\|\sup_{1\le j\le J}|\widetilde{T_{(\varphi_n\otimes\varphi_n)*\Phi_j|\mathbb{Z}^d}}(F_1,F_2)(x)|\Big\|_{L^{p,\infty}(\mathbb{T}^d,w)}\\&\le C\Pi_{j=1}^2\|F_j\|_{L^{p_j}(\mathbb{T}^d,w_j)}\end{aligned}$$

for any trigonometric polynomial F_1, F_2, where $C>0$ is a constant independent of φ, J and F_1, F_2. By applying Fatou's lemma again, we obtain that

$$\begin{aligned}\|\widetilde{T^*}(F_1,F_2)\|_{L^{p,\infty}(\mathbb{T}^d,w)}&\le\liminf_{J\to\infty}\Big\|\sup_{1\le j\le J}|\widetilde{T_{\Phi_{j|\mathbb{Z}^{2d}}}}(F1,F_2)(x)|\Big\|_{L^{p,\infty}(\mathbb{T}^d,w)}\\&\le C\Pi_{j=1}^2\|F_j\|_{L^{p_j}(\mathbb{T}^d,w_j)},\end{aligned}$$

where $C>0$ is a constant independent of F_1, F_2. Therefore, we get the desired result. □

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