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Weak Type Weighted Inequalities for the Commutators of the Multilinear Calderón-Zygmund Operators

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Abstract. For the commutators of multilinear Calderón-Zygmund singular integral operators with BMO functions, the weak type weighted norm inequalities with respect to $A_{\vec{p}}$ weights are obtained.

Key Words: Commutator, multilinear Calderón-Zygmund operator, multilinear maximal function, weight.

AMS Subject Classifications: 42B20, 42B25

1 Introduction

The multilinear Calderón-Zygmund theory originated in the works of Coifman and Meyer [2,3]. Later on the topic was retaken by several authors; including Christ and Journé [1] Kenig and Stein [7], and Grafakos and Torres [5,6]. We first recall the definitions of multilinear Calderón-Zygmund singular integral operators and commutators.

Let T be a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$T:\mathcal{S}(\mathbf{R}^n)\times\cdots\times\mathcal{S}(\mathbf{R}^n)\longrightarrow\mathcal{S}'(\mathbf{R}^n).$$

We say that T is an multilinear Calderón-Zygmund operator if, for some $1\le q_j<\infty$, it extends to a bounded multilinear operator from $L^{q_1}\times\cdots\times L^{q_m}$ to L^q, where $1/q=1/q_1+$

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$\cdots+1/q_m$, and if there exists a function K, defined off the diagonal $x=y_1=\cdots=y_m$ in $(\mathbf{R}^n)^{m+1}$, for $\vec{f}=(f_1,\cdots,f_m)$, satisfying

$$T(\vec{f})(x)=\int_{(\mathbf{R}^n)^m}K(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)d\vec{y}$$

for all $x\notin\bigcap_{j=1}^m \text{supp} f_j$, where $d\vec{y}=dy_1\cdots dy_m$ and $\vec{y}=(y_1,\cdots,y_m)$;

$$|K(y_0,y_1,\cdots,y_m)|\le\frac{A}{(\sum_{k,j=0}^m|y_k-y_j|)^{mn}} \tag{1.1}$$

and

$$|K(y_0,\cdots,y_j,\cdots,y_m)-K(y_0,\cdots,y_j',\cdots,y_m)|\le\frac{A|y_j-y_j'|^\gamma}{(\sum_{k,j=0}^m|y_k-y_j|)^{mn+\gamma}} \tag{1.2}$$

for some $\gamma>0$ and all $0\le j\le m$, whenever $|y_j-y_j'|\le\frac{1}{2}\max_{0\le k\le m}|y_j-y_k|$.

For a vector $\vec{b}=(b_1,\cdots,b_m)$ of locally integrable functions, we define the commutator of multilinear singular integral operators

$$T_{\vec{b}}\vec{f}(x)=\sum_{j=1}^m T_{b_j}^j\vec{f}(x)=\sum_{j=1}^m\int_{\mathbb{R}^n}(b_j(x)-b_j(y_j))K(x,y_1,\cdots,y_m)f_1(y_1)\cdots f_m(y_m)d\vec{y}.$$

Recently, Lerner, Ombrosi, Pérez and Trujillo-González [8] defined a new multilinear maximal function associated to the m-linear Calderón-Zygmund operator as

$$\mathcal{M}(\vec{f})(x)=\sup_{Q\ni x}\prod_{j=1}^m\frac{1}{|Q|}\int_Q|f_j(y_j)|dy_j,$$

and developed a $A_{\vec{P}}$ weighted theory for the multilinear maximal function and multilinear Calderón-Zygmund operators.

Let $1\le p_1,\cdots,p_m<\infty$, we will write p for the number given by $1/p=1/p_1+\cdots+1/p_m$, and $\vec{P}=(p_1,\cdots,p_m)$. Given $\vec{w}=(w_1,\cdots,w_m)$, w_j are nonnegative locally integrable functions on $\mathbb{R}^n$, $j=1,\cdots,m$, set $v_{\vec{w}}=\prod_{j=1}^m w_j^{p/p_j}$. We say that $\vec{w}$ satisfies the $A_{\vec{P}}$ condition if

$$\sup_Q\Big(\frac{1}{|Q|}\int_Q v_{\vec{w}}\Big)^{1/p}\prod_{j=1}^m\Big(\frac{1}{|Q|}\int_Q w_j^{1-p_j'}\Big)^{1/p_j'}<\infty.$$

When $p_j=1$, $(\frac{1}{|Q|}\int_Q w_j^{1-p_j'})^{1/p_j'}$ is understood as $(\inf_Q w_j)^{-1}$. For $m=1$, $A_{\vec{P}}$ is the Muckenhoupt weight class A_p. We denote $A_\infty=\cup_{p>1}A_p$.

Lerner, Ombrosi, Pérez and Trujillo-González [8] obtained the following weighted inequalities for the commutators of the multilinear singular integral operators.

Theorem 1.1. *Let* $1<p_j<\infty$, $j=1,\cdots,m$, $\vec{b}\in BMO^m$ *and* $\vec{w}\in A_{\vec{P}}$. *Then there exists* $C>0$ *such that*

$$\|T_{\vec{b}}(\vec{f})\|_{L^p(v_{\vec{w}})}\le C\prod_{j=1}^m\|f_j\|_{L^{p_j}(w_j)}.$$

Theorem 1.2. *Let* $\vec{w}\in A_{(1,\cdots,1)}$, $\vec{b}\in BMO^m$ *and* $\Phi(t)=t(1+\log^+t)$, $t>0$. *Then there exists* $C>0$ *such that*

$$v_{\vec{w}}(\{x\in\mathbb{R}^n:|T_{\vec{b}}(\vec{f})(x)|>t^m\})\le C\prod_{j=1}^m\Big(\int_{\mathbb{R}^n}\Phi\Big(\frac{|f_j(x)|}{t}\Big)w_j(x)dx\Big)^{1/m}.$$

In this paper, we consider the weak type weighted inequalities for the commutators of the multilinear singular integral operators when $1\le p_j<\infty$, $j=1,\cdots,m$, and at least one of the $p_j=1$. To state our results, we need some notation.

Let $\Psi(t):[0,\infty)\to[0,\infty)$ be a Young function. That is a continuous, convex, increasing function with $\Psi(0)=0$ and such that $\Psi(t)\to\infty$ as $t\to\infty$. The Luxemburg norm of a function f over a cube Q is defined by

$$\|f\|_{\Psi,Q}=\inf\Big\{\lambda>0:\frac{1}{|Q|}\int_Q\Psi\Big(\frac{|f(x)|}{\lambda}\Big)dx\le 1\Big\}.$$

Especially, for the Young function $\Phi(t)=t(1+\log^+t)$, the Luxemburg norm will be denoted by $\|f\|_{L(\log L),Q}$.

For $i=1,\cdots,m$, the multilinear maximal operators associated with Φ are defined as

$$\mathcal{M}^i_{L(\log L)}(\vec{f})(x)=\sup_{Q\ni x}\|f_i\|_{L(\log L),Q}\prod_{j\neq i}\frac{1}{|Q|}\int_Q|f_j(y_j)|dy_j.$$

Our main results can be state as follows.

Theorem 1.3. *Let* $1\le p_j<\infty$, $j=1,\cdots,m$, $\vec{b}\in BMO^m$ *and* $\vec{w}\in A_{\vec{P}}$. *Then there exists a constant* $C>0$ *such that*

$$v_{\vec{w}}(\{x\in\mathbb{R}^n:|T_{\vec{b}}(\vec{f})(x)|>t^m\})^{1/p}\le C\prod_{j=1}^m\Big(\int_{\mathbb{R}^n}\Phi\Big(\frac{|f_j(x)|}{t}\Big)^{p_j}w_j(x)dx\Big)^{1/p_j}\tag{1.3}$$

for all $\vec{f}$ *bounded with compact support.*

The proof of Theorem 1.3 will be based on the following result.

Theorem 1.4. *Let* $1\le p_j<\infty$, $j=1,\cdots,m$, *and* $\vec{w}\in A_{\vec{P}}$. *Then there exists a constant* $C>0$ *such that*

$$\begin{aligned}&v_{\vec{w}}(\{x\in\mathbb{R}^n:\mathcal{M}^i_{L\log L}(\vec{f})(x)>t^m\})^{1/p}\\&\le C\Big(\int_{\mathbb{R}^n}\Phi\Big(\frac{|f_i(x)|}{t}\Big)^{p_i}w_i(x)dx\Big)^{1/p_i}\prod_{j\neq i}\Big(\int_{\mathbb{R}^n}\Big(\frac{|f_j(x)|}{t}\Big)^{p_j}w_j(x)dx\Big)^{1/p_j}\end{aligned}\tag{1.4}$$

for $i=1,\cdots,m$.

2 Preliminaries

For $\delta>0$, let M_δ be the maximal function

$$M_\delta f(x)=\sup_{x\in Q}\Big(\frac{1}{|Q|}\int_Q |f(y)|^\delta dy\Big)^{\frac{1}{\delta}},$$

and $M^\sharp_\delta$ be the sharp maximal function of Fefferman and Stein,

$$M^\sharp_\delta f(x)=\sup_{x\in Q}\inf_c\Big(\frac{1}{|Q|}\int_Q \big||f(y)|^\delta-|c|^\delta\big|dy\Big)^{\frac{1}{\delta}}.$$

If $\delta=1$, M_1 and $M_1^\#$ are the Hardy-Littlewood maximal function and sharp maximal function, write as M and $M^\#$ respectively.

We will use the following form of the classical result of Fefferman and Stein [4]: Let $0<\delta<\infty$ and w be a weight in A_∞, if $\varphi:(0,\infty)\to(0,\infty)$ is doubling, then there exists a constant $C>0$, such that

$$\sup_{\lambda>0}\varphi(\lambda)w(\{y\in\mathbb{R}^n:M_\delta f(y)>\lambda\})\le C\sup_{\lambda>0}\varphi(\lambda)w(\{y\in\mathbb{R}^n:M^\sharp_\delta f(y)>\lambda\}) \tag{2.1}$$

for every function f such that the left-hand side is finite.

Lemma 2.1 (see [8])**.** *Let $\vec{w}=(w_1,\cdots,w_m)$ and $1\le p_1,\cdots,p_m<\infty$. Then $\vec{w}\in A_{\vec{P}}$ if and only if $w_j^{1-p'_j}\in A_{mp'_j}$, $j=1,\cdots,m$, and $v_{\vec{w}}\in A_{mp}$, where the condition $w_j^{1-p'_j}\in A_{mp'_j}$ in the case $p_j=1$ is understood as $w_j^{1/m}\in A_1$.*

Lemma 2.2 (see [8])**.** *Let $0<\delta\le 1/m$. Then there exists a constant $C>0$ such that*

$$M^\sharp_\delta(T(\vec{f}))(x)\le C\mathcal{M}(\vec{f})(x)$$

for all $\vec{f}$ of bounded measurable functions with compact support.

Lemma 2.3 (see [8])**.** *Let $\vec{b}\in BMO^m$, $0<\delta\le 1/m$, $\delta<\varepsilon$ and $j=1,\cdots,m$, then there exists a constant $C>0$ such that*

$$M^\sharp_\delta(T^j_{b_j})(\vec{f}))(x)\le C\|b_j\|_{BMO}(\mathcal{M}^j_{L(\log L)}(\vec{f})(x)+M_\varepsilon(T(\vec{f}))(x)) \tag{2.2}$$

for all $\vec{f}$ of bounded measurable functions with compact support.

Lemma 2.4 (see [8])**.** *Let $1<p_1,\cdots,p_m<\infty$. Then*

$$\Big(\int_{\mathbb{R}^n}\mathcal{M}(\vec{f})(x)^p v_{\vec{w}}(x)dx\Big)^{1/p}\le C\prod_{j=1}^m\Big(\int_{\mathbb{R}^n}|f_j(x)|^{p_j}w_j(x)dx\Big)^{1/p_j}$$

if and only if $\vec{w}\in A_{\vec{P}}$.

Theorem 2.1. *Let $1\le p_j<\infty$, $j=1,\cdots,m$, $w\in A_\infty$ and $\varphi(t)=\Phi(t)t^{m-1}$. Suppose that $\vec{b}\in BMO^m$. Then there exists a constant $C>0$, depending on the A_∞ constant of w, such that for $j=1,\cdots,m$,*

$$\begin{aligned}&\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:|T^j_{b_j}(\vec{f})(x)|>t^m\})^{1/p}\\ \le& C\varphi(\|b\|^{1/m}_{BMO})\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:\mathcal{M}^j_{L(\log L)}(\vec{f})(x)>t^m\})^{1/p}\end{aligned}\tag{2.3}$$

for all $\vec{f}$ bounded with compact support.

Proof. It is enough to consider $T_b(\vec{f})(x)=b(x)T(f_1,\cdots,f_m)(x)-T(bf_1,\cdots,f_m)(x)$ with $b\in BMO$. We can assume that the right-hand side of (2.3) is finite. It is easy to see that $\frac{1}{\varphi(1/t)}$ is doubling. By Lebesgue differentiation theorem, Fefferman-Stein inequality (2.1), Lemma 2.2 and Lemma 2.3 with exponents $0<\delta<\varepsilon<1/m$, we have

$$\begin{aligned}&\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:|T_b(\vec{f})(x)|>t^m\})^{1/p}\\ \le&\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:M_\delta(T_b(\vec{f}))(x)>t^m\})^{1/p}\\ \le& C\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:M^\sharp_\delta(T_b(\vec{f}))(x)>t^m\})^{1/p}\\ \le& C\varphi(\|b\|^{1/m}_{BMO})\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:\mathcal{M}^1_{L(\log L)}(\vec{f})(x)>t^m\})^{1/p}\\ &+C\varphi(\|b\|^{1/m}_{BMO})\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:M_\varepsilon(T(\vec{f})(x))>t^m\})^{1/p}\\ \le& C\varphi(\|b\|^{1/m}_{BMO})\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:\mathcal{M}^1_{L(\log L)}(\vec{f})(x)>t^m\})^{1/p}\\ &+C\varphi(\|b\|^{1/m}_{BMO})\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:M^\sharp_\varepsilon(T(\vec{f})(x))>t^m\})^{1/p}\\ \le& C\varphi(\|b\|^{1/m}_{BMO})\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:\mathcal{M}^1_{L(\log L)}(\vec{f})(x)>t^m\})^{1/p}\\ &+C\varphi(\|b\|^{1/m}_{BMO})\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:\mathcal{M}(\vec{f})(x)>t^m\})^{1/p}\\ \le& C\varphi(\|b\|^{1/m}_{BMO})\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:\mathcal{M}^1_{L(\log L)}(\vec{f})(x)>t^m\})^{1/p}.\end{aligned}$$

We need to verify now that

$$\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:M_\delta(T_b(\vec{f}))(x)>t^m\})^{1/p}<\infty,\tag{2.4}$$

and

$$\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:M_\varepsilon(T(\vec{f}))(x)>t^m\})^{1/p}<\infty. \tag{2.5}$$

We will only show (2.4) because the proof of (2.5) is very similar but easier. We may assume that ω is bounded. Note that $\omega_j=\min\{\omega,j\}\to\omega$ as $j\to\infty$, a.e. on $\mathbb{R}^n$, $\omega_j\in L^\infty$ and $[\omega_j]_{A_\infty}\le 2[\omega]_{A_\infty}$. The result for general ω will follow then by applying the Monotone Convergence Theorem.

Notice that $t^m\varphi(1/t)\ge 1$, $m\delta<1$ and the fact $M_{m\delta}:L^{mp,\infty}(\mathbb{R}^n)\to L^{mp,\infty}(\mathbb{R}^n)$, we have

$$\begin{aligned}&\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:M_\delta(T_b(\vec{f}))(x)>t^m\})^{1/p}\\ \le& C\sup_{t>0}t^m|\{x\in\mathbb{R}^n:M_{m\delta}(|T_b(\vec{f})|^{1/m})(x)>t\}|^{1/p}\\ \le& C\sup_{t>0}t^m|\{x\in\mathbb{R}^n:|T_b(\vec{f})(x)|^{1/m}>t\}|^{1/p}.\end{aligned}$$

Recalling that $\vec{f}$ has compact support, we may assume that $\text{supp}\vec{f}\subset B(0,R)$ for some $R>0$. Then

$$\begin{aligned}&\sup_{t>0}t^m|\{x\in\mathbb{R}^n:|T_b(\vec{f})(x)|^{1/m}>t\}|^{1/p}\\ \le& C\sup_{t>0}t^m|\{x\in B(0,2R):|T_b(\vec{f})(x)|^{1/m}>t\}|^{1/p}\\ &+C\sup_{t>0}t^m|\{x\in\mathbb{R}^n\setminus B(0,2R):|T_b(\vec{f})(x)|^{1/m}>t\}|^{1/p}=I+II.\end{aligned}$$

For I, we have

$$I\le C\int_{B(0,2R)}|T_b(\vec{f})(x)|^{1/p}dx\le CR^{(1-1/r)n}\Big(\int_{\mathbb{R}^n}|T_b(\vec{f})(x)|^{r/p}dx\Big)^{1/r}.$$

This last term is finite by the result in Theorem 1.1 if we choose r sufficiently large.

For II, if we assume that b is bounded, then for $x\in\mathbb{R}^n\setminus B(0,2R)$,

$$\begin{aligned}|T_b(\vec{f})(x)|&\le C\int_{(\mathbb{R}^n)^m}\frac{|b(x)-b(y_1)||f_1(y_1)|\cdots|f_m(y_m)|}{|x-y_1|^n\cdots|x-y_m|^n}d\vec{y}\\ &\le C\frac{1}{|x|^n}\int_{B(0,|x|)}|f_1(y)|dy\cdots\frac{1}{|x|^n}\int_{B(0,|x|)}|f_m(y)|dy\\ &\le C\mathcal{M}(\vec{f})(x).\end{aligned}$$

Thus, by Lemma 2.4, we have

$$\begin{aligned}II&\le C\sup_{t>0}t^m|\{x\in\mathbb{R}^n:\mathcal{M}(\vec{f})(x)>t^m\}|^{1/p}\\ &=C\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}}\le C\prod_{j=1}^m\|f_j\|_{L^{p_j}}<\infty,\end{aligned}$$

since the family $\vec{f}$ is bounded with compact support.

This proves (2.3) provided b is bounded. To obtain the result for a general b in BMO, we consider the sequence of functions $\{b_j\}$ given by

$$b_j(x)=\begin{cases} j, & b(x)>j, \\ b(x), & |b(x)|\le j, \\ -j, & b(x)<-j. \end{cases}$$

Note that the sequence converges pointwise to b and $\|b_j\|_{BMO}\le c\|b\|_{BMO}$. Thus

$$\begin{aligned}&\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:|T_b(\vec{f})(x)|>t^m\})^{1/p}\\ \le& C\varphi(\|b\|_{BMO}^{1/m})\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:\mathcal{M}^1_{L(\log L)}(\vec{f})(x)>t^m\})^{1/p},\end{aligned}$$

in which the constant C depending on the BMO norm of b. Since the family $\vec{f}$ is bounded with compact support and $T:L^{p_1}\times\cdots\times L^{p_m}\to L^{p,\infty}$, we have

$$\|T(b_jf_1,f_2,\cdots,f_m)-T(bf_1,f_2,\cdots,f_m)\|_{L^{p,\infty}}\to 0,\quad j\to\infty.$$

Thus for each compact set, an appropriate subsequence of $\{T_{b_j}(\vec{f})\}$ converges to $T_b(\vec{f})$ in measure. Hence for any $K>0$,

$$\begin{aligned}&\sup_{t>0}\frac{1}{\varphi(1/t)}w\big(\{x\in B(0,K):|T_b(\vec{f})(x)|>t^m\}\big)^{1/p}\\ \le& C\varphi(\|b\|_{BMO}^{1/m})\sup_{t>0}\frac{1}{\varphi(1/t)}w(\{x\in\mathbb{R}^n:\mathcal{M}^1_{L(\log L)}(\vec{f})(x)>t^m\})^{1/p},\end{aligned}$$

in which the constant C is independent of K. Finally taking the supremum over K completes the proof of the theorem. □

3 Proofs of Theorems

Proof of Theorem 1.4. Without loss of generality we may assume $i=1$. By homogeneity, we can assume that $t=1$ and $\vec{f}\ge 0$ bounded with compact support and satisfies $0<\|f_j\|_\infty\le 1$ for $j=2,\cdots,m$. Define the set

$$\Omega=\{x\in\mathbb{R}^n:\mathcal{M}^1_{L\log L}(\vec{f})(x)>1\}.$$

It is easy to see that Ω is open and we may assume that it is not empty. to estimate the size of Ω, it is enough to estimate the size of every compact set F contained in Ω. We can cover any such F by a finite family of cubes $\{Q_k\}$ for which

$$1<\|f_1\|_{\Phi,Q_k}\prod_{j=2}^m(f_j)_{Q_k},$$

where

$$(f_j)_{Q_k}=\frac{1}{|Q_k|}\int_{Q_k}f_j(y)dy.$$

Using Vitali's covering lemma, we can extract a subfamily of disjoint cubes $\{Q_i\}$ such that $F\subset\bigcup_i 3Q_i$. By homogeneity, we have

$$1<\|f_1\prod_{j=2}^m (f_j)_{Q_i}\|_{\Phi,Q_i}.$$

By the properties of the norm $\|\cdot\|_{\Phi,Q_i}$, using that $\Phi(at)\le a\Phi(t)$ for $0<a\le 1$, we have

$$1<\frac{1}{|Q_i|}\int_{Q_i}\Phi\Big(f_1\prod_{j=2}^m (f_j)_{Q_i}\Big)dy\le\frac{1}{|Q_i|}\int_{Q_i}\Phi(f_1(y))dy\prod_{j=2}^m\frac{1}{|Q_i|}\int_{Q_i}f_j(y)dy.$$

Finally by the condition on weights and Hölder's inequality, we have

$$\begin{aligned}
v_{\vec{w}}(F)\le& C\sum_i v_{\vec{w}}(Q_i)\le C\sum_i\int_{Q_i}v_{\vec{w}}(y)dy\Big(\frac{1}{|Q_i|}\int_{Q_i}\Phi(f_1(y))dy\prod_{j=2}^m\frac{1}{|Q_i|}\int_{Q_i}f_j(y)dy\Big)^p\\
\le& C\sum_i\int_{Q_i}v_{\vec{w}}(y)dy\Big(\frac{1}{|Q_i|}\int_{Q_i}\Phi(f_1(y))^{p_1}w_1(y)dy\Big)^{p/p_1}\Big(\frac{1}{|Q_i|}\int_{Q_i}w_1(y)^{1-p_1'}dy\Big)^{p/p_1'}\\
&\times\prod_{j=2}^m\Big(\frac{1}{|Q_i|}\int_{Q_i}f_j(y)^{p_j}w_j(y)dy\Big)^{p/p_j}\Big(\frac{1}{|Q_i|}\int_{Q_i}w_j(y)^{1-p_j'}dy\Big)^{p/p_j'}\\
\le& C\sum_i\Big(\int_{Q_i}\Phi(f_1(y))^{p_1}w_1(y)dy\Big)^{p/p_1}\prod_{j=2}^m\Big(\int_{Q_i}f_j(y)^{p_j}w_j(y)dy\Big)^{p/p_j}\\
\le& C\int_{\mathbb{R}^n}\Phi(f_1(y))^{p_1}w_1(y)dy\Big)^{p/p_1}\prod_{j=2}^m\Big(\int_{\mathbb{R}^n}f_j(y)^{p_j}w_j(y)dy\Big)^{p/p_j},
\end{aligned}$$

where $(\frac{1}{|Q|}\int_Q w_j^{1-p_j'})^{1/p_j'}$ is understood as $(\inf_Q w_j)^{-1}$ if $p_j=1$. This ends the proof. □

Proof of Theorem 1.3. By linearity it is enough to consider the operator T_b. By homogeneity it is enough to assume $t=1$ and to prove

$$\begin{aligned}
&v_{\vec{w}}(\{x\in\mathbb{R}^n:|T_b(\vec{f})(x)|>1\})^{1/p}\\
\le& C\Big(\int_{\mathbb{R}^n}\Phi\Big(|f_1(x)|\Big)^{p_1}w_1(x)dx\Big)^{1/p_1}\prod_{j=2}^m\Big(\int_{\mathbb{R}^n}|f_j(x)|^{p_j}w_j(x)dx\Big)^{1/p_j}.
\end{aligned}$$

Since Φ is submultiplicative, by Theorem 2.1 and Theorem 1.4, we have

$$\begin{aligned}
&v_{\vec{w}}(\{x\in\mathbb{R}^n:|T_b(\vec{f})(x)|>1\})^{1/p}\\
\le& C\sup_{t>0}\frac{t^{m-1}}{\Phi(1/t)}v_{\vec{w}}(\{x\in\mathbb{R}^n:|T_b(\vec{f})(x)|>t^m\})^{1/p}\\
\le& C\sup_{t>0}\frac{t^{m-1}}{\Phi(1/t)}v_{\vec{w}}(\{x\in\mathbb{R}^n:|\mathcal{M}^1_{L(\log L)}(\vec{f})(x)|>t^m\})^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&\le C\sup_{t>0}\frac{t^{m-1}}{\Phi(1/t)}\Big(\int_{\mathbb{R}^n}\Phi\Big(\frac{|f_1(x)|}{t}\Big)^{p_1}w_1(x)dx\Big)^{1/p_1}\prod_{j=2}^m\Big(\int_{\mathbb{R}^n}\Big(\frac{|f_j(x)|}{t}\Big)^{p_j}w_j(x)dx\Big)^{1/p_j}\\
&\le C\sup_{t>0}\frac{1}{\Phi(1/t)}\Big(\int_{\mathbb{R}^n}\Phi(|f_1(x)|)^{p_1}\Phi(1/t)^{p_1}w_1(x)dx\Big)^{1/p_1}\prod_{j=2}^m\Big(\int_{\mathbb{R}^n}|f_j(x)|^{p_j}w_j(x)dx\Big)^{1/p_j}\\
&\le C\Big(\int_{\mathbb{R}^n}\Phi(|f_1(x)|)^{p_1}w_1(x)dx\Big)^{1/p_1}\prod_{j=2}^m\Big(\int_{\mathbb{R}^n}|f_j(x)|^{p_j}w_j(x)dx\Big)^{1/p_j}.
\end{aligned}
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This ends the proof of Theorem 1.3.

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