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Unique Common Fixed Point for a Family of Mappings with a Nonlinear Quasi-Contractive Type Condition in Metrically Convex Spaces

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Abstract. A class Φ of 5-dimensional functions was introduced and an existence and uniqueness of common fixed points for a family of non-self mappings satisfying a ϕ_i-quasi-contractive condition and a certain boundary condition was given on complete metrically convex metric spaces, and from which, more general unique common fixed point theorems were obtained. Our main results generalize and improve many same type common fixed point theorems in references.

Key Words: Metrically convex space, a class Φ of 5-dimensional functions, ϕ_i-quasi-contraction, common fixed point, complete.

AMS Subject Classifications: 47H25, 54H25

1 Introduction and preliminaries

There have appeared many fixed point theorems for a self-single-valued self mapping on a closed subset of a Banach space. However, in many applications, the map under considerations is not a self-mapping on a closed set. In 1976, Assad [1] gave a sufficient condition for such a single valued mapping to obtain a fixed point result by proving a fixed point theorem for a Kannan type mapping on a Banach space and putting certain boundary conditions on the mapping. Similar results for multi-valued mappings were respectively given by Assad [2] and Assad and Kirk [3]. Late, some authors obtained and generalized the same type results on complete metrically convex metric spaces [4–7]. The all above results were discussed under certain contractive conditions or certain boundary conditions. Recently, we further discussed the existence problems of unique common fixed points for the mappings satisfying certain contractive or quasi-contractive conditions with linear property, see [8–10]. In order to generalize and improve these

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results, we will discuss the the existence problems of unique common fixed points for the mappings satisfying certain ϕ-quasi-contractive conditions with nonlinear property and boundary conditions, and give more general results.

Definition 1.1 (see [4,5]). A metric space (X,d) is said to be metrically convex, if for any $x,y\in X$ with $x\neq y$, there exists $z\in X$ with $z\neq x$ and $z\neq y$ such that $d(x,z)+d(z,y)=d(x,y)$.

Lemma 1.1 (see [3,4]). *If K is a nonempty closed subset of a complete metrically convex space (X,d), then for any $x\in K$ and $y\notin K$, there exists $z\in\partial K$ such that $d(x,z)+d(z,y)=d(x,y)$.*

Definition 1.2. $\phi\in\Phi$ if and only if $\phi:(\mathbb{R}^+)^5\to\mathbb{R}^+$ is continuous, non-decreasing for every variable, and there exists $k\in[0,1/2)$ such that $\lambda(t):=\phi(t,t,t,t,2t)<kt$ for all $t>0$, where $\mathbb{R}^+=[0,\infty)$. Here, k is said to be the number of ϕ.

Example 1.1. Define $\phi_1,\phi_2:(\mathbb{R}^+)^5\to\mathbb{R}^+$ as follows

$$\phi_1(t_1,t_2,t_3,t_4,t_5)=k_1t_1+k_2t_2+k_3t_3+k_4t_4+k_5t_5,$$

where $k_1,k_2,k_3,k_4,k_5\in[0,1)$ satisfying $k_1+k_2+k_3+k_4+2k_5<1/2$.

$$\phi_2(t_1,t_2,t_3,t_4,t_5)=\max\{k_1t_1,k_2t_2,k_3t_3,k_4t_4,k_5t_5\},$$

where $k_1,k_2,k_3,k_4,k_5\in[0,1)$ and $\max\{k_1,k_2,k_3,k_4,2k_5\}<1/2$. Then $\phi_1,\phi_2\in\Phi$.

2 Common fixed point theorems

Next theorem is the main result in this paper.

Theorem 2.1. *Let K be a nonempty closed subset of a complete metrically convex space (X,d) with $\text{int}K\neq\emptyset$, $\{T_i\}_{i\in\mathbb{N}}$ a family of self-mappings on X such that $K\subseteq T_i(X)$ for all $i\in\mathbb{N}$. Suppose that for any $x,y\in X$ and any $i,j\in\mathbb{N}$ with $i\leq j$,*

$$d(x,y)\leq\phi_i\big(d(T_ix,T_jy),d(x,T_ix),d(y,T_jy),d(x,T_jy),d(T_ix,y)\big),\tag{2.1}$$

where $\phi_i\in\Phi$ for all $i\in\mathbb{N}$. Suppose that $k=\sup_{i\in\mathbb{N}}k_i<1/2$, where k_i is the number of ϕ_i for all $i\in\mathbb{N}$. Furthermore, if $T_i(K^C)\cap\partial K=\emptyset$ for all $i\in\mathbb{N}$, and for any $i\in\mathbb{N}$ and $x\in K$, there exists $j\in\mathbb{N}$ such that $T_j(T_i(x))=x$, then $\{T_i\}_{i\in\mathbb{N}}$ have a unique common fixed point in K.

Proof. Take any element $x_0\in K$. We will construct two sequences $\{x_n\}$ and $\{x'_n\}$ in the following manner. Since $K\subseteq T_1(X)$, there exists $x'_1\in X$ such that $x_0=T_1x'_1$. If $x'_1\in K$, then put $x_1=x'_1$. If $x'_1\notin K$, then there exists $x_1\in\partial K$ such that $d(x_0,x_1)+d(x_1,x'_1)=d(x_0,x'_1)$ by Lemma 1.1. Similarly $K\subseteq T_2(X)$ implies that there exists $x'_2\in X$ such that $x_1=T_2x'_2$. If $x'_2\in K$, then put$x_2=x'_2$; if $x'_2\notin K$, then there exists $x_2\in\partial K$ such that $d(x_1,x_2)+d(x_2,x'_2)=d(x_1,x'_2)$ by Lemma 1.1. Continuing in this way, we obtain $\{x_n\}$ and $\{x'_n\}$ with the following properties:

(i) $x_{n-1}=T_n x'_n$;
(ii) if $x'_n \in K$, then put $x_n = x'_n$;
(iii) if $x'_n \notin K$, then there exists $x_n \in \partial K$ such that $d(x_{n-1},x_n)+d(x_n,x'_n)=d(x_{n-1},x'_n)$.

Let $P=\{x_i \in \{x_n\}: x_i = x'_i\}$ and $Q=\{x_i \in \{x_n\}: x_i \neq x'_i\}$. we have following result:

$$x_n \in Q \Longrightarrow x_{n-1}, \quad x_{n+1} \in P. \tag{2.2}$$

In fact, since $x_n \in Q$, so $x_n \neq x'_n$ and $x_n \in \partial K$ and $x'_n \notin K$ by (iii). If $x_{n-1} \in Q$, then $x_{n-1} \in \partial K$ by Q; on the other hand, since $x'_n \in K^C$, so $x_{n-1}=T_n x'_n \in T_n(K^C)$, but $T_n(K^C)\cap \partial K=\emptyset$, hence $x_{n-1}=T_n x'_n \notin \partial K$. This is a contradiction, so $x_{n-1}\in P$. Since $x_n \in \partial K$ and $x_n = T_{n+1}x'_{n+1}$ and $T_{n+1}(K^C)\cap\partial K=\emptyset$, so $x'_{n+1}\in K$, hence $x_{n+1}=x'_{n+1}$, which implies that $x_{n+1}\in P$.

Now, we will estimate $d(x_n,x_{n+1})$. In view of (2.2), we can divide the proof into three cases:

Case I. $x_n,x_{n+1}\in P$. In this case,

$$\begin{aligned}
&d(x_n,x_{n+1})=d(x'_n,x'_{n+1})\\
\le&\phi_n\big(d(T_n x'_n,T_{n+1}x'_{n+1}),d(x'_n,T_n x'_n),d(x'_{n+1},T_{n+1}x'_{n+1}),d(x'_n,T_{n+1}x'_{n+1}),d(T_n x'_n,x'_{n+1})\big)\\
=&\phi_n\big(d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}),0,d(x_{n-1},x_{n+1})\big)\\
\le&\phi_n\big(d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x_{n+1}),0,[d(x_n,x_{n+1})+d(x_{n-1},x_n)]\big).
\end{aligned}\tag{2.3}$$

If $d(x_{n-1},x_n)<d(x_n,x_{n+1})$, then by ϕ_n, (2.3) becomes

$$\begin{aligned}
d(x_n,x_{n+1})\le&\phi_n\big(d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_n,x_{n+1}),2d(x_n,x_{n+1})\big)\\
<&kd(x_n,x_{n+1}),
\end{aligned}$$

which is a contradiction. Hence $d(x_n,x_{n+1})\le d(x_{n-1},x_n)$, so (2.3) becomes

$$\begin{aligned}
d(x_n,x_{n+1})\le&\phi_n\big(d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_{n-1},x_n),2d(x_{n-1},x_n)\big)\\
\le&kd(x_{n-1},x_n).
\end{aligned}\tag{2.4}$$

Case II. $x_n\in P$, $x_{n+1}\in Q$. In this case, by (iii) and (2.2), we obtain

$$\begin{aligned}
&d(x_n,x_{n+1})\\
\le&d(x_n,x_{n+1})+d(x_{n+1},x'_{n+1})=d(x_n,x'_{n+1})=d(x'_n,x'_{n+1})\\
\le&\phi_n\big(d(T_n x'_n,T_{n+1}x'_{n+1}),d(x'_n,T_n x'_n),d(x'_{n+1},T_{n+1}x'_{n+1}),d(x'_n,T_{n+1}x'_{n+1}),d(T_n x'_n,x'_{n+1})\big)\\
=&\phi_n\big(d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x'_{n+1}),d(x_n,x_n),d(x_{n-1},x'_{n+1})\big)\\
\le&\phi_n\big(d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_n,x'_{n+1}),0,[d(x_{n-1},x_n)+d(x_n,x'_{n+1})]\big).
\end{aligned}\tag{2.5}$$

If $d(x_{n-1},x_n)<d(x_n,x'_{n+1})$, then (2.5) becomes

$$\begin{aligned}
d(x_n,x'_{n+1})\le&\phi_n\big(d(x_n,x'_{n+1}),d(x_n,x'_{n+1}),d(x_n,x'_{n+1}),d(x_n,x'_{n+1}),2d(x_n,x'_{n+1})\big)\\
<&kd(x_n,x'_{n+1}),
\end{aligned}$$

which is also a contradiction. Hence $d((x_{n-1},x_n)\geq d(x_n,x'_{n+1})$, therefore from (2.5),

$$\begin{aligned} d(x_n,x'_{n+1}) &\leq \phi_n\big(d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_{n-1},x_n),d(x_{n-1},x_n),2d(x_{n-1},x_n)\big)\\ &\leq kd(x_{n-1},x_n). \end{aligned} \tag{2.6}$$

But $d(x_n,x_{n+1})\leq d(x_n,x'_{n+1})$, hence combining (2.6), we obtain

$$d(x_n,x_{n+1})\leq kd(x_{n-1},x_n). \tag{2.7}$$

Case III. $x_n\in Q$, $x_{n+1}\in P$. In this case, suing (2.2) and (iii), we know $x_{n-1}\in P$, hence

$$\begin{aligned} &d(x_n,x_{n+1})\\ \leq& d(x_n,x'_n)+d(x'_n,x_{n+1})\leq d(x_{n-1},x_n)+d(x_n,x'_n)+d(x'_n,x_{n+1})\\ =&d(x_{n-1},x'_n)+d(x'_n,x_{n+1})=d(x_{n-1},x'_n)+d(x'_n,x'_{n+1})\\ \leq& d(x_{n-1},x'_n)\\ &+\phi_n\big(d(T_nx'_n,T_{n+1}x'_{n+1}),d(x'_n,T_nx'_n),d(x'_{n+1},T_{n+1}x'_{n+1}),d(x'_n,T_{n+1}x'_{n+1}),d(T_nx'_n,x'_{n+1})\big)\\ =&d(x_{n-1},x'_n)+\phi_n\big(d(x_{n-1},x_n),d(x_{n-1},x'_n),d(x_n,x_{n+1}),d(x_n,x'_n),d(x_{n-1},x_{n+1})\big). \end{aligned} \tag{2.8}$$

But $d(x_{n-1},x_n)+d(x_n,x'_n)=d(x_{n-1},x'_n)$, hence $d(x_{n-1},x_n)\leq d(x_{n-1},x'_n)$ and $d(x_n,x'_n)\leq d(x_{n-1},x'_n)$. Therefore, using (2.8), we obtain

$$\begin{aligned} &d(x_n,x_{n+1})\\ \leq& d(x_{n-1},x'_n)\\ &+\phi_n\big(d(x_{n-1},x'_n),d(x_{n-1},x'_n),d(x_n,x_{n+1}),d(x_{n-1},x'_n),[d(x_{n-1},x_n)+d(x_n,x_{n+1})]\big)\\ \leq& d(x_{n-1},x'_n)\\ &+\phi_n\big(d(x_{n-1},x'_n),d(x_{n-1},x'_n),d(x_n,x_{n+1}),d(x_{n-1},x'_n),[d(x_{n-1},x'_n)+d(x_n,x_{n+1})]\big). \end{aligned} \tag{2.9}$$

If $d(x_{n-1},x'_n)\leq d(x_n,x_{n+1})$, then (2.9) becomes

$$\begin{aligned} &d(x_n,x_{n+1})\\ \leq& d(x_{n-1},x'_n)+\phi_n\big(d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_n,x_{n+1}),d(x_n,x_{n+1}),2d(x_n,x_{n+1})\big)\\ \leq& d(x_{n-1},x'_n)+kd(x_n,x_{n+1}), \end{aligned}$$

hence

$$d(x_n,x_{n+1})\leq\Big(\frac{1}{1-k}\Big)d(x_{n-1},x'_n).$$

Therefore combining (2.6) in Case II, we obtain

$$d(x_n,x_{n+1})\leq\Big(\frac{1}{1-k}\Big)d(x_{n-1},x'_n)\leq\Big(\frac{k}{1-k}\Big)d(x_{n-2},x_{n-1}). \tag{2.10}$$

If $d(x_{n-1},x'_n)\geq d(x_n,x_{n+1})$, then (2.9) becomes

$$\begin{aligned}&d(x_n,x_{n+1})\\ \leq& d(x_{n-1},x'_n)+\phi_n\big(d(x_{n-1},x'_n),d(x_{n-1},x'_n),d(x_{n-1},x'_n),d(x_{n-1},x'_n),2d(x_{n-1},x'_n)\big)\\ \leq& d(x_{n-1},x'_n)+kd(x_{n-1},x'_n)=(1+k)d(x_{n-1},x'_n),\end{aligned}$$

hence combining (2.6) in Case II again, we obtain

$$d(x_n,x_{n+1})\leq(1+k)d(x_{n-1},x'_n)\leq(1+k)kd(x_{n-2},x_{n-1}). \tag{2.11}$$

Consequently, in two situations, we obtain

$$d(x_n,x_{n+1})\leq\max\Big\{\frac{k}{1-k},(1+k)k\Big\}d(x_{n-2},x_{n-1}).$$

Therefore, in all three cases (see (2.4), (2.7), (2.10) and (2.11)), we obtain

$$d(x_n,x_{n+1})\leq\max\Big\{k,\frac{k}{1-k},(1+k)k\Big\}\max\{d(x_{n-2},x_{n-1}),d(x_{n-1},x_n)\},\quad\forall n\in\mathbb{N},\quad n>2.$$

Let

$$M=\max\Big\{k,\frac{k}{1-k},(1+k)k\Big\},$$

then $0<M<1$ and

$$d(x_n,x_{n+1})<M\max\{d(x_{n-2},x_{n-1}),d(x_{n-1},x_n)\},\quad\forall n\in\mathbb{N},\quad n>2,$$

hence

$$d(x_n,x_{n+1})<(M)^{\frac{n}{2}-1}\max\{d(x_0,x_1),d(x_1,x_2)\},\quad\forall n\in\mathbb{N},\quad n\geq 2.$$

Put $\delta=(M)^{-1}\max\{d(x_0,x_1),d(x_1,x_2)\}$, then $d(x_n,x_{n+1})\leq(M^{\frac{1}{2}})^n\delta$ for all $n\in\mathbb{N}$ with $n\geq 2$. Hence for all $m>n>N$, $d(x_m,x_n)\leq\sum_{i=N}^{+\infty}d(x_i,x_{i+1})\leq\sum_{i=N}^{+\infty}(M^{\frac{1}{2}})^i\delta\to 0$ as $N\to+\infty$. This shows that $\{x_n\}$ is a Cauchy sequence. Let p be a limit of $\{x_n\}$, then $p\in K$ since K is closed and $x_n\in K$ for all $n\in\mathbb{N}$. And there exists an infinite subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfying $x_{n_j+1}\in P$ by (2.2), hence $x'_{n_j+1}=x_{n_j+1}$ and $T_{n_j+1}x'_{n_j+1}=x_{n_j}$.

For any fixed $n\in\mathbb{N}$, there exists $m\in\mathbb{N}$ such that $T_m(T_n(p))=p$. Take an enough large $j\in\mathbb{N}$ satisfying $\max\{n,m\}+1<n_j$, then

$$\begin{aligned}d(T_np,p)\leq& d(T_np,x'_{n_j+1})+d(x'_{n_j+1},p)=d(T_np,x'_{n_j+1})+d(x_{n_j+1},p)\\ \leq&\phi_m\big(d(T_m(T_n(p)),T_{n_j+1}x'_{n_j+1}),d(T_n(p),T_m(T_n(p)))),d(x'_{n_j+1},T_{n_j+1}x'_{n_j+1}),\\ &\quad d(T_np,T_{n_j+1}x'_{n_j+1}),d(T_m(T_n(p)),x'_{n_j+1})\big)+d(x_{n_j+1},p)\\ =&\phi_m\big(d(p,x_{n_j}),d(p,T_np),d(x_{n_j},x_{n_j+1}),d(T_np,x_{n_j}),d(p,x_{n_j+1})\big)+d(x_{n_j+1},p).\end{aligned} \tag{2.12}$$

Let $j\to+\infty$, then (2.12) becomes

$$\begin{aligned}d(T_np,p)\le&\phi_m\big(d(p,p),d(T_np,p),d(p,p),d(T_np,p),d(p,p)\big)+d(p,p)\\ \le&\phi_m\big(d(T_np,p),d(T_np,p),d(T_np,p),d(T_np,p),2d(T_np,p)\big)\\ \le&kd(T_np,p),\end{aligned}$$

hence $T_np=p$ for all $n\in\mathbb{N}$. Therefore, p is a common fixed point of $\{T_n\}_{n\in\mathbb{N}}$.

If u and v are all common fixed points of $\{T_n\}_{n\in\mathbb{N}}$, then using ϕ_1, we obtain

$$\begin{aligned}d(u,v)\le&\phi_1\big(d(T_1u,T_2v),d(u,T_1u),d(v,T_2v),d(u,T_2v),d(T_1u,v)\big)\\ \le&\phi_1\big(d(u,v),d(u,v),d(u,v),d(u,v),2d(u,v)\big)\\ \le&kd(u,v),\end{aligned}$$

hence $u=v$. Therefore, $\{T_n\}_{n\in\mathbb{N}}$ have a unique common fixed point $p\in K$. □

The following result is a very particular form of Theorem 2.1:

Corollary 2.1. Let K be a nonempty closed subset of a complete metrically convex space (X,d) with $\mathrm{int}K\neq\emptyset$, $\{T_i\}_{i\in\mathbb{N}}$ a family of self-mappings on X such that $K\subseteq T_i(X)$ for all $i\in\mathbb{N}$. suppose that for any $x,y\in X$ and $i,j\in\mathbb{N}$ with $i<j$,

$$d(x,y)\le k_{1,i}d(T_ix,T_jy)+k_{2,i}d(x,T_ix)+k_{3,i}d(y,T_jy)+k_{4,i}d(x,T_jy)+k_{5,i}d(T_ix,y),$$

where $\{k_{1,i},k_{2,i},k_{3,i},k_{4,i},k_{5,i}\}_{i=1}^{\infty}$ are non negative real numbers satisfying $\sup_{i\in\mathbb{N}}[k_{1,i}+k_{2,i}+k_{3,i}+k_{4,i}+2k_{5,i}]\le k<1/2$. If $T_i(K^C)\cap\partial K=\emptyset$ for all $i\in\mathbb{N}$, and for any $i\in\mathbb{N}$ and $x\in K$, there exists $j\in\mathbb{N}$ such that $T_j(T_i(x))=x$, then $\{T_i\}_{i\in\mathbb{N}}$ have a unique common fixed point in K.

Proof. Define for each $i\in\mathbb{N}$,

$$\phi_i(u_1,u_2,u_3,u_4,u_5)=k_{1,i}u_1+k_{2,i}u_2+k_{3,i}u_3+k_{4,i}u_4+k_{5,i}u_5,\quad\forall u_1,u_2,u_3,u_4,u_5\in[0,+\infty).$$

Then $\phi_i\in\Phi$ for all $i\in\mathbb{N}$. Hence the conclusion follows from Theorem 2.1. □

Remark 2.1. Corollary 2.1 is a common fixed point theorem for mappings satisfying an quasi-contractive condition with linear property, but this condition is very deferent from the linear conditions given in [9,10]. Hence Theorem 2.1 and Corollary 2.1 generalize and improve many corresponding common fixed point theorems.

Using Theorem 2.1, we obtain the following result.

Theorem 2.2. *Let K be a nonempty closed subset of a complete metrically convex space (X,d) with* int $K\neq\emptyset$*, $\{T_i\}_{i\in\mathbb{N}}$ a family of self-mappings on X and $\{m_i\}_{i\in\mathbb{N}}$ a family of positive integrals satisfying $K\subseteq T_i^{m_i}(X)$ for all $i\in\mathbb{N}$. Suppose that for any $x,y\in X$ and $i,j\in\mathbb{N}$ with $i\le j$,*

$$d(x,y)\le\phi_i\big(d(T_i^{m_i}x,T_j^{m_j}y),d(x,T_i^{m_i}x),d(y,T_j^{m_j}y),d(x,T_j^{m_j}y),d(T_i^{m_i}x,y)\big),$$

where $\phi_i\in\Phi$ for all $i\in\mathbb{N}$. Suppose that $k=\sup_{i\in\mathbb{N}}k_i<1/2$, where k_i is the number of ϕ_i for all $i\in\mathbb{N}$. Furthermore, if for each $i\in\mathbb{N}$, $T_i^{m_i}(K^C)\cap\partial K=\emptyset$, and for any given $i\in\mathbb{N}$ and $x\in X$, there exists $j\in\mathbb{N}$ such that $T_j^{m_j}(T_i^{m_i}(x))=x$. Then $\{T_i\}_{i\in\mathbb{N}}$ have a unique common fixed point in K.

Proof. Let $S_i=T_i^{m_i}$ for all $i\in\mathbb{N}$, then $\{S_i\}_{i\in\mathbb{N}}$ satisfy all conditions of Theorem 2.1, hence $\{S_i\}_{i\in\mathbb{N}}$ have a unique common fixed point $p\in K$.

For any fixed $i\in\mathbb{N}$, $S_i(T_i(p))=T_i^{m_i+1}(p)=T_i(S_i(p))=T_i(p)$, hence $T_i(p)$ is a fixed point of S_i for all $i\in\mathbb{N}$. For any $j\in\mathbb{N}$ with $j\neq i$, there exists $j'\in\mathbb{N}$ such that $S_{j'}(S_j(T_i(p)))=T_i(p)$. Consider S_j and $S_{j'}$. If $j\leq j'$, then

$$\begin{aligned}&d(T_i(p),S_j(T_i(p)))\\ \leq&\phi_j\big(d(S_j(T_i(p)),S_{j'}(S_j(T_i(p)))),d(T_i(p),S_j(T_i(p))),d(S_j(T_i(p)),S_{j'}(S_j(T_i(p)))),\\ &d(T_i(p),S_{j'}(S_j(T_i(p)))),d(S_j(T_i(p)),S_j(T_i(p)))\big)\\ =&\phi_j\big(d(S_j(T_i(p)),T_i(p))),d(T_i(p),S_j(T_i(p))),d(S_j(T_i(p)),T_i(p)),0,0)\big)\\ \leq&\phi_j\big(d(S_j(T_i(p)),T_i(p))),d(T_i(p),S_j(T_i(p))),d(S_j(T_i(p)),T_i(p)),\\ &d(S_j(T_i(p)),T_i(p)),2d(S_j(T_i(p)),T_i(p))\big)\\ \leq&kd(S_j(T_i(p)),T_i(p)).\end{aligned}$$

Hence $d(T_i(p),S_j(T_i(p)))=0$, so $T_i(p)=S_j(T_i(p))$. Similarly, we obtain the same result $T_i(p)=S_j(T_i(p))$ for case $j'<j$. Therefore $T_i(p)$ is a common fixed point of $\{S_j\}_{j\in\mathbb{N}}$ for all $i\in\mathbb{N}$. By the uniqueness of common fixed point of $\{S_j\}_{j\in\mathbb{N}}$, we obtain $T_i(p)=p$ for all $i\in\mathbb{N}$, hence p is a common fixed point of $\{T_i\}_{i\in\mathbb{N}}$. If u are v are all common fixed points of $\{T_i\}_{i\in\mathbb{N}}$, then they are also common fixed points of $\{S_i\}_{i\in\mathbb{N}}$, hence $u=v=p$. Therefore p is the unique common fixed point of $\{T_i\}_{i\in\mathbb{N}}$. □

Remark 2.2. $x\in X$ in the condition "for any given $i\in\mathbb{N}$ and $x\in X$, there exists $j\in\mathbb{N}$ such that $T_j^{m_j}(T_i^{m_i}(x))=x$" in Theorem 2.2 can not replaced by $x\in K$. In fact, even if $x\in K$, we cant not sure $T_i(x)\in K$. On the other hand, if $T_i(X)=K$ for all $i\in\mathbb{N}$, then $x\in X$ can be replaced by $x\in K$.

Using Theorem 2.2, we give a more general common fixed point theorem.

Theorem 2.3. *Let K be a nonempty closed subset of a complete metrically convex space (X,d) with* int$K\neq\emptyset$, *$\{T_{i,j}\}_{i,j\in\mathbb{N}}$ a family of self-mappings on X and $\{m_{i,j}\}_{i,j\in\mathbb{N}}$ a family of positive integrals such that $K\subseteq T_{i,j}^{m_{i,j}}(X)$ for all $i,j\in\mathbb{N}$. Suppost that for any $x,y\in X$ and $i,i',j\in\mathbb{N}$ with $i\leq i'$,*

$$d(x,y)\leq\phi_{i,j}\big(d(T_{i,j}^{m_{i,j}}x,T_{i',j}^{m_{i',j}}y),d(x,T_{i,j}^{m_{i,j}}x),d(y,T_{i',j}^{m_{i',j}}y),d(x,T_{i',j}^{m_{i',j}}y),d(T_{i,j}^{m_{i,j}}x,y)\big),$$

where $\phi_{i,j}\in\Phi$. Suppose that $k=\sup_{i,j\in\mathbb{N}}k_{i,j}<\frac{1}{2}$, where $k_{i,j}$ is the number of $\phi_{i,j}$ for all $i,j\in\mathbb{N}$. Furthermore, if the following conditions (a), (b), (c) hold, then $\{T_{i,j}\}_{i,j\in\mathbb{N}}$ have a unique common fixed point in K.

(a) $T_{i,j}^{m_{i,j}}(K^C)\cap\partial K=\emptyset$ for all $i,j\in\mathbb{N}$,

(b) for any given $j\in\mathbb{N}$ and $i\in\mathbb{N}$ and $x\in X$, there exists $i'\in\mathbb{N}$ such that $T_{i',j}^{m_{i'j}}(T_{i,j}^{m_{i,j}}(x))=x$,

(c) for each $i_1,i_2,j_1,j_2\in\mathbb{N}$ *and* $j_1\neq j_2$, $T_{i_1,j_1}T_{i_2,j_2}=T_{i_2,j_2}T_{i_1,j_1}$.
Then $\{T_{i,j}\}_{i,j\in\mathbb{N}}$ *have a unique common fixed point.*

Proof. For any fixed $j\in\mathbb{N}$, by (a), (b) and Theorem 2.2, $\{T_{i,j}\}_{i\in\mathbb{N}}$ have a unique common fixed point $p_j\in K$. Now, we will prove that $p_\mu=p_\nu$ for all $\mu,\nu\in\mathbb{N}$. In fact, for each $i_1,i_2,\mu,\nu\in\mathbb{N}$ with $\mu\neq\nu$, since $T_{i_1,\mu}(p_\mu)=p_\mu$ and $T_{i_2,\nu}(p_\nu)=p_\nu$, so $T_{i_1,\mu}(T_{i_2,\nu}(p_\nu))=T_{i_1,\mu}(p_\nu)$, hence $T_{i_2,\nu}(T_{i_1,\mu}(p_\nu))=T_{i_1,\mu}(T_{i_2,\nu}(p_\nu))=T_{i_1,\mu}(p_\nu)$ by (c). This means that $T_{i_1,\mu}(p_\nu)$ is a common fixed point of $\{T_{i_2,\nu}\}_{i_2\in\mathbb{N}}$ for all $i_1\in\mathbb{N}$. But $\{T_{i_2,\nu}\}_{i_2\in\mathbb{N}}$ have a unique common fixed point p_ν, hence $T_{i_1,\mu}(p_\nu)=p_\nu$ for all $i_1\in\mathbb{N}$, this shows that p_ν is a common fixed point of $\{T_{i_1,\mu}\}_{i_1\in\mathbb{N}}$, hence $p_\nu=p_\mu$. Put $p^*=p_j$, then p^* is the unique common fxied point of $\{T_{i,j}\}_{i,j\in\mathbb{N}}$. □

Remark 2.3. Theorem 2.3 states that, if the infinite matrix $(T_{i,j})_{i,j=1}^{\infty}$ is generated by all mappings $\{T_{i,j}\}_{i,j\in\mathbb{N}}$, then Theorem 2.3 only requires that any two mappings on the same line satisfy the quasi-contractive condition, and any two mappings on different lines fulfill the commutative property. In this case, we do not require the quasi-contractive condition or the commutative property for every two different mappings. Theorem 2.3 also shows that, no matter how we choose the initial data, the sequence constructed by an arbitrary family of mappings on the same line converges to the unique common fixed point of $\{T_{i,j}\}_{i,j\in\mathbb{N}}$. Theorem 2.2 requires the quasi-contractive condition for any different mappings. However, the commutative property is not needed. Therefore, the assumptions of Theorem 2.2 and Theorem 2.3 are quite different.

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