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Dyadic Bivariate Fourier Multipliers for Multi-Wavelets in $L^2(\mathbb{R}^2)$

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Abstract. The single 2 dilation orthogonal wavelet multipliers in one dimensional case and single A-dilation (where A is any expansive matrix with integer entries and $|detA|=2$) wavelet multipliers in high dimensional case were completely characterized by the Wutam Consortium (1998) and Z. Y. Li, et al. (2010). But there exist no more results on orthogonal multivariate wavelet matrix multipliers corresponding integer expansive dilation matrix with the absolute value of determinant not 2 in $L^2(\mathbb{R}^2)$. In this paper, we choose

$$2I_2=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

as the dilation matrix and consider the $2I_2$-dilation orthogonal multivariate wavelet $\Psi=\{\psi_1,\psi_2,\psi_3\}$, (which is called a dyadic bivariate wavelet) multipliers. We call the 3×3 matrix-valued function $A(s)=[f_{i,j}(s)]_{3\times 3}$, where $f_{i,j}$ are measurable functions, a dyadic bivariate matrix Fourier wavelet multiplier if the inverse Fourier transform of $A(s)(\widehat{\psi_1}(s),\widehat{\psi_2}(s),\widehat{\psi_3}(s))^\top=(\widehat{g_1}(s),\widehat{g_2}(s),\widehat{g_3}(s))^\top$ is a dyadic bivariate wavelet whenever (ψ_1,ψ_2,ψ_3) is any dyadic bivariate wavelet. We give some conditions for dyadic matrix bivariate wavelet multipliers. The results extended that of Z. Y. Li and X. L. Shi (2011). As an application, we construct some useful dyadic bivariate wavelets by using dyadic Fourier matrix wavelet multipliers and use them to image denoising.

Key Words: Multi-wavelets, Fourier multipliers, image denoising.

AMS Subject Classifications: 42C15, 46C05, 47B10

1 Introduction

One natural problem in wavelet theory concerns the construction of different wavelets. Naturally, one may attempt to construct new wavelets from a known one. This approach leads to the concept of wavelet multipliers [13, 30]. In the one dimensional case,

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the single 2 dilation orthogonal Fourier wavelet multipliers have been studied extensively and characterized completely [30]. In high dimensional case, any integer expansive matrix A with $|detA|=2$ dilation single orthogonal wavelet multipliers have been characterized completely [23–25]. The Parseval multi-wavelet frame multipliers in high dimensional case and single Parseval frame wavelet multipliers have been characterized completely [26,29]. One natural question is how to characterize orthogonal multivariate wavelet multipliers corresponding integer expansive dilation matrice A with $|detA|\neq 2$. Because there are many multivariate wavelets, this question becomes very complicated. We should find an easier case to discuss this problem. We know that characterizations of MRA wavelets play an key role in discussion on single wavelet multipliers in [23–25,30]. According to [22], the number of MRA wavelets corresponding dilation matrix A in high dimensional case equals $|detA|-1$. In this paper, we consider the uniform dilation matrix

$$2I_2=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

which is used as dilation matrix in most literature, dilation multivariate wavelet multipliers. There exist three $2I_2$-dilation MRA functions to construct a wavelet of $L^2(\mathbb{R}^2)$ (which is called a dyadic MRA bivariate wavelet by [22]). The main purpose of this paper is studying dyadic bivariate matrix Fourier wavelet multipliers in $L^2(\mathbb{R}^2)$. We will give some conditions for dyadic bivariate matrix Fourier wavelet multipliers and also give concrete examples. The results extend that in [27].

The rest of the paper is organized as follows. In the next section, we introduce notations and terms needed for this paper. Section 3 gives an example of a special dyadic MRA bivariate wavelet multiplier. In section 4, we discuss dyadic bivariate wavelet multipliers in $L^2(\mathbb{R}^2)$. Finally, we construct some useful dyadic bivariate wavelets by using multipliers and use them to image denoising.

2 Notations, definitions and preliminary results

Throughout, let

$$2I_2=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{s}=(s_1,s_2)\in\mathbb{R}^2.$$

Let $L^2(\mathbb{R}^2)$ be the set of all square Lebesgue integrable functions in $\mathbb{R}^2$. A $2I_2$-dilation orthogonal bivariate wavelet (or a dyadic bivariate wavelet or Parseval frame wavelet) is the function family $\Psi=\{\psi_1,\psi_2,\psi_3\}$ ($\psi_i\in L^2(\mathbb{R}^2)$, $i=1,2,3$) such that the set

$$\{2^n\psi_i(2^n\mathbf{t}-\ell): n\in\mathbb{Z},\quad \ell\in\mathbb{Z}^2,\quad i=1,2,3\}$$

forms an orthonormal basis or Parseval frame for $L^2(\mathbb{R}^2)$. For any function $f(t)\in L^1(\mathbb{R}^2)\bigcap L^2(\mathbb{R}^2)$, its Fourier transform is defined by

$$(\mathcal{F}f)(\mathbf{s})=\widehat{f}(\mathbf{s})=\frac{1}{2\pi}\int_{\mathbb{R}^2} f(\mathbf{t})e^{-i\mathbf{t}\circ\mathbf{s}}d\mu,$$

where μ denotes the Lebesgue measure in $\mathbb{R}^2$ and $\mathbf{t}\circ\mathbf{s}$ is the standard inner product of the vectors $\mathbf{s}$, $\mathbf{t}\in\mathbb{R}^2$. The inverse Fourier transform will be denoted by $\mathcal{F}^{-1}$.

We call the 3×3 matrix-valued function $A(s)=[f_{i,j}(s)]_{3\times 3}$, where $f_{i,j}$ are measurable functions, a dyadic bivariate matrix Fourier wavelet multiplier or dyadic bivariate Parseval matrix Fourier frame multiplier if the inverse Fourier transform of $A(s)(\widehat{\psi_1}(s),\widehat{\psi_2}(s),\widehat{\psi_3}(s))^\top=(\widehat{g_1}(s),\widehat{g_2}(s),\widehat{g_3}(s))^\top$ is a dyadic bivariate wavelet whenever (ψ_1,ψ_2,ψ_3) is a dyadic bivariate wavelet.

We denote $\phi_{j,\mathbf{n}}(\mathbf{s})=2^j\phi(2^j\mathbf{s}-\mathbf{n})$, $\forall j\in\mathbb{Z}$, $\mathbf{n}\in\mathbb{Z}^2$.

Definition 2.1. (see [15]) A sequence $\{V_j:j\in\mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}^2)$ is called a $2I_2$-dilation multi-resolution analysis (or dyadic MRA for short) if the following hold:
(i) $V_j\subset V_{j+1}$, $\forall j\in\mathbb{Z}$;
(ii) $\bigcap_{j\in\mathbb{Z}}V_j=\{0\}$, $\overline{\bigcup_{j\in\mathbb{Z}}V_j}=L^2(\mathbb{R}^2)$;
(iii) $f(\mathbf{t})\in V_j$ if and only if $f(2^{-j}\mathbf{t})\in V_0$ for $j\in\mathbb{Z}$;
(iv) There exists $\phi(\mathbf{t})$ in V_0 such that $\{\phi(\mathbf{t}-\ell):\ \ell\in\mathbb{Z}^2\}$ is an orthonormal basis for V_0.

The function $\phi(\mathbf{t})$ defined in (iv) above is called a dyadic scaling function for the MRA. In our case, it is known that three $2I_2$-dilation wavelet functions ψ_1,ψ_2 and ψ_3 can be derived from the above $2I_2$-dilation MRA by [22]. More precisely, the multiresolution structure of the V_j implies that the corresponding scaling function ϕ satisfies the following two scaling equation

$$\phi(\mathbf{s})=\sum_{\mathbf{n}\in\mathbb{Z}^2}h_{\mathbf{n}}(2\mathbf{s}-\mathbf{n})$$

for some sequence $(h_{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}^2}$. This scaling equation has the following frequency domain expression,

$$\widehat{\phi}(2\mathbf{s})=m_0(\mathbf{s})\widehat{\phi}(\mathbf{s}).$$

Orthonormality of the $\phi_{0,n}$ forces the trigonometric polynomial

$$m_0(\mathbf{s})=\frac{1}{2}\sum_{n\in\mathbb{Z}^2}h_{\mathbf{n}}e^{-i(\mathbf{n}\circ\mathbf{s})}$$

to satisfy

$$|m_0(\mathbf{s})|^2+|m_0(\mathbf{s}+(\pi,0))|^2+|m_0(\mathbf{s}+(0,\pi))|^2+|m_0(\mathbf{s}+(\pi,\pi))|^2=1.$$

$m_0(\mathbf{s})$ is called a dyadic low pass filter.

To construct an orthonormal basis of wavelets corresponding to this MRA, one has to find three functions ψ_1, ψ_2, ψ_3 in V_1, orthogonal to V_0 and such that the three spaces spanned by their respective integer translates are orthogonal; Moreover, the $\psi_i(\mathbf{s}-\mathbf{n})$ $(i=1,2,3)$ should also be orthonormal for each fixed i. This implies that

$$\widehat{\psi}_i(\mathbf{s})=m_i\Big(\frac{1}{2}\mathbf{s}\Big)\widehat{\phi}\Big(\frac{1}{2}\mathbf{s}\Big),\quad i=1,2,3,$$

where m_1, m_2, m_3 (are called dyadic high pass filters) are such that the matrix

$$\begin{pmatrix} m_0(\mathbf{s}) & m_1(\mathbf{s}) & m_2(\mathbf{s}) & m_3(\mathbf{s}) \\ m_0(\mathbf{s}+(\pi,0)) & m_1(\mathbf{s}+(\pi,0)) & m_2(\mathbf{s}+(\pi,0)) & m_3(\mathbf{s}+(\pi,0)) \\ m_0(\mathbf{s}+(0,\pi)) & m_1(\mathbf{s}+(0,\pi)) & m_2(\mathbf{s}+(0,\pi)) & m_3(\mathbf{s}+(0,\pi)) \\ m_0(\mathbf{s}+(\pi,\pi)) & m_1(\mathbf{s}+(\pi,\pi)) & m_2(\mathbf{s}+(\pi,\pi)) & m_3(\mathbf{s}+(\pi,\pi)) \end{pmatrix} \tag{2.1}$$

is unitary. We know that all the filters m_0, m_1, m_2, and m_3 are $2\pi\mathbb{Z}^2$ periodic functions by [15,22].

Here we explain the expression of the unitary matrix (2.1). Let $\mathcal{K}$ be the digit set (the representatives of co-sets) of group $\mathbb{Z}^2/2\mathbb{Z}^2$. It is obvious that $\mathcal{K}$ is not unique. For example $\mathcal{K}$ can be chosen $\{k_0=(0,0),k_1=(1,0),k_2=(0,1),k_3=(1,1)\}$, $\{k_0=(0,0),k_1=(1,0),k_2=(0,1),k_3=(-1,-1)\}$ or $\{k_0=(0,0),k_1=(1,1),k_2=(0,1),k_3=(1,2)\}$. So the general expression of the unitary matrix for some $\mathcal{K}$ is as follows

$$\begin{pmatrix} m_0(\mathbf{s}) & m_1(\mathbf{s}) & m_2(\mathbf{s}) & m_3(\mathbf{s}) \\ m_0(\mathbf{s}+\ell_1) & m_1(\mathbf{s}+\ell_1) & m_2(\mathbf{s}+\ell_1) & m_3(\mathbf{s}+\ell_1) \\ m_0(\mathbf{s}+\ell_2) & m_1(\mathbf{s}+\ell_2) & m_2(\mathbf{s}+\ell_2) & m_3(\mathbf{s}+\ell_2) \\ m_0(\mathbf{s}+\ell_3) & m_1(\mathbf{s}+\ell_3) & m_2(\mathbf{s}+\ell_3) & m_3(\mathbf{s}+\ell_3) \end{pmatrix},$$

where $\ell_1=\pi k_1$, $\ell_2=\pi k_2$, $\ell_3=\pi k_3$. Here we just choose $\mathcal{K}=\{k_0=(0,0),k_1=(1,0),k_2=(0,1),k_3=(1,1)\}$, which is as the same as in [15,22] for discussions. We can verify all the discussions are valid for a general $\mathcal{K}$.

Lemma 2.1 (see [4–6]). *$\Psi=\{\psi_1,\psi_2,\psi_3\}$ is a dyadic bivariate wavelet iff the following conditions hold*
(i) $\|\psi_i\|_2=1$, $i=1,2,3$;
(ii) $\sum_{i=1}^3\sum_{j\in\mathbb{Z}}|\widehat{\psi}_i(2^j\mathbf{s})|^2=1/(2\pi)^2$ *a.e.*
(iii) $\sum_{i=1}^3\sum_{j=0}^{\infty}\widehat{\psi}_i(2^j\mathbf{s})\overline{\widehat{\psi}_i(2^j(\mathbf{s}+2\pi\ell))}=0$ *a.e.* $\forall\ell\in\mathbb{Z}^2\backslash 2\mathbb{Z}^2$.

Lemma 2.2 (see [7–9]). *A dyadic bivariate wavelet $\Psi=\{\psi_1,\psi_2,\psi_3\}$ is a dyadic MRA bivariate wavelet with a single scaling function iff*

$$D_\Psi(\mathbf{s})=\sum_{i=1}^3\sum_{n=1}^{\infty}\sum_{\ell\in\mathbb{Z}^2}|\widehat{\psi}_i(2^n(\mathbf{s}+2\pi\ell))|^2=\frac{1}{(2\pi)^2}\ \ a.e. \tag{2.2}$$

Lemma 2.3 (see [15]). *ϕ is a dyadic scaling function for an MRA iff the following conditions hold*
(i) $\sum_{\ell\in\mathbb{Z}^2}|\widehat{\phi}(\mathbf{s}+2\pi\ell)|^2=1/(2\pi)^2$ *a.e.;*
(ii) $\lim_{j\to\infty}|\widehat{\phi}(2^{-j}\mathbf{s})|=1/2\pi$ *a.e.*
(iii) there exists a $2\pi\mathbb{Z}^2$ *periodic function* $m(\mathbf{s})\in L^2([-\pi,\pi)^2)$ *such that* $\widehat{\phi}(2\mathbf{s})=m(\mathbf{s})\widehat{\phi}(\mathbf{s})$.

3 Example of a dyadic bivariate matrix Fourier wavelet multiplier

Example 3.1. Let

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad (a_{ij}\in\mathbb{C}),$$

be unitary. Then we can easily check that A is a dyadic bivariate matrix Fourier multiplier.

In fact, let $(\psi_1(t),\psi_2(t),\psi_3(t))$ be any dyadic bivariate wavelet for $L^2(\mathbb{R}^2)$. We show that the inverse Fourier transform of $A(\widehat{\psi_1},\widehat{\psi_2},\widehat{\psi_3})^T$ is a dyadic bivariate wavelet for $L^2(\mathbb{R}^2)$.

In what follows we will use the same symbols of inner product $\langle,\rangle$ and norm $\|\cdot\|$ for both $L^2(\mathbb{R}^2)$ and $\mathbb{C}^2$ and the readers can easily distinguish them from the context.

By the definition of Parseval frame and [11] we have for every $f\in L^2(\mathbb{R})$ that

$$\begin{aligned}\|f\|^2&=\sum_{n,\ell\in\mathbb{Z}}|\langle f,\hat{D}^n\hat{T}^\ell\widehat{\psi_1}\rangle|^2+\sum_{n',\ell'\in\mathbb{Z}}|\langle f,\hat{D}^{n'}\hat{T}^{\ell'}\widehat{\psi_2}\rangle|^2+\sum_{n'',\ell''\in\mathbb{Z}}|\langle f,\hat{D}^{n''}\hat{T}^{\ell''}\widehat{\psi_3}\rangle|^2\\
&=\sum_{n,\ell\in\mathbb{Z}}\left\|\begin{pmatrix}\langle f,\hat{D}^n\hat{T}^\ell\widehat{\psi_1}\rangle\\ \langle f,\hat{D}^n\hat{T}^\ell\widehat{\psi_2}\rangle\\ \langle f,\hat{D}^n\hat{T}^\ell\widehat{\psi_3}\rangle\end{pmatrix}\right\|^2.\end{aligned}$$

Since A is a unitary matrix acting on $\mathbb{C}^2$, we obtain that

$$\begin{aligned}&\sum_{n,\ell\in\mathbb{Z}}|\langle f,a_{11}\hat{D}^n\hat{T}^\ell\widehat{\psi_1}+a_{12}\hat{D}^n\hat{T}^\ell\widehat{\psi_2}+a_{13}\hat{D}^n\hat{T}^\ell\widehat{\psi_3}\rangle|^2\\
&+\sum_{n',\ell'\in\mathbb{Z}}|\langle f,a_{21}\hat{D}^{n'}\hat{T}^{\ell'}\widehat{\psi_1}+a_{22}\hat{D}^{n'}\hat{T}^{\ell'}\widehat{\psi_2}+a_{23}\hat{D}^{n'}\hat{T}^{\ell'}\widehat{\psi_3}\rangle|^2\\
&+\sum_{n'',\ell''\in\mathbb{Z}}|\langle f,a_{31}\hat{D}^{n''}\hat{T}^{\ell''}\widehat{\psi_1}+a_{32}\hat{D}^{n''}\hat{T}^{\ell''}\widehat{\psi_2}+a_{33}\hat{D}^{n''}\hat{T}^{\ell''}\widehat{\psi_3}\rangle|^2\\
=&\sum_{n,\ell\in\mathbb{Z}}\left\|\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&2_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}\begin{pmatrix}\langle f,\hat{D}^n\hat{T}^\ell\widehat{\psi_1}\rangle\\ \langle f,\hat{D}^n\hat{T}^\ell\widehat{\psi_2}\rangle\\ \langle f,\hat{D}^n\hat{T}^\ell\widehat{\psi_3}\rangle\end{pmatrix}\right\|^2\\
=&\sum_{n,\ell\in\mathbb{Z}}\left\|\begin{pmatrix}\langle f,\hat{D}^n\hat{T}^\ell\widehat{\psi_1}\rangle\\ \langle f,\hat{D}^n\hat{T}^\ell\widehat{\psi_2}\rangle\\ \langle f,\hat{D}^n\hat{T}^\ell\widehat{\psi_3}\rangle\end{pmatrix}\right\|^2=\|f\|^2.\end{aligned}$$

Thus the inverse Fourier transform of $A(\widehat{\psi_1},\widehat{\psi_2},\widehat{\psi_3})^T$ is a dyadic bivariate wavelet for $L^2(\mathbb{R})$ and so A is a dyadic bivariate matrix Fourier multiplier for dyadic bivariate wavelets.

4 Dyadic bivariate matrix Fourier wavelet multipliers

In this section, we discuss the some conditions for Dyadic bivariate matrix Fourier wavelet Multipliers.

Theorem 4.1. *Let $A(s)=(f_{ij}(s))_{3\times 3}$ with $f_{ij}\in L^\infty(\mathbb{R}^2)$, $i,j=1,2,3$. Then A is a matrix Fourier dyadic bivariate multiplier for dyadic bivariate wavelets if*
(i) Each row and each column of $A(s)$ has only one non-zero entry $f_{ij}(s)$ with $|f_{ij}(s)|^2=1$ a.e. $s\in\mathbb{R}^2$;
(ii) $A(2s)A^*(s)$ is $2\pi\mathbb{Z}^2$-periodic;
(iii) $A^*(s)A(s+2\pi q)=\lambda(s)I_{3\times 3}$, where $|\lambda(s)|=1$ a.e. and $q\in\mathbb{Z}^2/2\mathbb{Z}^2$.

Proof. By the assumption of (i), A is unitary. We only to show that for any dyadic bivariate wavelet $(\psi_1,\psi_2,\psi_3)^\top$,

$$\begin{pmatrix}\widehat{g_1}(s)\\ \widehat{g_2}(s)\\ \widehat{g_3}(s)\end{pmatrix}=A(s)\begin{pmatrix}\widehat{\psi_1}(s)\\ \widehat{\psi_2}(s)\\ \widehat{\psi_3}(s)\end{pmatrix}=\begin{pmatrix}f_{1k_1}(s)\widehat{\psi_{k_1}}(s)\\ f_{2k_2}(s)\widehat{\psi_{k_2}}(s)\\ f_{1k_3}(s)\widehat{\psi_{k_3}}(s)\end{pmatrix}$$

satisfies the three equations in Lemma 2.1, where $k_1\neq k_2\neq k_3$.

(i) Since $\|\widehat{\psi}_i\|=\|\psi_i\|=1$, we have

$$\begin{aligned}\|g_i\|^2&=\|\hat{g}_i\|^2=\langle\hat{g}_i,\hat{g}_i\rangle=\int_{\mathbb{R}^2}f_{ik_i}(s)\overline{f_{ik_i}(s)}|\widehat{\psi_{k_i}}(s)|^2ds\\ &=\int_{\mathbb{R}^2}|\widehat{\psi_{k_i}}(s)|^2=\|\widehat{\psi_{k_i}}(s)\|^2=1,\quad i=1,2,3.\end{aligned}$$

So (i) of Lemma 2.1 holds.

(ii) For

$$\begin{aligned}\sum_{i=1}^{3}\sum_{j\in\mathbb{Z}}|\hat{g}_j(2^js)|^2&=\sum_{j\in\mathbb{Z}}\left\|\begin{pmatrix}f_{11}(2^js)&f_{12}(2^js)&f_{13}(2^js)\\ f_{21}(2^js)&f_{22}(2^js)&f_{23}(2^js)\\ f_{31}(2^js)&f_{32}(2^js)&f_{33}(2^js)\end{pmatrix}\begin{pmatrix}\hat{\psi}_1(2^js)\\ \hat{\psi}_2(2^js)\\ \hat{\psi}_3(2^js)\end{pmatrix}\right\|^2\\ &=\sum_{j\in\mathbb{Z}}\left\|A(2^js)\begin{pmatrix}\hat{\psi}_1(2^js)\\ \hat{\psi}_2(2^js)\\ \hat{\psi}_3(2^js)\end{pmatrix}\right\|^2\\ &=\sum_{j\in\mathbb{Z}}\left\|\begin{pmatrix}\hat{\psi}_1(2^js)\\ \hat{\psi}_2(2^js)\\ \hat{\psi}_3(2^js)\end{pmatrix}\right\|^2=\sum_{i=1}^{3}\sum_{j\in\mathbb{Z}}|\hat{\psi}_i(2^js)|^2=\frac{1}{(2\pi)^2}.\end{aligned}$$

Thus equation (ii) of Lemma 2.1 holds.

Now we verify equation (iii) of Lemma 2.1. Set

$$\Delta(s,q)=\sum_{i=1}^{3}\sum_{j\geq 0}\hat{g}_i(2^js)\overline{\hat{g}_i(2^j(s+2\pi\ell))}=\sum_{j\geq 0}\left\langle\begin{pmatrix}\hat{g}_1(2^js)\\ \hat{g}_2(2^js)\\ \hat{g}_3(2^js)\end{pmatrix},\begin{pmatrix}\hat{g}_1(2^j(s+2\pi\ell))\\ \hat{g}_2(2^j(s+2\pi\ell))\\ \hat{g}_3(2^j(s+2\pi\ell))\end{pmatrix}\right\rangle.$$

Then

$$\begin{aligned}\Delta(s,q)=&\sum_{j\geq 0}\left\langle A(2^js)\begin{pmatrix}\hat{\psi}_1(2^js)\\ \hat{\psi}_2(2^js)\\ \hat{\psi}_3(2^js)\end{pmatrix},A(2^j(s+2\pi\ell))\begin{pmatrix}\hat{\psi}_1(2^j(s+2\pi\ell))\\ \hat{\psi}_2(2^j(s+2\pi\ell))\\ \hat{\psi}_3(2^j(s+2\pi\ell))\end{pmatrix}\right\rangle\\ =&\sum_{j\geq 0}\left\langle \begin{pmatrix}\hat{\psi}_1(2^js)\\ \hat{\psi}_2(2^js)\\ \hat{\psi}_3(2^js)\end{pmatrix},A^*(2^js)A(2^j(s+2\pi\ell))\begin{pmatrix}\hat{\psi}_1(2^j(s+2\pi\ell))\\ \hat{\psi}_2(2^j(s+2\pi\ell))\\ \hat{\psi}_3(2^j(s+2\pi\ell))\end{pmatrix}\right\rangle.\end{aligned}$$

For $j\geq 1$, set $t=2^{j-1}s$. By the assumption (ii) of the theorem, we get

$$\begin{aligned}A(2^js)A^*(2^{j-1}s)=&A(2(2^{j-1}s))A^*(2^{j-1}s)=A(2t)A^*(t)\\ =&A(2(t+2^{j-1}2\pi\ell))A^*(t+2^{j-1}2\pi\ell)\\ =&A(2(2^{j-1}s+2^{j-1}\cdot 2\pi\ell)A^*(2^{j-1}s+2^{j-1}\cdot 2\pi\ell)\\ =&A(2^j(s+2\pi\ell))A^*(2^{j-1}(s+2\pi\ell)).\end{aligned}$$

This implies that

$$A^*(2^js)A(2^j(s+2\pi\ell))=A^*(2^{j-1}s)A(2^{j-1}(s+2\pi\ell))\cdots=A^*(s)A(s+2\pi\ell).$$

By the assumption (iii), we obtain

$$\begin{aligned}\Delta(s,q)=&\sum_{j\geq 0}\sum_{j\geq 0}\left\langle \begin{pmatrix}\hat{\psi}_1(2^js)\\ \hat{\psi}_2(2^js)\\ \hat{\psi}_3(2^js)\end{pmatrix},A^*(s)A(s+2\pi\ell)\begin{pmatrix}\hat{\psi}_1(2^j(s+2\pi\ell))\\ \hat{\psi}_2(2^j(s+2\pi\ell))\\ \hat{\psi}_3(2^j(s+2\pi\ell))\end{pmatrix}\right\rangle\\ =&\overline{\lambda}(s)\sum_{j\geq 0}\langle\cdots,\cdots\rangle\\ =&\overline{\lambda}(s)\sum_{i=1}^{3}\sum_{j\geq 0}\hat{\psi}_i(2^js)\overline{\hat{\psi}_i(2^j(s+2\pi\ell))}=0\end{aligned}$$

for a.e. $s\in\mathbb{R}^2$ and all $\ell\in\mathbb{Z}^2\backslash 2\mathbb{Z}^2$, and hence $A(s)$ is a dyadic bivariate matrix Fourier multiplier, as claimed. □

Remark 4.1. If a unitary matrix $A(s)$ satisfies (ii) and (iii) in Theorem 4.1, by using the same argument of Theorem 1.4 in [26], $A(s)$ is a Parseval dyadic bivariate matrix Fourier frame multiplier.

Remark 4.2. The above result extends the result in [27].

We call the 3×3 matrix-valued function $A(s)=[f_{i,j}(s)]_{3\times 3}$, where $f_{i,j}$ are measurable functions, a dyadic bivariate matrix Fourier MRA wavelet multiplier if the inverse Fourier transform of $A(s)(\widehat{\psi_1}(s),\widehat{\psi_2}(s),\widehat{\psi_3}(s))^\top=(\widehat{g_1}(s),\widehat{g_2}(s),\widehat{g_3}(s))^\top$ is a dyadic bivariate MRA wavelet whenever (ψ_1,ψ_2,ψ_3) is a dyadic bivariate MRA wavelet.

Theorem 4.2. *Let $A(s)=(f_{ij}(s))_{3\times 3}$ with $f_{ij}\in L^\infty(\mathbb{R}^2)$, $i,j=1,2,3$. Then A is a dyadic bivariate matrix Fourier MRA wavelet multiplier for dyadic bivariate MRA wavelets if*
(i) Each row and each column of $A(s)$ has only one non-zero entry $f_{ij}(s)$ with $|f_{ij}(s)|^2=1$ a.e. $s\in\mathbb{R}^2$;
(ii) $A(2s)A^*(s)$ is $2\pi\mathbb{Z}^2$-periodic;
(iii) $A^*(s)A(s+2\pi q)=\lambda(s)I_{3\times 3}$, where $|\lambda(s)|=1$ a.e. and $q\in\mathbb{Z}^2/2\mathbb{Z}^2$.

Proof. Let (ψ_1,ψ_2,ψ_3) be any dyadic bivariate MRA wavelet. By Theorem 4.1, $(g_1(s),g_2(s),g_3(s))^T$ is defined by

$$\begin{pmatrix} \widehat{g_1}(s) \\ \widehat{g_2}(s) \\ \widehat{g_3}(s) \end{pmatrix} = A(s) \begin{pmatrix} \widehat{\psi_1}(s) \\ \widehat{\psi_2}(s) \\ \widehat{\psi_3}(s) \end{pmatrix}$$

is a dyadic bivariate wavelet. It sufficiency to show that it is a dyadic bivariate MRA wavelet. By the assumption of (i),

$$\begin{aligned}\Delta(s) &= \sum_{i=1}^{3}\sum_{n=1}^{\infty}\sum_{\ell\in\mathbb{Z}^2} |\widehat{g_i}(2^n(\mathbf{s}+2\pi\ell))|^2 \\ &= \sum_{n=1}^{\infty}\sum_{\ell\in\mathbb{Z}^2}\left\| A(2^n(s+2\pi\ell)) \begin{pmatrix} \hat{\psi}_1(2^n(s+2\pi\ell)) \\ \hat{\psi}_2(2^n(s+2\pi\ell)) \\ \hat{\psi}_3(2^n(s+2\pi\ell)) \end{pmatrix} \right\|^2 \\ &= \sum_{n=1}^{\infty}\sum_{\ell\in\mathbb{Z}^2}\left\| \begin{pmatrix} \hat{\psi}_1(2^n(s+2\pi\ell)) \\ \hat{\psi}_2(2^n(s+2\pi\ell)) \\ \hat{\psi}_3(2^n(s+2\pi\ell)) \end{pmatrix} \right\|^2 = \frac{1}{(2\pi)^2}.\end{aligned}$$

By Lemma 2.2, $(g_1(s),g_2(s),g_3(s))^T$ is a dyadic bivariate MRA wavelet. $\square$

5 Dyadic bivariate wavelets examples and applications to image denoising

Finally we provide several examples demonstrating that matrix Fourier multipliers can be applied to a simple dyadic bivariate wavelet to obtain some more interesting and nice ones. In particular, we hope that these examples may indicate good potentials of using matrix Fourier multipliers to improve the smoothness (in the frequency domain) of generating functions for a dyadic bivariate wavelet.

We will use the Dyadic MRA Shannon Type wavelet to construct different dyadic bivariate (or Parseval) wavelets and apply them to image denoising.

Example 5.1. The Dyadic MRA Shannon Type wavelet.

Let $\Omega=[-\pi,\pi]^2$. Let

$$\hat{\varphi}(\mathbf{s})=\frac{1}{2\pi}\chi_\Omega.$$

Define a $2\pi\mathbb{Z}^2$ periodic function $m_0(\mathbf{s})$ as $m_0(\mathbf{s})|_\Omega = \chi_{F_0}$. By Lemma 2.3, φ is an A-dilation scaling function satisfying

$$\hat{\varphi}(\mathbf{s}) = m_0((A^\tau)^{-1}\mathbf{s})\hat{\varphi}((A^\tau)^{-1}\mathbf{s}).$$

Let

$$\hat{\psi}_1(\mathbf{s}) = \frac{1}{2\pi}\chi_I, \quad \hat{\psi}_2(\mathbf{s}) = \frac{1}{2\pi}\chi_{II}, \quad \hat{\psi}_3(\mathbf{s}) = \frac{1}{2\pi}\chi_{III}.$$

Define three $2\pi\mathbb{Z}^2$ periodic functions $m_1(\mathbf{s})$, $m_2(\mathbf{s})$, $m_3(\mathbf{s})$ as $m_1(\mathbf{s})|_\Omega = \chi_{F_1}, m_2(\mathbf{s})|_\Omega = \chi_{F_2}, m_3(\mathbf{s})|_\Omega = \chi_{F_3}$. Where $F_0 = (A^\tau)^{-1}\Omega$, the central square part of $[-\pi,\pi]^2$ in Fig. 1 and $F_1 = (A^\tau)^{-1}I$, $F_2 = (A^\tau)^{-1}II$, $F_3 = (A^\tau)^{-1}III$ (see Fig. 1).

We can verify that m_0, m_1, m_2 and m_3 satisfy the condition that matrix (2.1) is unitary. Moreover, the family $\{\psi_1,\psi_2,\psi_3\}$ is a dyadic bivariate wavelet with scaling function $\varphi(\mathbf{s})$ (which is called a dyadic Shannon type wavelet). We call m_0 a dyadic Shannon type low pass filter.

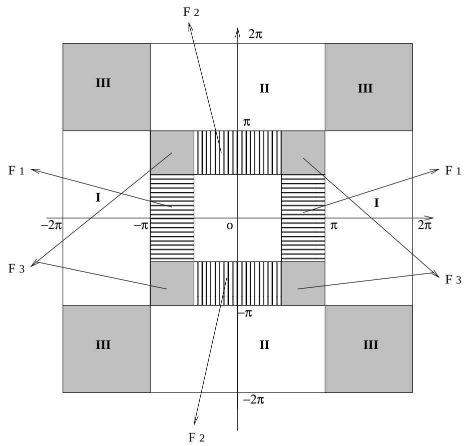

Figure 1: The supports of the dyadic Shannon type filters m_i $(i=0,1,2,3)$ in a simple period and the supports of $\hat{\varphi}$, $\widehat{\psi}_i$, $(i=1,2,3)$.

We use u_1, u_2 and u_3 denote $(1,0)^\top$, $(0,1)^\top$ and $(1,1)^\top$.

Example 5.2. Let

$$A_1(\mathbf{s}) = \begin{pmatrix} \sin(\mathbf{s}\cdot u_1) & 0 & \cos(\mathbf{s}\cdot u_1) \\ 0 & 1 & 0 \\ \cos(\mathbf{s}\cdot u_1) & 0 & -\sin(\mathbf{s}\cdot u_1) \end{pmatrix} = \begin{pmatrix} \sin(s_1) & 0 & \cos(s_1) \\ 0 & 1 & 0 \\ \cos(s_1) & 0 & -\sin(s_1) \end{pmatrix}.$$

Then $A_1(\mathbf{s})$ is a dyadic bivariate Parseval wavelet multiplier by Remark 4.1. Thus we can obtain a dyadic bivariate Parseval wavelet (η_1,η_2,η_3) by using the Dyadic MRA Shannon

Type wavelet as follows (see Fig. 2)

$$\begin{pmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \\ \hat{\eta}_3 \end{pmatrix} = A_1(\mathbf{s}) \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \\ \hat{\psi}_3 \end{pmatrix} = \begin{pmatrix} \sin(s_1)\hat{\psi}_1 + \cos(s_1)\hat{\psi}_3 \\ \hat{\psi}_2 \\ \cos(s_1)\hat{\psi}_1 - \sin(s_1)\hat{\psi}_3 \end{pmatrix}.$$

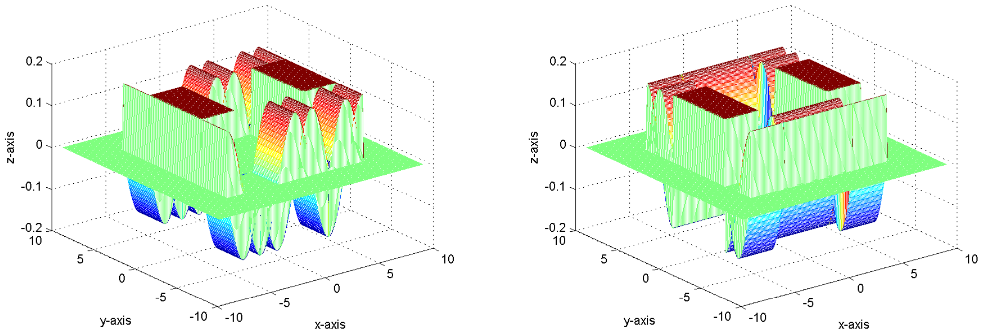

Figure 2: Wavelet $(\hat{\eta}_1,\hat{\eta}_2,\hat{\eta}_3)$ and wavelet $(\widehat{u_1},\widehat{u_2},\widehat{u_3})$.

Example 5.3. Let

$$A_2(\mathbf{s}) = \begin{pmatrix} \sin(\mathbf{s}\cdot u_2) & 0 & \cos(\mathbf{s}\cdot u_2) \\ 0 & 1 & 0 \\ \cos(\mathbf{s}\cdot u_2) & 0 & -\sin(\mathbf{s}\cdot u_2) \end{pmatrix} = \begin{pmatrix} \sin(s_2) & 0 & \cos(s_2) \\ 0 & 1 & 0 \\ \cos(s_2) & 0 & -\sin(s_2) \end{pmatrix}.$$

Then $A_2(\mathbf{s})$ is a dyadic bivariate Parseval wavelet multiplier by Remark 4.1. Thus (u_1,u_2,u_3) (see Fig. 2) is a dyadic bivariate Parseval wavelet defined by the following

$$\begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{pmatrix} = A_2(\mathbf{s}) \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \\ \hat{\psi}_3 \end{pmatrix} = \begin{pmatrix} \sin(s_2)\hat{\psi}_1 + \cos(s_2)\hat{\psi}_3 \\ \hat{\psi}_2 \\ \cos(s_2)\hat{\psi}_1 - \sin(s_2)\hat{\psi}_3 \end{pmatrix}.$$

Example 5.4. Let

$$A_3(\mathbf{s}) = \begin{pmatrix} \sin(\mathbf{s}\cdot u_3) & 0 & \cos(\mathbf{s}\cdot u_3) \\ 0 & 1 & 0 \\ \cos(\mathbf{s}\cdot u_3) & 0 & -\sin(\mathbf{s}\cdot u_3) \end{pmatrix} = \begin{pmatrix} \sin(s_1+s_2) & 0 & \cos(s_1+s_2) \\ 0 & 1 & 0 \\ \cos(s_1+s_2) & 0 & -\sin(s_1+s_2) \end{pmatrix}.$$

Then $A_3(\mathbf{s})$ is a dyadic bivariate Parseval wavelet multiplier Remark 4.1. So we obtain a dyadic bivariate Parseval wavelet (k_1,k_2,k_3) (see Fig. 3) defined by the following

$$\begin{pmatrix} \hat{k}_1 \\ \hat{k}_2 \\ \hat{k}_3 \end{pmatrix} = A_3(\mathbf{s}) \begin{pmatrix} \hat{\psi}_1 \\ \hat{\psi}_2 \\ \hat{\psi}_3 \end{pmatrix} = \begin{pmatrix} \sin(s_2)\hat{\psi}_1 + \cos(s_2)\hat{\psi}_3 \\ \hat{\psi}_2 \\ \cos(s_2)\hat{\psi}_1 - \sin(s_2)\hat{\psi}_3 \end{pmatrix}.$$

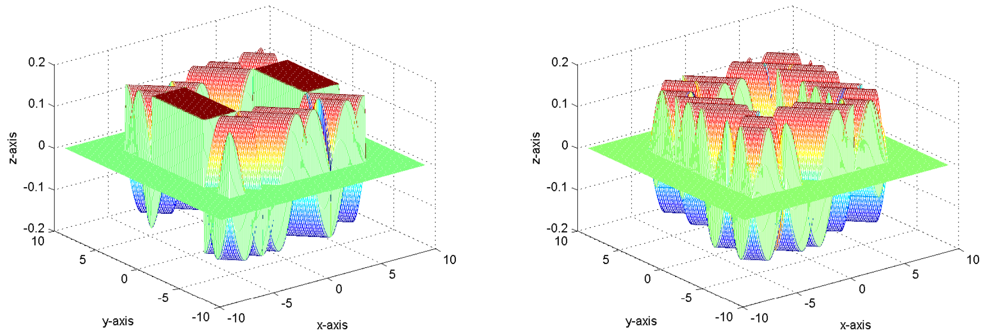

Figure 3: Wavelet $(\widehat{k_1},\widehat{k_2},\widehat{k_3})$ and wavelet $(\hat{r_1},\hat{r_2},\hat{r_3})$.

Example 5.5. Let

$$A_4(\mathbf{s})=\begin{pmatrix} \sin(\mathbf{s}\cdot u_3) & \cos^2(\mathbf{s}\cdot u_3) & \cos(\mathbf{s}\cdot u_3)\sin(\mathbf{s}\cdot u_3) \\ 0 & \sin(\mathbf{s}\cdot u_3) & -\cos(\mathbf{s}\cdot u_3) \\ -\cos(\mathbf{s}\cdot u_3) & \cos(\mathbf{s}\cdot u_3)\sin(\mathbf{s}\cdot u_3) & \sin^2(\mathbf{s}\cdot u_3) \end{pmatrix}$$
$$=\begin{pmatrix} \sin(s_1+s_2) & \cos^2((s_1+s_2) & \cos(s_1+s_2)\sin(s_1+s_2) \\ 0 & \sin(s_1+s_2) & -\cos(s_1+s_2) \\ -\cos(s_1+s_2) & \cos(s_1+s_2)\sin(s_1+s_2) & \sin^2(s_1+s_2) \end{pmatrix}.$$

Then $A_4(\mathbf{s})$ is a dyadic bivariate Parseval wavelet multiplier by Remark 4.1. So we obtain a dyadic Parseval wavelet (r_1,r_2,r_3) (see Fig. 3) defined by the following

$$\begin{pmatrix} \hat{r_1} \\ \hat{r_2} \\ \hat{r_3} \end{pmatrix}=A_4(\mathbf{s})\begin{pmatrix} \hat{\psi_1} \\ \hat{\psi_2} \\ \hat{\psi_3} \end{pmatrix}$$
$$=\begin{pmatrix} \sin(s_1+s_2)\hat{\psi_1}+\cos^2((s_1+s_2)\hat{\psi_2}+\cos(s_1+s_2)\sin(s_1+s_2)\hat{\psi_3} \\ \sin(s_1+s_2)\hat{\psi_2}-\sin(s_1+s_2)\hat{\psi_3} \\ -\cos(s_1+s_2)\hat{\psi_1}+\cos(s_1+s_2)\sin(s_1+s_2)\hat{\psi_2}+\sin^2(s_1+s_2)\hat{\psi_3} \end{pmatrix}.$$

Example 5.6. Let

$$A_5(\mathbf{s})=\begin{pmatrix} 0 & 0 & e^{i(\mathbf{s}\cdot u_1)} \\ 0 & 1 & 0 \\ e^{-i(\mathbf{s}\cdot u_3)} & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & e^{is_1} \\ 0 & 1 & 0 \\ e^{-i(s_1+s_2)} & 0 & 0 \end{pmatrix}.$$

Then $A_5(\mathbf{s})$ is a dyadic bivariate wavelet multiplier by Theorem 4.1. Thus we can get the dyadic bivariate wavelet (v_1,v_2,v_3) (see Fig. 4), which is also a MRA dyadic bivariate

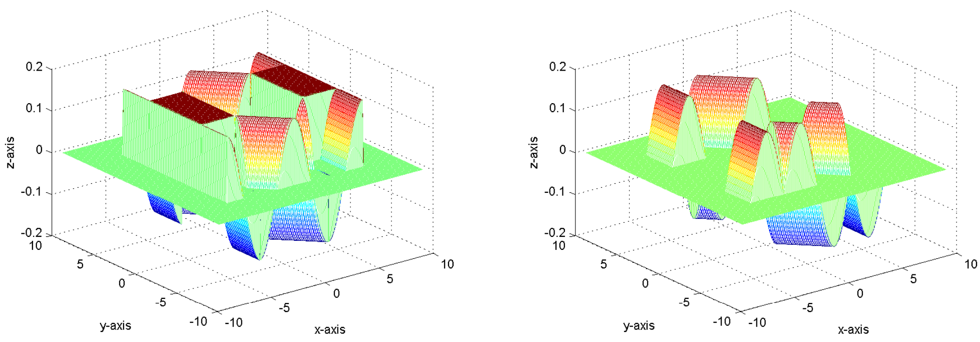

Figure 4: The real part of wavelet $(\hat{v}_1,\hat{v}_2,\hat{v}_3)$ and the imaginary part of wavelet $(\hat{v}_1,\hat{v}_2,\hat{v}_3)$.

wavelet by Theorem 4.2, by the following

$$\begin{aligned}\begin{pmatrix}\hat{v}_1\\ \hat{v}_2\\ \hat{v}_3\end{pmatrix} &= A_5(\mathbf{s})\begin{pmatrix}\hat{\psi}_1\\ \hat{\psi}_2\\ \hat{\psi}_3\end{pmatrix} = \begin{pmatrix}e^{is_1}\hat{\psi}_3\\ \hat{\psi}_2\\ e^{-i(s_1+s_2)}\hat{\psi}_1\end{pmatrix}\\ &= \begin{pmatrix}\cos(s_1)\hat{\psi}_3\\ \hat{\psi}_2\\ \cos(s_1+s_2)\hat{\psi}_1\end{pmatrix} + i\begin{pmatrix}\sin(s_1)\hat{\psi}_3\\ 0\\ -\sin(s_1+s_2)\hat{\psi}_1\end{pmatrix}.\end{aligned}$$

Example 5.7. Now we use dyadic bivariate wavelets (η_1,η_2,η_3), (u_1,u_2,u_3) and (k_1,k_2,k_3) to image denoising. Then we contrast the processed results with the results by using Sym4 wavelet.

Table 1: The comparative results.

Wavelet	RMSE	SNR
Sym4	0.0936	6.4349
(η_1,η_2,η_3)	0.0631	9.8652
(u_1,u_2,u_3)	0.0642	9.7089
(k_1,k_2,k_3)	0.0674	9.2927

The quantity is measured by the root mean squared error (RMSE), the smaller it is, the better the quantity is

$$RMSE = \frac{1}{MN}\sqrt{\sum_{i=1}^{M}\sum_{i=1}^{N}(X'(i,j)-X(i,j))^2}.$$

The signal-to-noise ratio(SNR) with decibels [db] as unit. The higher it is, the better the quantity is

$$SNR = 10\lg\frac{\sum_{i=1}^{M}\sum_{i=1}^{N}(X(i,j)-\overline{X(i,j)})^2}{\sum_{i=1}^{M}\sum_{i=1}^{N}(X'(i,j)-X(i,j))^2}.$$

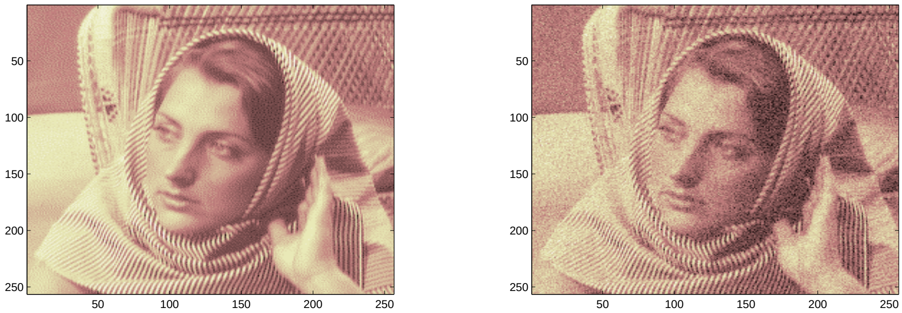

Figure 5: Original image and the noised image.

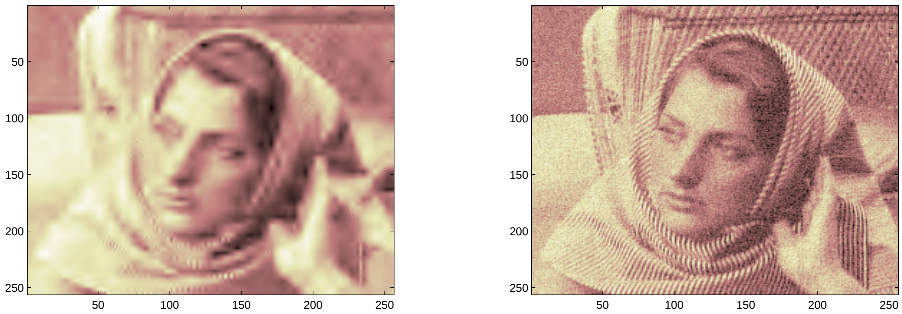

Figure 6: The denoising image by Sym4 and the denoising image by (η_1,η_2,η_3).

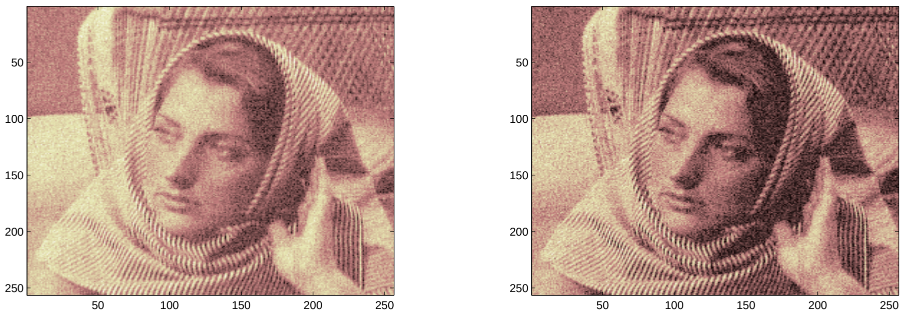

Figure 7: The denoising image by (u_1,u_2,u_3) and the denoising image by (k_1,k_2,k_3).

Where X is the original signal data and X' is the denoising signal data.

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References

[1] M. Bownik, A characterization of affine dual frames in $L^2(\mathbb{R}^n)$, Appl. Comput. Harmon. Anal., 8 (2000), 203–221.
[2] M. Bownik, Connectivity and density in the set of framelets, Math. Res. Lett., 14 (2007), 285–293.
[3] M. Bownik, The closure of the set of tight frame wavelets, Acta Appl. Math., 107 (2009), 195–201.
[4] M. Bownik, On Characterization of multiwavelets in $L^2(\mathbb{R}^n)$, Proc. Amer. Soc., 129(11) (2001), 3265–3274.
[5] B. Han, On dual wavelet tight frames, Appl. Comput. Harmon. Anal., 4 (1997), 380–413.
[6] B. Han, Wavelets, MSc Thesis at Institute of Mathematics, The Chinese Academy of Sciences, 1994.
[7] B. Han, Some applications of projection operators in wavelets, Acta Math. Sinica, 1 (1995), 105–112.
[8] M. Bownik, On the existence of multiresoltion analysis for framelets, Math. Ann., 332 (2005), 705–720.
[9] M. Bownik and Z. Rzeszotnik, The spectral function of shift-invariant spaces, Michigan Math. J., 51 (2003), 387–414.
[10] C. Chui, W. Czaja, M. Maggioni and G. Weiss, Characterization of general tight wavelet frames with matrix dilations and tightness preserving oversampling, J. Fourier Anal. Appl., 8 (2002), 173–200.
[11] X. Dai, Y. Diao and Q. Gu, Frame wavelet sets in $\mathbb{R}$, Proc. Amer. Math. Soc., 7(129) (2000), 2045–2055.
[12] X. Dai, Y. Diao, Q. Gu and D. Han, Frame wavelets in subspaces of $L^2(\mathbb{R}^d)$, Proc. Amer. Math. Soc., 130(11) (2002), 3259–3267.
[13] X. Dai and D. Larson, Wandering vectors for unitary systems and orthogonal wavelets, Memoirs. Amer. Math. Soc., 134(640) (1998).
[14] X. Dai, D. R. Larson and D. M. Speegle, Wavelet sets in $\mathbb{R}^n$, J. Fourier Anal. Appl., 3(2) (1997), 451–456.
[15] I. Daubechies, Ten lecture on wavelets, CBMS Lecture Notes, SIAM, 61 (1992).
[16] M. Frazier, G. Garrigós, K. Wang and G. Weiss, A characterization of functions that generate wavelet and related expansion, J. Fourier Anal. Appl., 3 (1997), 883–906.
[17] B. Han, On dual wavelet tight frames, Appl. Comput. Harmonic Anal., 4 (1997), 380–413.
[18] D. Han and D. R. Larson, Wandering vector multipliers for unitary groups, Trans. Amer. Math. Soc., 352(8) (2001), 3347–3370.
[19] D. Han and D. Larson, On the orthogonality of frames and the density and connectivity of wavelet frames, Acta Appl. Math., 107 (2009), 211–222.
[20] D. Han and D. Larson, Unitary Systems and Bessel Generator Multipliers, Wavelets and Multiscale Analysis–Theory and Applications, Springer, 2011.
[21] E. Hernándes and G. Weiss, A First Course on Wavelets, CRC Press, Boca Raton, FL, 1996.
[22] Y. Meyer, Wavelets and Operators, Cambridge Studies in Advanced Mathematics, 37, Cambridge University Press, Cambridge, 1992.
[23] Y. Li, On a class of bidimensional nonseparable wavelet multipliers, J. Math. Anal. Appl., 270 (2002), 543–560.
[24] Z. Y. Li, X. Dai, Y. Diao and J. Xin, Multipliers, phases and connectivity of MRA wavelets in $L^2(\mathbb{R}^2)$, J. Fourier Anal. Appl., 16 (2010), 155–176.

[25] Z. Y. Li, X. Dai, Y. Diao and W. Huang, The path-Connectivity of MRA Wavelets in $L^2(\mathbb{R}^d)$, Illinois J. Math., (54)(2) (2010), 601–620.

[26] Z. Y. Li and D. Han, *Matrix Fourier multipliers for Parseval multi-wavelet frames,* Appl. Comput. Harm. Anal., 35 (2013), 407–418.

[27] Z. Y. Li and X. L. Shi, Dyadic bivariate wavelet multipliers in $L^2(\mathbb{R}^2)$, Acta Math. Sinica, (27)(8) (2011), 500–512.

[28] M. Paluszyhski, Hrvoje Sikid, Guido Weiss and Shaoliang Xiao, Generalized low pass filters and MRA frame wavelets, The J. Geometric Anal., 11(2) (2001), 311–342.

[29] Z. Y. Li and X. L. Shi, Parseval frame wavelet multipliers in $L^2(\mathbb{R}^d)$, China Ann. Math., 33(6) (2012), 949–960.

[30] The Wutam Consortium, Basic properties of wavelets, J. Fourier Anal. Appl., 4(4) (1998), 575–594.