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On an Inequality of Pual Turan Concerning Polynomials-II

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Abstract. Let $P(z)$ be a polynomial of degree n and for any complex number α, let $D_\alpha P(z)=nP(z)+(\alpha-z)P'(z)$ denote the polar derivative of the polynomial $P(z)$ with respect to α. In this paper, we obtain inequalities for the polar derivative of a polynomial having all zeros inside a circle. Our results shall generalize and sharpen some well-known results of Turan, Govil, Dewan et al. and others.

Key Words: Polar derivative, polynomials, inequalities, maximum modulus, growth.

AMS Subject Classifications: 30A10, 30C10, 30C15

1 Introduction and statement of results

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative. Then according to the well-known Bernstein's inequality [4] on the derivative of a polynomial, we have

$$\max_{|z|=1}|P'(z)|\le n\max_{|z|=1}|P(z)|. \tag{1.1}$$

The equality holds in (1.1) if and only if $P(z)$ has all its zeros at the origin.

For the class of polynomials $P(z)$ having all zeros in $|z|\le 1$, Turan [11] proved that

$$\max_{|z|=1}|P'(z)|\ge \frac{n}{2}\max_{|z|=1}|P(z)|. \tag{1.2}$$

The inequality (1.2) is best possible and becomes equality for $P(z)=\alpha z^n+\beta$ where $|\alpha|=|\beta|$.

In the literature, there already exists some refinements and generalizations of the inequality (1.2), for example see Aziz and Dawood [3], Govil [5], Dewan and Mir [6], Dewan, Singh and Mir [7], Mir, Dar and Dawood [10] etc.

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Inequality (1.2) was refined by Aziz and Dawood [3] and they proved under the same hypothesis that

$$\max_{|z|=1}|P'(z)|\geq \frac{n}{2}\Big\{\max_{|z|=1}|P(z)|+\min_{|z|=1}|P(z)|\Big\}. \tag{1.3}$$

As an extension of (1.3), it was shown by Govil [5], that if $P(z)$ has all its zeros in $|z|\leq k$, $k\leq 1$, then

$$\max_{|z|=1}|P'(z)|\geq \frac{n}{1+k}\Big\{\max_{|z|=1}|P(z)|+\frac{1}{k^{n-1}}\min_{|z|=k}|P(z)|\Big\}. \tag{1.4}$$

For the class of polynomials

$$P(z)=a_n z^n+\sum_{\nu=\mu}^{n} a_{n-\nu}z^{n-\nu}, \quad 1\leq \mu\leq n,$$

of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, Aziz and Shah [2] proved

$$\max_{|z|=1}|P'(z)|\geq \frac{n}{1+k^{\mu}}\Big\{\max_{|z|=1}|P(z)|+\frac{1}{k^{n-\mu}}\min_{|z|=k}|P(z)|\Big\}. \tag{1.5}$$

For $\mu=1$, inequality (1.5) reduces to (1.4).

Let $D_\alpha P(z)$ denote the polar derivative of the polynomial $P(z)$ of degree n with respect to α, then

$$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z).$$

Recently Dewan, Singh and Mir [7] besides proving some other results, also proved the following interesting generalization of (1.5).

Theorem 1.1. *If*

$$P(z)=a_n z^n+\sum_{\nu=\mu}^{n} a_{n-\nu}z^{n-\nu}, \quad 1\leq \mu\leq n,$$

is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, and δ is any complex number with $|\delta|\leq 1$, then for $|z|=1$,

$$|D_\delta P(z)|\leq n\Big(\frac{k^{\mu}+|\delta|}{1+k^{\mu}}\Big)\max_{|z|=1}|P(z)|-n\Big(\frac{1-|\delta|}{k^{n-\mu}(1+k^{\mu})}\Big)\min_{|z|=k}|P(z)|. \tag{1.6}$$

In this paper, we shall first prove a result which gives certain generalizations of the inequality (1.4) by considering polynomials having all zeros in $|z|\leq k$, $k\leq 1$ with s-fold zeros at $z=0$. We shall also present a refinement of Theorem 1.1. We first prove the following result.

Theorem 1.2. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k\le 1$ with s-fold zeros at $z=0$, then for $\alpha,\beta\in\mathbb{C}$ with $|\alpha|\ge k$, and $|\beta|\le 1$,*

$$\begin{aligned}&\min_{|z|=1}\Big|zD_\alpha P(z)+\frac{\beta(n+sk)(|\alpha|-k)}{1+k}P(z)\Big|\\ \ge&\frac{|z|^n}{k^n}\Big|n\alpha+\frac{\beta(n+sk)(|\alpha|-k)}{1+k}\Big|\min_{|z|=k}|P(z)| \quad \textit{for} \ \ |z|\ge 1. \end{aligned} \tag{1.7}$$

Remark 1.1. According to the Lemma 2.1, we have for $|z|=1$,

$$|zD_\alpha P(z)|\ge\frac{(|\alpha|-k)(n+sk)}{1+k}|P(z)|,$$

then for suitable argument of β, we have

$$\Big|zD_\alpha P(z)+\frac{\beta(n+sk)(|\alpha|-k)}{1+k}P(z)\Big|=|zD_\alpha P(z)|-\frac{|\beta|(n+sk)(|\alpha|-k)}{1+k}|P(z)|. \tag{1.8}$$

For this choice of β, we have from (1.7) and (1.8) that for $|z|=1$,

$$\begin{aligned}&|zD_\alpha P(z)|-\frac{|\beta|(n+sk)(|\alpha|-k)}{1+k}|P(z)|\\ =&\Big|zD_\alpha P(z)+\frac{\beta(n+sk)(|\alpha|-k)}{1+k}P(z)\Big|\\ \ge&\min_{|z|=1}\Big|zD_\alpha P(z)+\frac{\beta(n+sk)(|\alpha|-k)}{1+k}P(z)\Big|\\ \ge&\frac{1}{k^n}\Big|n\alpha+\frac{\beta(n+sk)(|\alpha|-k)}{1+k}\Big|\min_{|z|=k}|P(z)|\\ \ge&\frac{1}{k^n}\Big\{n|\alpha|-\frac{|\beta|(n+sk)(|\alpha|-k)}{1+k}\Big\}\min_{|z|=k}|P(z)|.\end{aligned}$$

Equivalently

$$|zD_\alpha P(z)|\ge\frac{|\beta|(n+sk)(|\alpha|-k)}{1+k}|P(z)|+\frac{1}{k^n}\Big\{n|\alpha|-\frac{|\beta|(n+sk)(|\alpha|-k)}{1+k}\Big\}\min_{|z|=k}|P(z)|, \tag{1.9}$$

for $|z|=1$, $|\beta|\le 1$ and $|\alpha|\ge k$. Making $|\beta|\to 1$ in (1.9), we get the following

Corollary 1.1. If $P(z)$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k\le 1$, with s-fold zeros at $z=0$, then for every complex α with $|\alpha|\ge k$ and $|z|=1$,

$$|D_\alpha P(z)|\ge\frac{(n+sk)(|\alpha|-k)}{1+k}\max_{|z|=1}|P(z)|+\frac{(n-s)|\alpha|+(n+sk)}{(1+k)k^{n-1}}\min_{|z|=k}|P(z)|. \tag{1.10}$$

Remark 1.2. Dividing both sides of (1.10) by $|\alpha|$ and let $|\alpha|\to\infty$ and take $s=0$, we get (1.4). For $k=1$ and $s=0$, Theorem 1.2 reduces to a result of Liman, Mohapatra and Shah [8].

Finally, we prove the following refinement of Theorem 1.1.

Theorem 1.3. *If*

$$P(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu},\quad 1\le\mu\le n,$$

is a polynomial of degree n having all its zeros in $0<|z|\le k$, $k\le 1$ and γ is any complex number with $|\gamma|\le 1$, then

$$\max_{|z|=1}|D_\gamma P(z)|\le\frac{n(A_\mu+|\gamma|)}{1+A_\mu}\max_{|z|=1}|P(z)|-\frac{n(1-|\gamma|)A_\mu}{(1+A_\mu)k^n}m, \tag{1.11}$$

where

$$A_\mu=\frac{n(|a_n|-\frac{m}{k^n})k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n(|a_n|-\frac{m}{k^n})k^{\mu-1}+\mu|a_{n-\mu}|} \tag{1.12}$$

and $m=\min_{|z|=k}|P(z)|$.

Remark 1.3. Since by Lemma 2.4, we have $A_\mu\le k^\mu$, $1\le\mu\le n$. Also when $P(z)$ has all its zeros in $|z|\le k$, $k\le 1$, it is easy to verify, for example by the derivative test and Lemma 2.5, that for every α with $|\alpha|\le 1$, the function

$$\frac{n(x+|\alpha|)}{1+x}\max_{|z|=1}|P(z)|-\frac{n(1-|\alpha|)x}{k^n(1+x)}m$$

is a non-decreasing in x. Hence Theorem 1.3 is a refinement of Theorem 1.1.

Remark 1.4. If we take $\gamma=0$ in (1.11), we get for $|z|=1$,

$$\big|nP(z)-zP'(z)\big|\le\frac{nA_\mu}{1+A_\mu}\Big\{\max_{|z|=1}|P(z)|-\frac{m}{k^n}\Big\}. \tag{1.13}$$

If $\max_{|z|=1}|P(z)|=|P(e^{i\phi})|$, $0\le\phi<2\pi$, we get (1.13), that

$$|P'(e^{i\phi})|\ge\Big(\frac{n}{1+A_\mu}\Big)\max_{|z|=1}|P(z)|+\frac{nA_\mu}{k^n(1+A_\mu)}m. \tag{1.14}$$

Since $\max_{|z|=1}|P'(z)|\ge|P'(e^{i\phi})|$, $0\le\phi<2\pi$, then from (1.14), we immediately get a result of Mir, Dar and Dawood [10].

2 Lemmas

We need the following lemmas to prove our theorems.

Lemma 2.1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k\le 1$, with s-fold zeros at the origin, then for every complex α with $|\alpha|\ge k$, we have for $|z|=1$,*

$$|D_\alpha P(z)|\ge \frac{(|\alpha|-k)(n+ks)}{1+k}\max_{|z|=1}|P(z)|, \tag{2.1}$$

where $0\le s\le n$.

The above lemma is due to Dewan and Mir [6].

Lemma 2.2. *If $P(z)$ is a polynomial of degree n and α is any non-zero complex number and $Q(z)=z^n\overline{P(1/\overline{z})}$, then*

$$|D_\alpha Q(z)|=|n\bar{\alpha}P(z)+(1-\bar{\alpha}z)P'(z)|=|\alpha||D_{\frac{1}{\bar{\alpha}}}P(z)| \quad \textit{for} \ \ |z|=1. \tag{2.2}$$

The above lemma is an implicit in Aziz [1]. The following three lemmas are due to Dewan, Singh and Mir [7].

Lemma 2.3. *If*

$$P(z)=a_0+\sum_{\nu=t}^{n}a_\nu z^\nu, \quad 1\le t\le n,$$

is a polynomial of degree n having no zeros in $|z|<k$, $k\ge 1$, then for every complex α with $|\alpha|\ge 1$,

$$\max_{|z|=1}|D_\alpha P(z)|\le \frac{n}{1+s_0}\Big\{(|\alpha|+s_0)\max_{|z|=1}|P(z)|-(|\alpha|-1)m\Big\}, \tag{2.3}$$

where

$$s_0=k^{t+1}\left(\frac{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|-m}k^{t-1}+1}{\left(\frac{t}{n}\right)\frac{|a_t|}{|a_0|-m}k^{t+1}+1}\right)$$

and $m=\min_{|z|=k}|P(z)|$.

Lemma 2.4. *If*

$$P(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu}, \quad 1\le \mu\le n,$$

is a polynomial of degree n having all zeros in $|z|\le k$, $k\le 1$, then

$$A_\mu=\frac{n(|a_n|-\frac{m}{k^n})k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n(|a_n|-\frac{m}{k^n})k^{\mu-1}+\mu|a_{n-\mu}|}\le k^\mu, \tag{2.4}$$

where $m=\min_{|z|=k}|P(z)|$.

Lemma 2.5. *If*

$$P(z)=\sum_{\nu=0}^{n}a_\nu z^\nu$$

is a polynomial of degree n having all its zeros in $|z|\leq k$, $k>0$, *then* $|Q(z)|\geq m/k^n$ *for* $|z|\leq 1/k$, *and in particular*

$$|a_n|>\frac{m}{k^n},$$

where $m=\min_{|z|=k}|P(z)|$ *and* $Q(z)=z^n\overline{P(1/\overline{z})}$.

3 Proof of theorems

Proof of Theorem 1.2. If $P(z)$ has a zero on $|z|=k$, then the theorem is trivial. Therefore, assume that $P(z)$ has all its zeros in $|z|<k$, $k\leq 1$. Let $m=\min_{|z|=k}|P(z)|$, then $m>0$ and hence for every complex number γ with $|\gamma|<1$, we have

$$|\frac{\gamma_m z^n}{k^n}|<|P(z)| \quad \text{for } |z|=k.$$

It follows by Rouche's theorem, that the polynomial

$$G(z)=P(z)-\frac{\gamma_m z^n}{k^n}$$

of degree n has all its zeros in $|z|<k$, $k\leq 1$. On applying Lemma 2.1 to $G(z)$, we have for every complex number α with $|\alpha|\geq k$ and $|z|=1$,

$$|zD_\alpha G(z)|\geq\frac{(n+ks)(|\alpha|-k)}{1+k}|G(z)|.$$

Equivalently

$$|zD_\alpha P(z)-\frac{\alpha\gamma_{mn}z^n}{k^n}|\geq\frac{(n+ks)(|\alpha|-k)}{1+k}\Big|P(z)-\frac{\gamma_m z^n}{k^n}\Big| \quad \text{for } |z|=1. \tag{3.1}$$

Since by Laguerre's theorem (see [9, pp. 52]), the polynomial

$$D_\alpha G(z)=D_\alpha P(z)-\frac{\alpha\gamma_{mn}z^{n-1}}{k^n},$$

has all zeros in $|z|<k$, $k\leq 1$, for every complex α with $|\alpha|\geq k$, therefore for every complex β with $|\beta|<1$, the polynomial

$$\begin{aligned}T(z)=&\Big\{zD_\alpha P(z)-\frac{\alpha\gamma_{mn}z^n}{k^n}\Big\}+\frac{\beta(n+ks)(|\alpha|-k)}{1+k}\Big\{P(z)-\frac{\gamma_m z^n}{k^n}\Big\}\\ &\Big\{zD_\alpha P(z)+\frac{\beta(n+ks)(|\alpha|-k)}{1+k}P(z)\Big\}-\frac{\gamma_m z^n}{k^n}\Big\{n\alpha+\frac{\beta(n+ks)(|\alpha|-k)}{1+k}\Big\}\\ &\neq 0 \quad \text{for } |z|\geq k.\end{aligned} \tag{3.2}$$

Since $k\le 1$, we have $T(z)\neq 0$ for $|z|\ge 1$ as well.

Now choosing the argument of γ in (3.2) suitably and letting $|\gamma|\to 1$, we get for $|z|\ge 1$ and $|\beta|<1$,

$$\Big|zD_\alpha P(z)+\frac{\beta(n+ks)(|\alpha|-k)}{1+k}P(z)\Big|\ge\Big|\frac{mz^n}{k^n}\Big\{n\alpha+\frac{\beta(n+ks)(|\alpha|-k)}{1+k}\Big\}\Big|,$$

or

$$\Big|zD_\alpha P(z)+\frac{\beta(n+ks)(|\alpha|-k)}{1+k}P(z)\Big|\ge\frac{|z|^n}{k^n}\Big|n\alpha+\frac{\beta(n+ks)(|\alpha|-k)}{1+k}\Big|\min_{|z|=k}|P(z)|.$$

For β with $|\beta|=1$, the above inequality holds by continuity. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Since

$$P(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu},\quad 1\le\mu\le n,$$

has all its zeros in $0<|z|\le k$, $k\le 1$, therefore the polynomial $Q(z)=z^n\overline{P(1/\overline{z})}$ has no zeros in $|z|<1/k$, $1/k\ge 1$. On applying Lemma 2.3 to $Q(z)$, we get for every complex number α with $|\alpha|\ge 1$ and $|z|=1$,

$$|D_\alpha Q(z)|\le\frac{n}{1+\psi_0}\Big\{(|\alpha|+\psi_0)\max_{|z|=1}|Q(z)|-(|\alpha|-1)m'\Big\},\tag{3.3}$$

where

$$m'=\min_{|z|=\frac{1}{k}}|Q(z)|=\frac{m}{k^n}$$

and

$$\begin{aligned}\psi_0&=\Big(\frac{1}{k}\Big)^{\mu+1}\left\{\frac{\big(\frac{\mu}{n}\big)\big(\frac{|a_{n-\mu}|}{|a_n|-m'}\big)\big(\frac{1}{k}\big)^{\mu-1}+1}{\big(\frac{\mu}{n}\big)\big(\frac{|a_{n-\mu}|}{|a_n|-m'}\big)\big(\frac{1}{k}\big)^{\mu+1}+1}\right\}\\&=\frac{\mu|a_{n-\mu}|+n\big(|a_n|-\frac{m}{k^n}\big)k^{\mu-1}}{n\big(|a_n|-\frac{m}{k^n}\big)k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}\\&=\frac{1}{A_\mu}.\end{aligned}$$

Hence from (3.3) it follows that for every α with $|\alpha|\ge 1$ and $|z|=1$,

$$\begin{aligned}|D_\alpha Q(z)|&\le\frac{n}{1+(\frac{1}{A_\mu})}\Big\{\Big(|\alpha|+\frac{1}{A_\mu}\Big)\max_{|z|=1}|P(z)|-(|\alpha|-1)\frac{m}{k^n}\Big\}\\&=\Big(\frac{nA_\mu}{1+A_\mu}\Big)\Big\{\frac{(|\alpha|A_\mu+1)}{A_\mu}\max_{|z|=1}|P(z)|-(|\alpha|-1)\frac{m}{k^n}\Big\}.\end{aligned}\tag{3.4}$$

Using (2.2) of Lemma 2.2 in (3.4), we get for $|\alpha|\geq 1$ and $|z|=1$,

$$|\alpha||D_{\frac{1}{\overline{\alpha}}}P(z)|\leq\frac{n(|\alpha|A_\mu+1)}{1+A_\mu}\max_{|z|=1}|P(z)|-\frac{nA_\mu(|\alpha|-1)m}{k^n(1+A_\mu)}. \tag{3.5}$$

Replacing $1/\overline{\alpha}$ by γ, we obtain for $|\gamma|\leq 1$ and $|z|=1$,

$$|D_\gamma P(z)|\leq\frac{n(A_\mu+|\gamma|)}{1+A_\mu}\max_{|z|=1}|P(z)|-\frac{nA_\mu(1-|\gamma|)m}{k^n(1+A_\mu)},$$

which is (1.11) and this completes the proof of Theorem 1.3. □

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