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Optimal Recovery of Functions on the Sphere on a Sobolev Spaces with a Gaussian Measure in the Average Case Setting

Zexia Huang[1] and Heping Wang[2,*]

[1] *School of Science, Xihua University, Chengdu 610039, China*
[2] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*

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Abstract. In this paper, we study optimal recovery (reconstruction) of functions on the sphere in the average case setting. We obtain the asymptotic orders of average sampling numbers of a Sobolev space on the sphere with a Gaussian measure in the $L_q(\mathbb{S}^{d-1})$ metric for $1\le q\le\infty$, and show that some worst-case asymptotically optimal algorithms are also asymptotically optimal in the average case setting in the $L_q(\mathbb{S}^{d-1})$ metric for $1\le q\le\infty$.

Key Words: Optimal recovery on the sphere, average sampling numbers, optimal algorithm, Gaussian measure.

AMS Subject Classifications: 41A25, 41A35

1 Introduction

This paper is devoted to studying the optimal recovery (reconstruction) of functions on the sphere on a Sobolev space with a Gaussian measure in the average case setting. Let F be a Banach space of functions defined on D, G be a normed linear spaces with norm $\|\cdot\|_G$, and let γ be a centered Gaussian probability measure on F. We want to reconstruct functions f from F using finitely many arbitrary function values $f(x)$ for some $x\in D$. It is well known that, in the average case setting with the average being respect to a centered Gaussian measure, adaptive choice of the above function values as well as nonlinear algorithms do not essentially help, see [20, 24]. Hence, we can restrict our attention to linear algorithms, i.e., algorithms of the form

$$A_N(f):=\sum_{j=1}^{N} f(x_j)h_j, \quad h_j\in G, \quad x_j\in D. \tag{1.1}$$

*Corresponding author. *Email addresses:* `xhu_hzx@163.com` (Z. X. Huang), `wanghp@cnu.edu.cn` (H. P. Wang)

For $0<p<\infty$, the p-average error of an algorithm A_N in G with respect to the measure γ is defined by

$$e^{\rm avg}(A_N,\gamma,G)_p:=\Big(\int_F \|f-A_N(f)\|_G^p\gamma(df)\Big)^{\frac{1}{p}}.$$

We define the p-average sampling numbers of F in G by

$$g_N^{(a)}(F,\gamma,G)_p:=\inf_{x_j\in D,\ h_j\in G,\ j=1,\cdots,N} e^{\rm avg}(A_N,\gamma,G)_p.$$

We stress that for a centered Gaussian measure, the averaging parameter p is irrelevant up to a constant (see [11, Theorem 1.2] or [27, Corollary 1]).

There are a few papers devoted to studying average case approximation, see for example, [4, 5, 9, 10, 12–17, 20, 22–29]. However, much less attention has been devoted to average sampling numbers; for exceptions see, e.g., [13, 14, 23]. In [23] and [14], among others, the authors obtained upper bounds for average sampling numbers on the Wiener space in the uniform norm and on the weighted Korobov spaces in the L_2 metric, respectively. In [13] the authors investigate average sampling numbers of a periodic Sobolev space with a Gaussian measure in the L_q metric for $1\le q\le\infty$, and obtain their asymptotic orders. More information about average case setting results can be found in [20] and [24].

In the paper, we shall investigate average sampling numbers in the L_q metric for $1\le q\le\infty$ on a Sobolev space on the sphere with a Gaussian measure, and obtain the asymptotic orders. We show that some worst-case asymptotically optimal algorithms are also asymptotically optimal in the average case setting.

2 Main results

Let $\mathbb{S}^{d-1}=\{x\in\mathbb{R}^d:|x|=1\}$ be the unit sphere of $\mathbb{R}^d$ endowed with the usual rotation variant measure $d\sigma(x)$, and let $d(x,y)=\arccos(x\cdot y)$ be the geodesic distance between two points $x,y\in\mathbb{S}^{d-1}$, where $x\cdot y$ is the usual inner product and $|x|=(x\cdot x)^{1/2}$ is the Euclidean norm. For $1\le q\le\infty$, denote by $L_q\equiv L_q(\mathbb{S}^{d-1})$ the collection of real measurable functions f on $\mathbb{S}^{d-1}$ with finite norm

$$\|f\|_q=\Big(\int_{\mathbb{S}^{d-1}}|f(x)|^q d\sigma(x)\Big)^{\frac{1}{q}},\quad 1\le q<\infty,$$

and for $q=\infty$, the essential supremum is understood instead of the integral. We denote by Π_n^d the space of all spherical polynomials of degree at most n, and by $\mathcal{H}_l^d$ the space of all spherical harmonics of degree l on $\mathbb{S}^{d-1}$. It is well known that the spaces $\mathcal{H}_l^d$, $l=0,1,\cdots$, are just the eigenspaces corresponding to the eigenvalues $-l(l+d-2)$ of the Laplace-Beltrami operator Δ on the sphere and are mutually orthogonal with respect to the inner product

$$\langle f,g\rangle:=\int_{\mathbb{S}^{d-1}}f(x)g(x)d\sigma(x),$$

and

$$\Pi_n^d=\sum_{l=0}^{n}\bigoplus\mathcal{H}_l^d,\quad L_2=\sum_{l=0}^{\infty}\bigoplus\mathcal{H}_l^d.$$

So, for each $f\in L_2$, the series $\sum_{l=0}^{\infty}H_l(f)$, which is called the Fourier-Laplace series of f, converges to f in L_2-metric, where $H_l(f)$ denotes the orthogonal projection of f onto $\mathcal{H}_l^d$. We denote by $N(d,l)$ the dimension of $\mathcal{H}_l^d$, and by

$$\{Y_{lk}:k=1,\cdots,N(d,l)\}$$

a fixed L_2-orthonormal system in $\mathcal{H}_l^d$. Then

$$N(d,l):=\dim\mathcal{H}_l^d=\begin{cases}1, & \text{if } l=0,\\ \dfrac{(2l+d-2)\Gamma(l+d-2)}{\Gamma(d-1)\Gamma(l+1)}, & \text{if } l=1,2,\cdots,\end{cases}\asymp(l+1)^{d-2},$$

and for each $f\in L_2$,

$$H_l(f,x)=\int_{\mathbb{S}^{d-1}}f(y)Z_l(x,y)d\sigma(y),\quad x\in\mathbb{S}^{d-1},\tag{2.1}$$

where $Z_l(\cdot,\cdot)$ is the reproducing kernel of $\mathcal{H}_l^d$ defined by

$$Z_l(x,y)=\sum_{k=1}^{N(d,l)}Y_{lk}(x)Y_{lk}(y)=\frac{N(d,l)}{|\mathbb{S}^{d-1}|}\frac{P_l^{(\frac{d-3}{2},\frac{d-3}{2})}(x\cdot y)}{P_l^{(\frac{d-3}{2},\frac{d-3}{2})}(1)},\quad x,y\in\mathbb{S}^{d-1},\tag{2.2}$$

$|\mathbb{S}^{d-1}|$ is the surface area of $\mathbb{S}^{d-1}$, and $P_k^{(\frac{d-3}{2},\frac{d-3}{2})}$ is the Jacobi polynomial of degree k as defined in [21, pp. 58].

Thus, for any $f\in L_2$, f can be expressed by its Fourier-Laplace series:

$$f=\sum_{l=0}^{\infty}H_l(f)=\sum_{l=0}^{\infty}\sum_{k=1}^{N(d,l)}\hat{f}_{lk}Y_{lk},$$

where

$$\hat{f}_{lk}=\langle f,Y_{lk}\rangle=\int_{\mathbb{S}^{d-1}}f(x)Y_{lk}(x)d\sigma(x)$$

are the Fourier coefficients of f. More information about the harmonic analysis on the sphere may be found in the books [7] and [30].

Given $r\in\mathbb{R}$, we define the r-th-order Laplace-Beltrami operator $(-\Delta)^r$ on $\mathbb{S}^{d-1}$ in a distributive sense by

$$H_0((-\Delta)^r(f))=0,\quad H_l((-\Delta)^r(f))=(l(l+d-2))^rH_l(f),\quad l=1,2,\cdots,$$

where f is a distribution on $\mathbb{S}^{d-1}$. It then follows that for $f\in L_2$, $r\in\mathbb{R}$, the r-th-order derivative of the distribution f is

$$f^{(r)}:=(-\Delta)^{\frac{r}{2}}f=\sum_{l=1}^{\infty}(l(l+d-2))^{\frac{r}{2}}H_l(f). \tag{2.3}$$

For $r>0$, the Sobolev space $W_2^r\equiv W_2^r(\mathbb{S}^{d-1})$ is defined by

$$\begin{aligned}W_2^r:=\Big\{f(x)=\sum_{l=1}^{\infty}H_l(f)=\sum_{l=1}^{\infty}\sum_{k=1}^{N(d,l)}\hat{f}_{lk}Y_{lk}:\|f\|_{W_2^r}^2:=\langle f^{(r)},f^{(r)}\rangle\\ =\sum_{l=1}^{\infty}(l(l+d-2))^r\|H_l(f)\|_2^2=\sum_{l=1}^{\infty}(l(l+d-2))^r\sum_{k=1}^{N(d,l)}|\hat{f}_{lk}|^2<\infty\Big\}\end{aligned}$$

with inner product

$$\langle f,g\rangle_r:=\langle f^{(r)},g^{(r)}\rangle. \tag{2.4}$$

Obviously, $(W_2^r,\langle\cdot,\cdot\rangle_r)$ is a Hilbert space. We equip W_2^r, $r>(d-1)/2$ with a Gaussian measure μ whose mean is zero and whose correlation operator $\mathcal{C}_\mu$ has eigenfunctions Y_{lk} and eigenvalues

$$\lambda_l=(l(l+d-2))^{-\frac{s}{2}},\quad s>d-1,\quad l=1,2,\cdots, \tag{2.5}$$

that is,

$$\mathcal{C}_\mu Y_{lk}=\lambda_l Y_{lk},\quad l=1,2,\cdots,\quad k=1,2,\cdots,N(d,l). \tag{2.6}$$

From the properties of Gaussian measure (see [1, pp. 48-49]), we know that

$$\langle\mathcal{C}_\mu f,g\rangle_r=\int_{W_2^r}\langle f,v\rangle_r\langle g,v\rangle_r\mu(dv). \tag{2.7}$$

This paper is devoted to studying the average sampling numbers of W_2^r with respect to the measure μ in the L_q metric. Our main results are as follows.

Theorem 2.1. *Let $1\le q\le\infty$ and $0<p<\infty$. Then*

$$g_N^{(a)}(W_2^r,\mu,L_q)_p\asymp\begin{cases}N^{-(r+\frac{s}{2})/(d-1)+\frac{1}{2}}, & 1\le q<\infty,\\ N^{-(r+\frac{s}{2})/(d-1)+\frac{1}{2}}\sqrt{\ln N}, & q=\infty,\end{cases} \tag{2.8}$$

where $A(N)\asymp B(N)$ means that $A(N)\ll B(N)$ and $B(N)\ll A(N)$, and $A(N)\ll B(N)$ means that there exists a constant $c>0$ independent of N such that $A(N)\le cB(N)$.

Remark 2.1. From the proof of Theorem 2.1 we know that the worst-case asymptotically optimal algorithms I_n constructed in Section 3 are also asymptotically optimal in the average case setting.

Remark 2.2. For $0<p<\infty$, the p-average linear N-width of (W_2^r,μ) in L_q is defined by

$$\lambda_N^{(a)}(W_2^r,\mu,L_q)_p=\inf_{L_N}\Big(\int_{W_2^r}\|f-L_N(f)\|_q^p\mu(df)\Big)^{1/p},$$

where L_N runs over all bounded linear operators from W_2^r to L_q with rank at most N. Obviously, algorithms A_N defined by (1.1) are the bounded linear operators from W_2^r to L_q with rank at most N. This means that

$$g_N^{(a)}(W_2^r,\mu,L_q)_p\geq\lambda_N^{(a)}(W_2^r,\mu,L_q)_p. \tag{2.9}$$

It follows from [25] that for $0<p<\infty$,

$$\lambda_N^{(a)}(W_2^r,\mu,L_q)_p\asymp\begin{cases} N^{-(r+\frac{s}{2})/(d-1)+\frac{1}{2}}, & 1\leq q<\infty,\\ N^{-(r+\frac{s}{2})/(d-1)+\frac{1}{2}}\sqrt{\ln N}, & q=\infty.\end{cases} \tag{2.10}$$

Comparing (2.8) with (2.10), we know that

$$g_N^{(a)}(W_2^r,\mu,L_q)_p\asymp\lambda_N^{(a)}(W_2^r,\mu,L_q)_p.$$

So, in order to prove Theorem 2.1 it suffices to prove the upper estimates of $g_N^{(a)}(W_2^r,\mu,L_q)_p$.

3 Proof of Theorem 2.1

Let η be a C^∞-function on $[0,\infty)$ with the properties that $\eta(x)=1$ for $0\leq x\leq 1$ and $\eta(x)=0$ for $x>2$. We define for an integer $n\geq 1$,

$$K_{n,\eta}(x,y)=\sum_{k=0}^{2n}\eta\Big(\frac{k}{n}\Big)Z_k(x,y),\quad x,y\in\mathbb{S}^{d-1}, \tag{3.1}$$

where $Z_k(x,y)$ is given in (2.2). Correspondingly, we define the operators $V_{n,\eta}$, $n=1,2,\dots$, by

$$V_{n,\eta}(f,x)=\int_{\mathbb{S}^{d-1}}f(y)K_{n,\eta}(x,y)d\sigma(y)=\sum_{k=0}^{2n}\eta(\frac{k}{n})H_k(f)(x),\quad x\in\mathbb{S}^{d-1},$$

where $f\in L_p$, $1\leq p\leq\infty$. Then, for any $f\in\Pi_n^d$,

$$f(x)=V_{n,\eta}(f,x)=\int_{\mathbb{S}^{d-1}}f(y)K_{n,\eta}(x,y)d\sigma(y). \tag{3.2}$$

For $f\in L_p$, $1\leq p<\infty$ or $f\in C(\mathbb{S}^{d-1})$ if $p=\infty$, we define

$$A_0(f)=V_{1,\eta}(f),\quad A_j(f)=V_{2^j,\eta}(f)-V_{2^{j-1},\eta}(f)\quad\text{for } j\geq 1.$$

Then $A_j(f)\in\Pi^d_{2^{j+1}}$, and $\sum_{j=0}^{\infty}A_j(f)$ converges to f in the L_p metric due to the fact that $V_{n,\eta}(f)$ converges to f as $n\to\infty$. Also for $j\ge 1$, we have

$$A_j(f)=\sum_{l=2^{j-1}+1}^{2^j}\Big(1-\eta\Big(\frac{l}{2^{j-1}}\Big)\Big)H_l(f)+\sum_{l=2^j+1}^{2^{j+1}}\eta\Big(\frac{l}{2^j}\Big)H_l(f). \tag{3.3}$$

Now we construct asymptotically optimal sampling operators. Given N, we need a positive quadrature rule Q_n on $\mathbb{S}^{d-1}$ defined by

$$Q_n(f)=\sum_{\omega\in\Lambda_n}\lambda_\omega f(\omega),\quad \min_{\omega\in\Lambda_n}\lambda_\omega>0,$$

and satisfying

$$\int_{\mathbb{S}^{d-1}}f(x)d\sigma(x)=Q_n(f)\quad\text{for all }\ f\in\Pi^d_{3n}, \tag{3.4}$$

and $\#\Lambda_n\le N\asymp n^{d-1}$, where $\#\Lambda_n$ is the cardinality of $\Lambda_n\subset\mathbb{S}^{d-1}$. The existence of such positive quadrature rule is assured theoretically for uniformly distributed points (see [2, 18, 19]). An explicit expression is given in [8] for $d=3$ and [3] for $d=4$. For example, we have

$$\int_{\mathbb{S}^2}f(x)d\sigma(x)=\frac{2\pi}{(2n+1)(n+2)}\sum_{j=1}^{2n+1}\sum_{k=1}^{4n+1}v_jf\Big(\frac{j\pi}{2n+2},\frac{k\pi}{2n+1}\Big),\quad f\in\Pi^3_{4n},$$

where $f(x)=f(\theta,\varphi)$, $x_1=\cos\theta$, $x_2=\sin\theta\cos\varphi$, $x_3=\sin\theta\sin\varphi$, and

$$v_j=\sin\Big(\frac{j\pi}{2n+2}\Big)\sum_{l=1}^{n+1}\frac{1}{2l-1}\sin\Big(\frac{(2l-1)j\pi}{2n+2}\Big),\quad j=1,\cdots,2n+1.$$

Note that the above points are not uniformly distributed.

Suppose that Q_n is a positive quadrature rule satisfying the above properties. Then (3.2) and (3.4) imply that for any $f\in\Pi^d_n$,

$$f(x)=\sum_{\omega\in\Lambda_n}\lambda_\omega f(\omega)K_{n,\eta}(x,\omega).$$

We define sampling operators as follows:

$$I_n(f,x)=\sum_{\omega\in\Lambda_n}\lambda_\omega f(\omega)K_{n,\eta}(x,\omega),\quad f\in C(\mathbb{S}^{d-1}).$$

Then for any $f\in\Pi^d_n$,

$$I_n(f,x)=f(x),\quad \forall x\in\mathbb{S}^{d-1}. \tag{3.5}$$

We remark that the operators I_n are asymptotically optimal sampling operators in the worst case setting. We shall show that I_n are also asymptotically optimal in the average case setting.

In order to prove this and Theorem 2.1, it suffices to prove the upper bounds of $e^{\rm avg}(I_n,\mu,L_q)_p$ for $1\le q\le\infty$ and $0<p<\infty$ due to the fact that

$$g_N^{(a)}(W_2^r,\mu,L_q)_p\le e^{\rm avg}(I_n,\mu,L_q)_p. \tag{3.6}$$

The following lemmas play an important role in the estimate of upper bounds of $e^{\rm avg}(I_n,\mu,L_q)_p$.

Lemma 3.1. *Let $1\le q<\infty$, and $l=0,1,2,\cdots$. Then*

$$Q_{l,n}:=\int_{W_2^r}\|A_l(f)-I_n(A_l(f))\|_q^q\mu(df)=C(q)\int_{\mathbb{S}^{d-1}}|R_{l,n}(x)|^q d\sigma(x), \tag{3.7}$$

where $C(q)=\pi^{-\frac{1}{2}}2^{\frac{q}{2}}\Gamma(\frac{q+1}{2})$,

$$\begin{aligned}R_{l,n}^2(x)=&V_l(x,x)-2\sum_{\omega\in\Lambda_n}\lambda_\omega K_{n,\eta}(x,\omega)V_l(x,\omega)\\&+\sum_{\omega\in\Lambda_n}\sum_{\xi\in\Lambda_n}\lambda_\omega\lambda_\xi K_{n,\eta}(x,\omega)K_{n,\eta}(x,\xi)V_l(\xi,\omega),\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}V_l(x,x')&:=\int_{W_2^r}A_l(f,x)A_l(f,x')\mu(df)\\&=\sum_{k=2^{l-1}+1}^{2^l}\Big(1-\eta\Big(\frac{k}{2^{l-1}}\Big)\Big)^2\frac{Z_k(x,x')}{(k(k+d-2))^{r+\frac{s}{2}}}+\sum_{k=2^l+1}^{2^{l+1}}\Big(\eta\Big(\frac{k}{2^l}\Big)\Big)^2\frac{Z_k(x,x')}{(k(k+d-2))^{r+\frac{s}{2}}}.\end{aligned} \tag{3.9}$$

Proof. For a fixed $x\in\mathbb{S}^{d-1}$, set

$$L_l(f,x):=A_l(f,x)-I_n(A_l(f),x).$$

Then $L_l(f,x)$ is a bounded linear functional on W_2^r. Since the measure μ is symmetric Gaussian on W_2^r, we know that $L_l(f,x)$, as a random variable on the measurable space (W_2^r,μ), obeys the normal distribution $N(0,R_{l,n}^2(x))$, where

$$\begin{aligned}R_{l,n}^2(x)&=\int_{W_2^r}\big|L_l(f,x)\big|^2\mu(df)=\int_{W_2^r}\Big|A_l(f,x)-\sum_{\omega\in\Lambda_n}\lambda_\omega A_l(f,\omega)K_{n,\eta}(x,\omega)\Big|^2\mu(df)\\&=\int_{W_2^r}\Big(A_l^2(f,x)-2\sum_{\omega\in\Lambda_n}\lambda_\omega K_{n,\eta}(x,\omega)A_l(f,\omega)A_l(f,x)\\&\quad+\sum_{\omega\in\Lambda_n}\sum_{\xi\in\Lambda_n}\lambda_\omega\lambda_\xi K_{n,\eta}(x,\omega)K_{n,\eta}(x,\xi)A_l(f,\omega)A_l(f,\xi)\Big)\mu(df)\\&=V_l(x,x)-2\sum_{\omega\in\Lambda_n}\lambda_\omega K_{n,\eta}(x,\omega)V_l(x,\omega)\\&\quad+\sum_{\omega\in\Lambda_n}\sum_{\xi\in\Lambda_n}\lambda_\omega\lambda_\xi K_{n,\eta}(x,\omega)K_{n,\eta}(x,\xi)V_l(\xi,\omega),\end{aligned}$$

where $V_l(x,x') := \int_{W_2^r} A_l(f,x)A_l(f,x')\mu(df)$. Then $L_l(f,x)/R_{l,n}(x)$, as a random variable on the measurable space (W_2^r,μ), obeys the normal distribution $N(0,1)$. This means that

$$\begin{aligned}&\int_{W_2^r}|A_l(f,x)-I_n(A_l(f),x)|^q\mu(df)=|R_{l,n}(x)|^q\int_{W_2^r}\Big|\frac{L_l(f,x)}{R_{l,n}(x)}\Big|^q\mu(df)\\ =&|R_{l,n}(x)|^q\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}|t|^q e^{-\frac{t^2}{2}}dt=\pi^{-\frac{1}{2}}2^{\frac{q}{2}}\Gamma\Big(\frac{q+1}{2}\Big)|R_{l,n}(x)|^q=C(q)|R_{l,n}(x)|^q.\end{aligned}$$

Using the Fubini theorem, we get

$$\begin{aligned}&\int_{W_2^r}\|A_l(f)-I_n(A_l(f))\|_q^q\mu(df)\\ =&\int_{W_2^r}\int_{\mathbb{S}^{d-1}}|A_l(f,x)-I_n(A_l(f),x)|^q d\sigma(x)\mu(df)\\ =&\int_{\mathbb{S}^{d-1}}\int_{W_2^r}|A_l(f,x)-I_n(A_l(f),x)|^q\mu(df)d\sigma(x)\\ =&C(q)\int_{\mathbb{S}^{d-1}}|R_{l,n}(x)|^q d\sigma(x).\end{aligned}$$

Finally, we compute $V_l(x,x')$. We use the method in [28]. Denote

$$I_{lm}(x,x')=\int_{W_2^r}H_l(f,x)H_m(f,x')\mu(df).$$

By the addition formula (2.2) for the spherical harmonics of degree l we have

$$\begin{aligned}I_{lm}(x,x')=&\sum_{i=1}^{N(d,l)}\sum_{j=1}^{N(d,m)}Y_{li}(x)Y_{mj}(x')\int_{W_2^r}\langle f,Y_{li}\rangle\langle f,Y_{mj}\rangle\mu(df)\\ =&\sum_{i=1}^{N(d,l)}\sum_{j=1}^{N(d,m)}Y_{li}(x)Y_{mj}(x')(l(l+d-1))^{-r-s/2}\delta_{lm}\delta_{ij}\\ =&\delta_{lm}(l(l+d-1))^{-r-s/2}\sum_{i=1}^{N(d,l)}Y_{li}(x)Y_{li}(x')\\ =&\delta_{lm}(l(l+d-1))^{-r-s/2}Z_l(x,x'),\end{aligned}$$

where

$$\delta_{lm}=\begin{cases}1, & l=m,\\ 0, & l\neq m,\end{cases}$$

and in the second equality we used the equality (see [28, pp. 370])

$$\int_{W_2^r}\langle f,Y_{li}\rangle\langle f,Y_{mj}\rangle\mu(df)=(l(l+d-1))^{-r-s/2}\delta_{lm}\delta_{ij}.$$

Hence, by (3.3) we get

$$\begin{aligned} V_l(x,x') &= \int_{W_2^r} A_l(f,x)A_l(f,x')\mu(df) \\ &= \sum_{k=2^{l-1}+1}^{2^l} \Big(1-\eta\Big(\frac{k}{2^{l-1}}\Big)\Big)^2 \int_{W_2^r} H_k(f,x)H_k(f,x')\mu(df) \\ &\quad + \sum_{k=2^l+1}^{2^{l+1}} \Big(\eta\Big(\frac{k}{2^l}\Big)\Big)^2 \int_{W_2^r} H_k(f,x)H_k(f,x')\mu(df) \\ &= \sum_{k=2^{l-1}+1}^{2^l} \Big(1-\eta\Big(\frac{k}{2^{l-1}}\Big)\Big)^2 \frac{Z_k(x,x')}{(k(k+d-2))^{r+\frac{s}{2}}} + \sum_{k=2^l+1}^{2^{l+1}} \Big(\eta\Big(\frac{k}{2^l}\Big)\Big)^2 \frac{Z_k(x,x')}{(k(k+d-2))^{r+\frac{s}{2}}}. \end{aligned}$$

This completes the proof of Lemma 3.1. □

Next we estimate $|R_{l,n}(x)|$.

Lemma 3.2. *For any* $x,x'\in\mathbb{S}^{d-1}$*,*

$$|V_l(x,x')| \ll 2^{-(2r+s)l+(d-1)l}, \tag{3.10}$$

where $V_l(x,x')$ *is given in* (3.9).

Proof. We use the following well-known property of Jacobi polynomials [21, pp. 168], For all $l\in\mathbb{N}$ and any $t\in[-1,1]$,

$$\big|P_l^{(\frac{d-3}{2},\frac{d-3}{2})}(t)\big| \le \big|P_l^{(\frac{d-3}{2},\frac{d-3}{2})}(1)\big|.$$

It then follows from (2.2) that

$$|Z_k(x,x')| = \frac{N(d,k)}{|\mathbb{S}^{d-1}|}\Big|\frac{P_k^{(\frac{d-3}{2},\frac{d-3}{2})}(x\cdot x')}{P_k^{(\frac{d-3}{2},\frac{d-3}{2})}(1)}\Big| \ll k^{d-2}.$$

Therefore, from (3.9), we obtain

$$|V_l(x,x')| \ll \sum_{k=2^{l-1}+1}^{2^l} \frac{k^{d-2}}{(k(k+d-2))^{r+\frac{s}{2}}} + \sum_{k=2^l+1}^{2^{l+1}} \frac{k^{d-2}}{(k(k+d-2))^{r+\frac{s}{2}}} \ll 2^{-(2r+s)l+(d-1)l}.$$

So, we complete the proof. □

For any given n, there exist a unique v such that $2^{v-1}\le n<2^v$. It follows from (3.5) and (3.7) that for $l\le v-1$, $R_{l,n}(x)=0$. Now we estimate $R_{l,n}^2(x)$ for $l\ge v$.

Lemma 3.3. *Let* $l\ge v$ *and* $x\in\mathbb{S}^{d-1}$*. Then we have*

$$|R_{l,n}^2(x)| \ll 2^{-(2r+s)l+(d-1)l}. \tag{3.11}$$

Proof. It follows from [6, Theorem 2.1]) that for any $x\in\mathbb{S}^{d-1}$,

$$\sum_{\omega\in\Lambda_n}\lambda_\omega|K_{n,\eta}(x,\omega)|\le c\int_{\mathbb{S}^{d-1}}|K_{n,\eta}(x,y)|d\sigma(y)\ll 1. \tag{3.12}$$

Combining (3.8), (3.10) with (3.12), we obtain

$$\begin{aligned}|R^2_{l,n}(x)|\le&|V_l(x,x)|+2\sum_{\omega\in\Lambda_n}\lambda_\omega|K_{n,\eta}(x,\omega)||V_l(x,\omega)|\\&+\sum_{\omega\in\Lambda_n}\sum_{\xi\in\Lambda_n}\lambda_\omega\lambda_\xi|K_{n,\eta}(x,\omega)||K_{n,\eta}(x,\xi)||V_l(\xi,\omega)|\\ \ll&2^{-(2r+s)l+(d-1)l}.\end{aligned}$$

The proof of Lemma 3.3 is finished. □

Proof of Theorem 2.1. From [11, Theorem 1.2] or [27, Corollary 1], we have for $1\le q\le\infty$ and $0<p<\infty$,

$$e^{\rm avg}(I_n,\mu,L_q)_p\asymp e^{\rm avg}(I_n,\mu,L_q)_1,$$

where the equivalent constants depend only on p. So in order to prove Theorem 2.1, by (3.6) it is sufficient to give the upper estimates of $e^{\rm avg}(I_n,\mu,L_q)_1$. It follows by the triangle inequality that

$$\begin{aligned}e^{\rm avg}(I_n,\mu,L_q)_1=&\int_{W_2^r}\Big\|\sum_{l=v}^{\infty}\big(A_l(f)-I_n(A_l(f))\big)\Big\|_q\mu(df)\\ \le&\sum_{l=v}^{\infty}\int_{W_2^r}\|A_l(f)-I_n(A_l(f))\|_q\mu(df).\end{aligned} \tag{3.13}$$

If $1\le q<\infty$, then by the Hölder inequality and Lemma 3.1 we get

$$\begin{aligned}e^{\rm avg}(I_n,\mu,L_q)_1\le&\sum_{l=v}^{\infty}\Big(\int_{W_2^r}\|A_l(f)-I_n(A_l(f))\|_q^q\mu(df)\Big)^{\frac1q}\\ =&\sum_{l=v}^{\infty}\Big(C(q)\int_{\mathbb{S}^{d-1}}|R_{l,n}(x)|^qd\sigma(x)\Big)^{\frac1q},\end{aligned}$$

where $|R^2_{l,n}(x)|$ is given in (3.8). It follows from (3.11) that

$$e^{\rm avg}(I_n,\mu,L_q)_1\ll\sum_{l=v}^{\infty}2^{-(r+\frac s2)l+\frac{d-1}{2}l}\ll n^{-(r+\frac s2)+\frac{d-1}{2}}.$$

If $q=\infty$, we remark that $A_l(f)-I_n(A_l(f))$ is a trigonometric polynomial of degree at most 2^{l+1}. By the Nikolskii inequality, we get for any $1\le p_l<\infty$,

$$\|A_l(f)-I_n(A_l(f))\|_\infty\le c2^{\frac{(d-1)l}{p_l}}\|A_l(f)-I_n(A_l(f))\|_{p_l},$$

where c is a positive constant independent of p_l and l.

Applying the Hölder inequality, we have

$$\begin{aligned}&\int_{W_2^r}\|A_l(f)-I_n(A_l(f))\|_\infty\mu(df)\\ \le& c2^{\frac{(d-1)l}{p_l}}\int_{W_2^r}\|A_l(f)-I_n(A_l(f))\|_{p_l}\mu(df)\\ \le& c2^{\frac{(d-1)l}{p_l}}\Big(\int_{W_2^r}\|A_l(f)-I_n(A_l(f))\|_{p_l}^{p_l}\mu(df)\Big)^{\frac{1}{p_l}}\\ =& c2^{\frac{(d-1)l}{p_l}}\Big(C(p_l)\int_{\mathbb{S}^{d-1}}|R_{l,n}(x)|^{p_l}d\sigma(x)\Big)^{\frac{1}{p_l}}\\ \ll& c2^{\frac{(d-1)l}{p_l}}C(p_l)^{\frac{1}{p_l}}2^{-(r+\frac{s}{2})l+\frac{d-1}{2}l},\end{aligned}$$

where

$$C(p_l)=\pi^{-\frac{1}{2}}2^{\frac{p_l}{2}}\Gamma\Big(\frac{p_l+1}{2}\Big).$$

By Stirling's formula

$$\lim_{x\to+\infty}\frac{\Gamma(x)}{\sqrt{2\pi}x^{x-\frac{1}{2}}\exp(-x)}=1,$$

we have

$$\Big(\Gamma\Big(\frac{x+1}{2}\Big)\Big)^{\frac{1}{x}}\ll(\sqrt{2\pi})^{\frac{1}{x}}\Big(\frac{x+1}{2}\Big)^{(\frac{x+1}{2}-\frac{1}{2})\frac{1}{x}}\exp\Big(-\frac{x+1}{2x}\Big)\ll x^{\frac{1}{2}}.$$

Now letting $p_l=l$, we obtain

$$\int_{W_2^r}\|A_l(f)-I_n(A_l(f))\|_\infty\mu(df)\ll l^{\frac{1}{2}}2^{-(r+\frac{s}{2})l+\frac{d-1}{2}l},$$

and therefore, by (3.13)

$$e^{\rm avg}(I_n,\mu,L_\infty)_1\ll\sum_{l=v}^{\infty}l^{\frac{1}{2}}2^{-(r+\frac{s}{2})l+\frac{d-1}{2}l}\ll n^{-(r+\frac{s}{2})+\frac{d-1}{2}}\sqrt{\ln n}.$$

This completes the proof of Theorem 2.1. □

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References

[1] V. I. Bogachev, Gaussian Measures, Mathematical Surveys and Monographs, Amer, Math. Soc., 62 (1998).

[2] G. Brown and F. Dai, Approximation of smooth functions on compact two-point homogeneous spaces, J. Funct. Anal., 220 (2005), 401–423.

[3] G. Brown, F. Dai,and Y. Sun, Kolmogorov width of classes of smooth functions on the sphere $\mathbb{S}^{d-1}$, J. Complexity, 18(4) (2002), 1001–1023.

[4] Guanggui Chen and Gensun Fang, Probabilistic and average widths of multivariate Sobolev spaces with mixed derivative equipped with the Gaussian measure, J. Complexity, 20(6) (2004), 858–875.

[5] Guanggui Chen and Gensun Fang, Linear widths of a multivariate function space equipped with a Gaussian measure, J. Approx. Theory, 132 (2005), 77–96.

[6] F. Dai, On generalized hyperinterpolation on the sphere, Proc. Am. Math. Soc., 134(10) (2006), 2931–2941.

[7] F. Dai and Y. Xu, Approximation Theroy and Harmonic Analysis on Spheres and Balls, Springer, 2012.

[8] J. R. Driscoll and D. M. Healy, Computing Fourier transforms and convolutions on the 2-sphere, Adv. Appl. Math., 15 (1994), 202–250.

[9] Gensun Fang and Peixin Ye, Probabilistic and average linear widths of Sobolev space with Gaussian measure, J. Complexity, 19 (2003), 73–84.

[10] Gensun Fang and Peixin Ye, Probabilistic and average linear widths of Sobolev space with Gaussian measure in L_∞-norm, Constr. Approx., 20 (2004), 159–172.

[11] X. Fernique, Integrabilite des vecteurs gaussiens, C. R. Acad. Sci. Paris Ser. A, 270 (1970), 1698–1699.

[12] F. Hickernell, G. W. Wasilkowski and H. Woźniakowski, Tractability of linear multivariate problems in the average case setting, in: Monte Carlo and Quasi-Monte Carlo Methods 2006, A. Keller, S. Heinrich, H. Niederreiter (eds.), 461-494, Springer, Berlin, 2008.

[13] Zexia Huang and Heping Wang, Sampling numbers of periodic Sobolev spaces with a Gaussian measure in the average case setting, J. Approx. Theory, 167(1) (2013), 109–120.

[14] F. Y. Kuo, I. H. Sloan, and H. Woźniakowski, Lattice rule algorithms for multivariate approximation in the average case setting, J. Complexity, 24(2) (2008), 283–323.

[15] V. E. Maiorov, Linear widths of function spaces equipped with the Gaussian measure, J. Approx. Theory, 77(1) (1994), 74–88.

[16] V. E. Maiorov, About n-widths of the Wiener space in L_q-norm, J. Complexity, 12(1) (1996), 47–57.

[17] V. E. Maiorov and G. W. Wasilkowski, Probabilistic and average linear widths in L_∞-norm with respect to r-fold Wiener measure, J. Approx. Theory, 84(1) (1996), 31–40.

[18] H. N. Mhaskar, F. J. Narcowich and J. D. Ward, Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature, Math. Comput., 70 (2001), 1113–1130 (Corrigendum: Math. Comput., 71 (2001), 453–454).

[19] F. J. Narcowich, P. Petrushev and J. D. Ward, Localized tight frames on spheres, SIAM J. Math. Anal., 38(2) (2006), 574–594.

[20] K. Ritter, Average-Case Analysis of Numerical Problems, Lecture Notes in Math, Vol. 1733, Springer-Verlag, 2000.

[21] G. Szegö, Orthogonal Polynomials, Amer. math. Soc, Colloq. Publ., Vol. 23, Providence, 4thedition, 1975.

[22] Yongsheng Sun and Chengyong Wang, μ-average, n-widths on the Wiener space, J. Complexity, 10(4) (1994), 428–436.

[23] Yongsheng Sun and Chengyong Wang, Average error bounds of best approximation of continuous functions on the Wiener space, J. Complexity, 11 (1995), 74–104.

[24] J. F. Traub, G.W. Wasilkowski, H. Woźniakowski, Information-Based Complexity, Academic Press, New York, 1988.

[25] Heping Wang, Probabilistic and average widths of Sobolev spaces on compact two-point homogeneous spaces equipped with a Gaussian measure, Constr. Approx., 39 (2014), 485–516.

[26] Heping Wang, Weigang Jiang and Xuebo Zhai, Approximation of multivariate periodic functions on the L_2 space with a Gaussian measure, J. Math. Anal. Appl., 388(2) (2012), 929–941.

[27] Heping Wang and Xuebo Zhai, Best approximation of functions on the ball on the weighted Sobolev space equipped with a Gaussian measure, J. Approx. Theory, 162 (2010), 1160–1177.

[28] Heping Wang, Xuebo Zhai and Yanwei Zhang, Approximation of functions on the sphere in the average case setting, J. Complexity, 25 (2009), 362–376.

[29] Heping Wang, Yanwei Zhang and Xuebo Zhai, Approximation of periodic functions on the Sobolev space with a Gaussion measure, Sci. China Ser. A, 53(2) (2010), 373–384.

[30] Kunyang Wang and Luoqing Li, Harmonic Analysis and Approximation on the Unit Sphere, Science Press, Beijing, 2000.